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Analytical Solutions for Confined Aquifers with non constant Pumping using Computer Algebra

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Abstract: - Using Computer Algebra Software (CAS) certain work about symbolic computational hydrogeology is realized. The explicit solutions for the boundary value problems that correspond to a confined aquifer with various forms of non constant pumping rates, are derived. Many different cases are considered and the corresponding solutions are given in terms of Whittaker functions which are generalizations of the usual Theis well function. The method of solution is the Laplace Transform Technique. This method is implemented at fully using only CAS. The solutions are obtained by means of certain algorithm. As a result we obtain the explicit forms of the drawdown profiles for the various pumping regimes that were considered. and also we do some exercises with our solutions applying the notion of image well. From our results is possible to derive new protocols for estimating aquifer properties.

Key-Words: - Symbolic Computational Hydrology, Confined Aquifers, Pumping Tests, Laplace Transform. Whittaker functions, Computer Algebra.

1. Introduction

Mathematical Hydrology is a source of very interesting computational problems at such way that is possible to speak about a new emergent scientific discipline named Computational Hydrology (CH). The CH have two faces. The first face is the Numeric Computational Hydrology (NCH) which refers to hydrologic problems that are solved using numerical analysis and software for numerical computations (Matlab). The second face is named Symbolic Computational Hydrology (SCH) and consists in to obtain analytical solutions for hydrologic problems using mathematical analysis implemented by Computer Algebra Software (CAS)(Mathematica, Maple) [1,2]. The NCH is actually intensively studied but the SCH is a domain that remains practically unexplored. The object of the present work is to explore the lands of the SCH. We intend to present, some examples of problems on SCH. Such problems are concerned with the computation of the analytical solutions for the case of a confined aquifer with a well which is working according with a pumping rate variable on time. We obtain some generalizations of the well known Theis solution [3]. Such generalizations are given in terms of Whittaker functions [4].

The mathematical method that is used in this work is the Laplace Transform Technique fully implemented using CAS (Maple). It is verified that Maple is a very powerful tool for the development of the SCH.

2 The Mathematical Problem

We consider a confined aquifer with a well pumping at a rate that is changing with the time according to an arbitrary function. This configuration is a generalization of the configuration originally studied and solved by Theis for the case of constant pumping rate [3]. We assume as valid the same assumptions originally introduced by Theis, with the only difference that here the pumping rate is an arbitrary function of time. The mathematical model that was used by Theis is a diffusion equation with a sink of a form of Dirac delta function, that represents a well of infinitesimal radius. Such diffusion equation with Dirac term is complemented with adequate initial conditions without other boundary conditions. But the Dirac term can be extracted from the equation and can be converted in a new boundary condition and such way that the mathematical model to consider here is the following general boundary value problem [5,6]

$$\left(\frac{\partial}{\partial t}S(r,t)\right)-\frac{k\left(\left(\frac{\partial}{\partial r}S(r,t)\right)+r\left(\frac{\partial^2}{\partial r^2}S(r,t)\right)\right)}{r}=0 \qquad (1)$$

$$S(r,0)=0 \qquad (2)$$

$$\lim_{r\to\infty} S(r,t)=0 \qquad (3)$$

$$\lim_{r\to 0} 2\,\pi\,k_1\,r\left(\frac{\partial}{\partial r}S(r,t)\right)=-Q(t) \qquad (4)$$

where k is the hydraulic diffusivity, S(r,t) is the observed drawdown at a distance *r* from the pumping well, k_1 is the aquifer transmissivity and Q(t) gives the variation on time of the pumping rate. The equation (1) is the diffusion equation for the drawdown, the equation (2) represents the initial condition of a non perturbed aquifer, the equation (3) gives the natural condition at infinite and the equation (4) is the condition for a extracting well of very small radius with a pumping rate Q(t).

We consider here a general form for Q(t), namely

$$Q(t)=\sum_{n=0}^{\infty} Q_n\,t^n \qquad (5)$$

where Q_n are constants. Some concrete examples that we want to explore here are:

$$Q(t)=Q \qquad (6)$$

$$Q(t)=Q\,t \qquad (7)$$

$$Q(t)=Q\,t^2 \qquad (8)$$

$$Q(t)=Q\,t^n \qquad (9)$$

$$Q(t)=Q\,e^{(\varepsilon\,t)} \qquad (10)$$

$$Q(t)=Q\,t\,e^{(\varepsilon\,t)} \qquad (11)$$

$$Q(t)=Q\,t^m\,e^{(\varepsilon\,t)} \qquad (12)$$

The case of (6) is the original problem considered by Theis [3]. The case (10) with ε = i w, was studied in [5].

