The Navier-Stokes Equations with Lagrangian Differences

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Abstract: The motion of a viscous incompressible fluid flow in bounded domains with a smooth boundary can be described by the nonlinear Navier-Stokes equations (N). This description corresponds to the so-called Eulerian approach. We develop a new approximation method for (N) in both the stationary and the nonstationary case by a suitable coupling of the Eulerian and the Lagrangian representation of the flow, where the latter is defined by the trajectories of the particles of the fluid. The method leads to a sequence of uniquely determined approximate solutions with a high degree of regularity, which contains a convergent subsequence with limit function \( v \) such that \( v \) is a weak solution on (N).

Key–Words: Navier-Stokes equations, Lagrangian representation, weak solutions

1 Introduction

For the description of fluid flow there are in principle two approaches, the Eulerian approach and the Lagrangian approach. The first one describes the flow by its velocity \( v = (v_1(t, x), v_2(t, x), v_3(t, x)) = v(t, x) \) at time \( t \) in every point \( x = (x_1, x_2, x_3) \) of the domain \( G \) containing the fluid. The second one uses the trajectory \( x = (x_1(t), x_2(t), x_3(t)) = x(t) = X(t, 0, x_0) \) of a single particle of fluid, which at initial time \( t = 0 \) is located at some point \( x_0 \in G \). The second approach is of great importance for the numerical analysis and computation of fluid flow also involving different media with interfaces [2, 3, 5, 8], while the first one has also often been used in connection with theoretical questions [4, 6, 7, 9].

It is the aim of the present note to develop a new approximation method for the nonlinear Navier-Stokes equations by coupling both the Lagrangian and the Eulerian approach. The method avoids fixpoint considerations and leads to a sequence of approximate systems, whose solution is unique and has a high degree of regularity, important at least for numerical purposes. Moreover, we can show that our method allows the construction of global weak solutions of the Navier-Stokes equations (compare [2, 4] for a local theory): The sequence of approximate solutions has at least one accumulation point satisfying the Navier-Stokes equations in a weak sense [6].

2 The Stationary Navier - Stokes Equations

We consider the stationary motion of a viscous incompressible fluid in a bounded domain \( G \subset \mathbb{R}^3 \) with a sufficiently smooth boundary \( S \). Because for steady flow the streamlines and the trajectories of the fluid particles coincide, both approaches mentioned above are correlated by the autonomous system of characteristic ordinary differential equations

\[
x'(t) = v(x(t)), \quad x(0) = x_0 \in G,
\]

which is an initial value problem for

\[
t \rightarrow x(t) = X(t, 0, x_0) = X(t, x_0)
\]

if the velocity field \( x \rightarrow v(x) \) is known in \( G \).

To determine the velocity, in the present case we have to solve the steady-state nonlinear equations

\[
-\nu \Delta v + v \cdot \nabla v + \nabla p = F \quad \text{in} \quad G, \\
\text{div} v = 0 \quad \text{in} \quad G, \quad v = 0 \quad \text{on} \quad S
\]

of Navier-Stokes. Here \( x \rightarrow p(x) \) is an unknown kinematic pressure function. The constant \( \nu > 0 \) (kinematic viscosity) and the external force density \( F \) are given data. The incompressibility of the fluid is expressed by \( \text{div} v = 0 \), and on the boundary \( S \) we require the no-slip condition \( v = 0 \).
3 The Lagrangian Approach

Let us start recalling some facts, which concern existence and uniqueness for the solution of the initial value problem (1): If the function \( v \) belongs to the space \( C^0_0(G) \) of vector fields being Lipschitz continuous in the closure \( \overline{G} = G \cup S \) and vanishing on the boundary \( S \), then for all \( x_0 \in G \) the solution

\[
t \mapsto x(t) = X(t, x_0)
\]

is uniquely determined and exists for all \( t \in \mathbb{R} \) (because \( v = 0 \) on the boundary \( S \), the trajectories remain in \( G \) for all times). Due to the uniqueness, the set of mappings

\[
\mathcal{R} = \{ X(t, \cdot) : G \to G | t \in \mathbb{R} \}
\]

defines a commutative group of \( C^1 \)-diffeomorphisms on \( G \). In particular, for \( t \in \mathbb{R} \) the inverse mapping \( X(t, \cdot)^{-1} \) of \( X(t, \cdot) \) is given by \( X(-t, \cdot) \), i.e.

\[
X(t, \cdot) \circ X(-t, \cdot) = X(t, X(-t, \cdot)) = X(0, \cdot) = id,
\]

or, equivalently,

\[
X(t, X(-t, x)) = x
\]

for all \( x \in G \). Moreover we obtain \( \det \nabla X(t, x) = 1 \) if

\[
v \in C^0_0(G) = \{ u \in C^0_0(G) | \text{div } u = 0 \},
\]

in addition. This important measure preserving property implies

\[
\langle f, g \rangle = \langle f \circ X(t, \cdot), g \circ X(t, \cdot) \rangle
\]

for all functions \( f, g \in L^2(G) \), where \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( L^2(G) \).