Then the mathematical problem that is proposed and solved in this work consist in to obtain the analytical solution of the equations (1)-(4) with the specifications (5)-(12). The Theis solution is re-derived and alternative formulas respect to that are given at [5,6], are obtained for the case of sinusoidal pumping rate.

3 Problem Solution

The boundary value problem (1)-(4) with the specifications (5)-(12) is a linear problem with computable analytical solution [6]. But such solution can not be obtained by means of the method of separation of variables. It is necessary to apply the Laplace Transform method. Since the necessary manipulations to solve the equations (1)-(4) with (5)-(12) are too voluminous as for to be realized by hand with pencil and paper, it is necessary to apply some type of system of computer algebra that allows symbolic computation [1,2]. In this work the method of Laplace transform was implemented at fully using only CAS with an appropriate algorithm. A very useful mathematical tools for this work was the theory of Whittaker functions [4]. In the following subsection our method of solution is described more detailed.

3.1. Method of Solution

A sketch of the algorithm that we have used to solve (1)-(4) for all specifications (5)-(12) is as follows. The inputs of the algorithm are: *Eq*, that represents the equation (1); *I.C.* that represents the initial conditions (2) and *B.C.* that represents the boundary conditions (3)-(4).

The output for the algorithm is the explicit profile for the drawdown it is to say the analytical form of the function S(r,t). Our algorithm operates at the following way:

The inputs *Eq*., *I.C.*, and *B.C,* by means of a *Laplace Transformer* are turned into a transformed equation denoted *T.Eq*. and a transformed boundary condition denoted *T.B.C*. Then, *T.Eq* and *T.B.C*. are processed by a certain *Dsolver* that generates the transformed solution denoted *Tsol*.

Next, *Tsol* is processed by means of an *inverser* , and we obtain the explicit form of the solution, denoted *sol*.

We implement such algorithm using Maple [2]. There are three ways to do that: using a worksheet, using a procedure and using a package. In this work we use a procedure which is constructed with Maple. Such procedure is the following:

```
> restart:        (B1)
        ↓
>Hydro:=proc(Q):(B2)
        ↓
>with(inttrans):(B3)
        ↓
> eq:=diff(S(r,t),t)-
k/r*(diff(S(r,t),r)+r*dif
f(S(r,t),`$`(r,2)))=0:  B4
        ↓
```

```
>eqp:=subs({laplace(S(r,t),t,s)=V(r),S(r,0)=0},laplace(eq,t,s)):    (B5)
↓
>assume(s>0,k>0):

>assume(m::positive):(B6)
↓
>sol:=dsolve(eqp,V(r)):B7
↓
>aux:=_C2=solve(limit(2*pi*k[1]*r*diff(rhs(sol),r),r=0)=subs({laplace(t^n,t,s)=n!/s^(n+1)},laplace(Q,t,s)),_C2):    (B8)
↓
>sol2:=subs({_C1=0,aux},sol):    (B9)
↓
>S(r,t)=invlaplace(rhs(sol2),s,t);    (B10)
↓
> end proc:    B(11)
```

This Maple procedure, is named Hydro as we can see in the block (B2). The block (B3) loads the Maple package for integral transforms. In the block (B4) the equation (1) is introduced inside the Maple environment . In the block (B5) the laplace transformation of the equation (1) with (2) is made. The block (B6) introduces the necessary assumptions for computations and the block (B7) is the Dsolver of the transformed equation. The block (B8) applies the boundary condition (4) and the block (B9) introduces the boundary condition (3). Finally the block (B10), makes the inverse laplace transformation and then the analytical form of S(r,t) is obtained. The block (B11) is the control command for the finalization of the procedure.

Our procedure Hydro, can be visualized as a black box that transform certain pumping rate variable on time Q(t), in to the analytical form of S(r,t).

For the applications, being previously loaded the procedure Hydro, the activation command is:

```
> Hydro(Q(t));
```

3.2. Results of Computations

Here we present the results of computations for pumping rates corresponding to the equations (5)-(12).

Explicitly, for Q(t) given by (6), we use the command

```
> Hydro(Q);
```

and the result is

$$S(r,t) = -\frac{1}{4}\frac{Q\,\mathrm{Ei}\left(-\frac{r^2}{4kt}\right)}{\pi k_1} \qquad (13)$$

where Ei(x) is the Exponential integral function [7]. This is justly the original Theis solution for a constant pumping rate of a well within a confined aquifer [3].

Now for Q(t) given by (7), we apply the command

```
> Hydro(Q*t);
```

and the answer is

$$S(r,t) = \frac{1}{2}\frac{Q\sqrt{k}\,t^{(3/2)}\,e^{\left(-\frac{r^2}{8kt}\right)}\,\mathrm{W}\left(\frac{-3}{2}, 0, \frac{r^2}{4kt}\right)}{\pi k_1 r} \qquad (14)$$

where W(u,v,x) is the Whittaker function with parameters u and v and with argument x [4].

When the pumping rate are changing on time as (8), we use the command

```
> Hydro(Q*t^2);
```

and the result is

$$S(r,t) = \frac{Q\sqrt{k}\,t^{(5/2)}\,e^{\left(-\frac{r^2}{8kt}\right)}\,\mathrm{W}\left(\frac{-5}{2}, 0, \frac{r^2}{4kt}\right)}{\pi k_1 r} \qquad (15)$$

where the Whittaker functions appear again.

More generally for the pumping rate given by (9) the command is

```
> Hydro(Q*t^n);
```

and the result is

$$S(r,t) = \frac{1}{2}\frac{Q\,n!\,\sqrt{k}\,t^{(n+1/2)}\,e^{\left(-\frac{r^2}{8kt}\right)}\,\mathrm{W}\left(-\frac{1}{2}-n, 0, \frac{r^2}{4kt}\right)}{\pi k_1 r} \qquad (16)$$

Now for the general Q(t) with the form (5) we use the command

```
>Hydro(Sum(Q[n]*t^n,n=0..infinity);
```

and the result is

$$S(r,t) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\frac{Q_n\,n!\,\sqrt{k}\,t^{(n+1/2)}\,e^{\left(-\frac{r^2}{8kt}\right)}\,\mathrm{W}\left(-\frac{1}{2}-n, 0, \frac{r^2}{4kt}\right)}{\pi r k_1}\right) \qquad (17)$$

Similarly for the pumping rate of the form (10) we use the command

```
>Hydro(Sum(Q*(epsilon*t)^n/n!,n=0..infinity));
```

and the result is

$$S(r,t) = \frac{1}{2}\,\frac{\displaystyle\sum_{n=0}^{\infty} \frac{Q\varepsilon^{n}\sqrt{k}\,t^{(n+1/2)}\,e^{\left(-\frac{r^2}{8kt}\right)}\,W\!\left(-\frac{1}{2}-n,0,\frac{r^2}{4kt}\right)}{r}}{\pi k_1} \tag{18}$$

For the pumping rate of the form (11) we use the expansion

$$Q\,t\,e^{(\varepsilon t)} = \sum_{n=1}^{\infty} \frac{Q\,(\varepsilon t)^{(n-1)}\,t}{(n-1)!} \tag{19}$$

and then we apply the command

```
>simplify(Hydro(Sum(Q*(epsilon*t)^(n-1)*t/(n-1)!,n=1..infinity)));
```

for hence the result is

$$S(r,t) = \frac{1}{2}\,\frac{\displaystyle\sum_{n=1}^{\infty} \frac{Q\varepsilon^{(n-1)}\,n\sqrt{k}\,t^{(n+1/2)}\,e^{\left(-\frac{r^2}{8kt}\right)}\,W\!\left(-\frac{1}{2}-n,0,\frac{r^2}{4kt}\right)}{r}}{\pi k_1} \tag{20}$$

Finally for the more general pumping rate that is considered here, given by (12); we use the expansion

$$Q\,t^{m}\,e^{(\varepsilon t)} = \sum_{n=m}^{\infty} \frac{Q\,(\varepsilon t)^{(n-m)}\,t^{m}}{(n-m)!} \tag{21}$$

and we can use the following command

```
> Hydro(Sum(Q*(epsilon)^(n-m)*t^n/(n-m)!,n=m..infinity));
```

which produces the result

$$S(r,t) = \frac{1}{2}\,\frac{\displaystyle\sum_{n=m}^{\infty} \frac{Q\varepsilon^{(n-m)}\,n!\sqrt{k}\,t^{(n+1/2)}\,e^{\left(-\frac{r^2}{8kt}\right)}\,W\!\left(-\frac{1}{2}-n,0,\frac{r^2}{4kt}\right)}{(n-m)!\,r}}{\pi k_1} \tag{22}$$

3.3. Analysis of Results

It is clear that (22) contains as particular cases all the results (13)-(20). In particular, the Theis solution can be obtained from (22) with m=0 and ε=0 . From the other side the case m=0 and ε=iw, was considered in [5] and the result was established in terms of Bessel functions [7]. Here we have an alternative formula in terms of Whittaker functions [4].

Now we want to show the effective expansions for the solutions (13)-(22) when $t \to \infty$. The results are displayed at Table 1. In this table the equations (23)-(30) are respectively the approximations of the exact equations (13)-(18), (20) and (22). In the expansions of the Table 1, γ is the Euler'constant [7], namely, γ = 0.5772 and Ψ(x) is the Digamma function [8]. As the reader can note, the leading term of (23) is obtained from (26) with n=0 and with Ψ(1)=-γ. Also, the leading term of (23) is obtained from (30) with m=0, ε = 0 and again with Ψ(1)=-γ. As we can observe for all expansions (23)-(30), the leading term is logarithmical respect to the distance *r* and the time *t*.

3.3. Applications of Results

All the results that were derived, the equations (13)-(22) are generalizations of the original Theis equation and hence these equations are valid only for infinite and confined aquifers. In the case of semi-infinite aquifers with boundaries it is possible to use the method of images. For example in the case of an aquifer bounded by a river and with a well with pumping rate given by (12), the total drawdown is the sum of the drawdown that is produced by the pumping well and the drawdown corresponding to the image well. The total drawdown for this configuration is showed at Figure 1. The result is presented inside a Maple environment using the procedure Hydro(Q) and the results (22) and (30). In the Figure 1, r_1 is the distance from the observation point to the pumping well and r_2 is the distance from the observation point to the image well. We observe in Figure 1 that the total drawdown $S(r_1, r_2, t)$ is quasi-stationary when $t \to \infty$.

Another possible application of our results (14)-(22) is concerned with the elaboration of a protocol for the experimental determination of the geo-hydraulic properties of aquifers using pumping tests [5]. This issue can be the object of a possible future work. Another line of future research is the extension of the results that were obtained here for the case of semi-confined or leaky aquifers [9].

4 Conclusions

This work was an intend to bring the reader some of the taste of Symbolic Computational Hydro-geology. The example that was chosen corresponds to the case of a confined aquifer with a non constant pumping rate. The authors believe that some of the results that were presented here are new in Mathematical Hydro-geology. Such results can be considered as generalizations of the Theis formula. Our results can have practical applications in hydrogeology, particularly in the field of new protocols for characterization of aquifers. We confirm the high importance of Computer Algebra Software, particularly Maple, inside the domain of Mathematical Hydrology. Using Maple all the results that were presented, are easily obtained. From the work that was done we can note the great relevance of the theory of Whittaker functions for the Mathematical Hydrogeology. Finally we conclude that the new emergent discipline, named, Symbolic Computational Hydro-geology has a very prominent future.

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Table 1. Expansions for the solutions (13)-(22)

$$S(r,t) = -\frac{1}{4}\frac{Q\left(\gamma+\ln\left(\frac{r^2}{4kt}\right)\right)}{\pi k_1} + \frac{1}{4}\frac{Q}{\pi k_1}\left(\frac{r^2}{4kt}\right) - \frac{1}{16}\frac{Q}{\pi k_1}\left(\frac{r^2}{4kt}\right)^2 + \frac{1}{72}\frac{Q}{\pi k_1}\left(\frac{r^2}{4kt}\right)^3 + O\left(\left(\frac{r^2}{4kt}\right)^4\right) \quad (23)$$

$$S(r,t) = \frac{1}{4}\frac{Qt\left(\ln\left(\frac{4kt}{r^2}\right)-1-\gamma\right)}{\pi k_1} + O\left(\frac{\sqrt{4}\left(\frac{r^2}{kt}\right)^{(3/2)}}{16}\right) \quad (24)$$

$$S(r,t) = \frac{1}{2}\frac{Qt^2\left(\frac{1}{2}\ln\left(\frac{4kt}{r^2}\right)-\frac{3}{4}-\frac{\gamma}{2}\right)}{\pi k_1} + O\left(\frac{\sqrt{4}\left(\frac{r^2}{kt}\right)^{(3/2)}}{16}\right) \quad (25)$$

$$S(r,t) = -\frac{1}{4}\frac{Qt^n\left(-\ln\left(\frac{4kt}{r^2}\right)+\Psi(1+n)+2\gamma\right)}{\pi k_1} + O\left(\frac{\sqrt{4}\left(\frac{r^2}{kt}\right)^{(3/2)}}{16}\right) \quad (26)$$

$S(r,t)=\sum_{n=0}^{\infty}\left(-\frac{1}{4}\,\frac{Q_n\,t^n\left(-\ln\left(\frac{4\,k\,t}{r^2}\right)+\Psi(1+n)+2\,\gamma\right)}{\pi\,k_1}\right)$	(27)
$S(r,t)=\sum_{n=0}^{\infty}\left(-\frac{1}{4}\,\frac{\varepsilon^n\,t^n\,Q\left(-\ln\left(\frac{4\,k\,t}{r^2}\right)+\Psi(1+n)+2\,\gamma\right)}{n!\,\pi\,k_1}\right)$	(28)
$S(r,t)=\sum_{n=1}^{\infty}\left(-\frac{1}{4}\,\frac{\varepsilon^{(n-1)}\,t^n\,Q\left(-\ln\left(\frac{4\,k\,t}{r^2}\right)+\Psi(1+n)+2\,\gamma\right)}{(n-1)!}\right)$	(29)
$S(r,t)=\sum_{n=m}^{\infty}\left(-\frac{1}{4}\,\frac{Q\,\varepsilon^{(n-m)}\,t^n\left(-\ln\left(\frac{4\,k\,t}{r^2}\right)+\Psi(1+n)+2\,\gamma\right)}{(n-m)!}\right)$	(30)

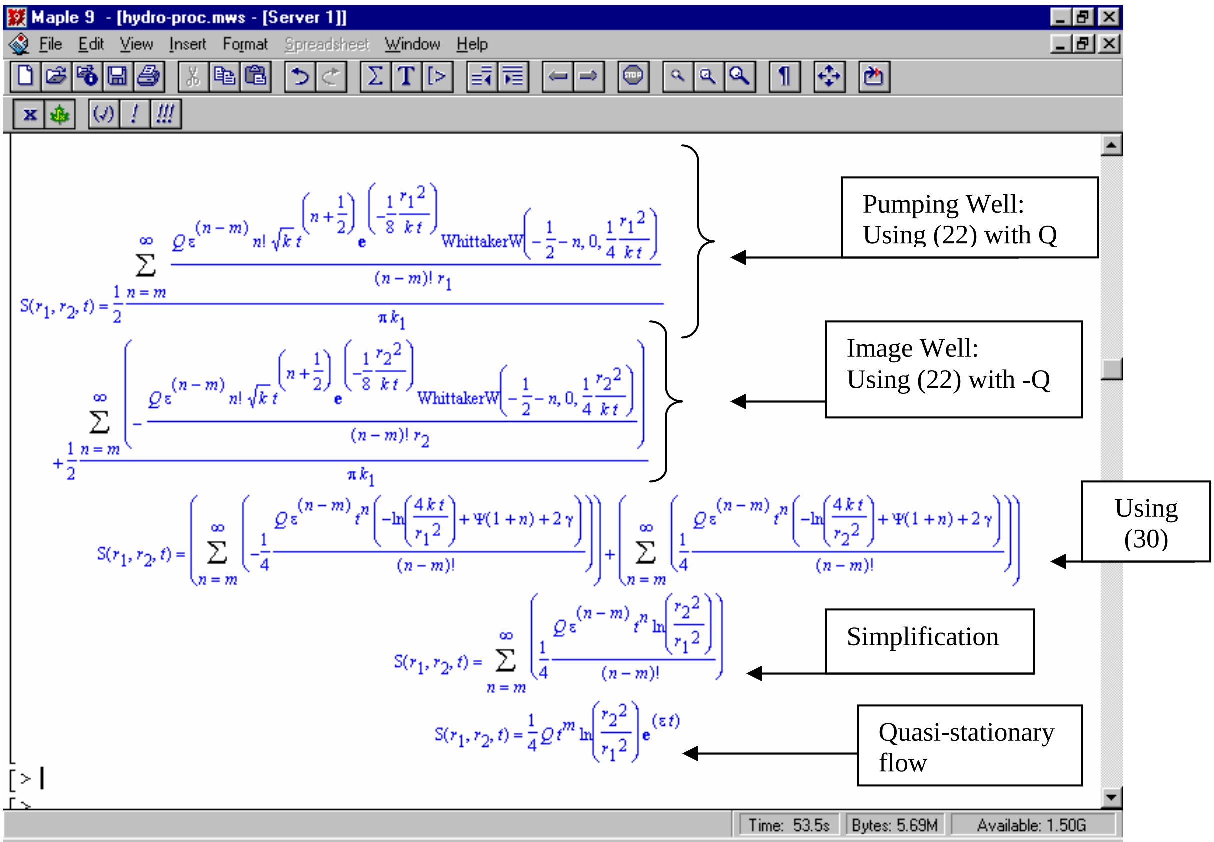

Figure 1. Total drawdown for a semi-infinite confined aquifer bounded by a river and with a well with pumping rate of the form (12). The method of image well is applied.