4 The Eulerian Approach

Next let us consider the Navier-Stokes boundary value problem (2). It is well known that, given \( F \in L^2(G) \), there is at least one function \( v \) satisfying (2) in some weak sense [6]. To define such a weak solution we need the space \( V(G) \), being the closure of \( \mathcal{C}^0_0 \) (smooth divergence free vector functions with compact support in \( G \)) with respect to the Dirichlet-norm \( \| \nabla u \| = (\nabla u, \nabla u) \), where we define

\[
\langle \nabla u, \nabla v \rangle = \sum_{i,j=1}^3 \langle D_j u_i, D_j v_i \rangle.
\]

Let us recall the following

Definition 1 Let \( F \in L^2(G) \) be given. A function \( v \in V(G) \) satisfying for all \( \Phi \in \mathcal{C}^\infty_0(G) \) the identity

\[
\nu \langle \nabla v, \nabla \Phi \rangle - \langle v \cdot \nabla \Phi, v \rangle = \langle F, \Phi \rangle
\]

is called a weak solution of the Navier-Stokes equations (2), and (3) is called the weak form of (2).

For a suitable approximation of the nonlinear term let us keep in mind its physical deduction. It is a convective term arising from the total or substantial derivative of the velocity vector \( v \). Thus it seems to be reasonable to use a total difference quotient for its approximation.

To do so, let \( v \in C^0_0(G) \) be given. Then for any \( \epsilon \in \mathbb{R} \) the mapping \( X(\epsilon, \cdot) : G \to G \) and its inverse \( X(-\epsilon, \cdot) \) are well defined. Consider for some \( u \in \mathcal{C}^1(G) \cap C^m(G) \) is the space of continuous functions having continuous partial derivatives up to and including order \( m \in \mathbb{N} \) in \( G \) and \( x \in G \) the one-sided Lagrangian difference quotients

\[
L^+_{\epsilon} u(x) = \frac{1}{\epsilon} [u(X(\epsilon, \cdot)) - u(x)],
\]

\[
L^-_{\epsilon} u(x) = \frac{1}{\epsilon} [u(x) - u(X(-\epsilon, \cdot))],
\]

and the central Lagrangian difference quotient

\[
L^{\epsilon} u(x) = \frac{1}{2} (L^+_{\epsilon} u(x) + L^-_{\epsilon} u(x)).
\]

Since for sufficiently regular functions

\[
L^-_{\epsilon} u(x) \longrightarrow v(x) \cdot \nabla u(x)
\]

and

\[
L^+_{\epsilon} u(x) \longrightarrow v(x) \cdot \nabla u(x)
\]

as \( \epsilon \to 0 \), the above quotients can all be used for the approximation of the convective term \( v \cdot \nabla v \). There is, however, an important advantage of the central quotient (4) with respect to the conservation of the energy:

Let \( v \in C^0_0(G) \) and \( u, w \in L^2(G) \). Let \( X(\epsilon, \cdot) \) and \( X(-\epsilon, \cdot) \) denote the mappings constructed from the solution of (1). Then, using the measure preserving property from above, we obtain only for the central quotient the orthogonality relation

\[
\langle L^\epsilon u, u \rangle = 0.
\]
5 The Stationary Approximate System

To establish an approximation procedure we assume that some approximate velocity field \( v^n \) has been found. To construct \( v^{n+1} \) we proceed as follows:

1) Construct \( X^n = X \left( \frac{1}{n} \cdot \right) \) and its inverse \( X^{-n} = X \left( -\frac{1}{n} \cdot \right) \) from the initial value problem

\[
x'(t) = v^n(x(t)), \quad x(0) = x_0 \in G.
\]

2) Construct \( v^{n+1} \) and \( p^{n+1} \) from the boundary value problem

\[
-\nu \Delta v^{n+1} + \frac{n}{2} [v^{n+1} \circ X^n - v^{n+1} \circ X^{-n}] + \nabla \nu^{n+1} = F \text{ in } G,
\]

\[
\nabla \cdot v^{n+1} = 0 \text{ in } G,
\]

\[
\nabla \cdot v^{n+1} = 0 \text{ on } S.
\]

Concerning the existence and uniqueness for the solution (6) and (7) we need the usual Sobolev Hilbert spaces \( H^m(G), m \in \mathbb{N} \), which denote the closure of \( C^m(G) \) with respect to the norm \( \| \cdot \|_{H^m} \) (see [1]). A main result is now stated in the following

**Theorem 2** a) Assume \( v^n \in H^3(G) \cap V(G) \) and \( F \in H^1(G) \). Then for all \( x_0 \in G \) the initial value problem (6) is uniquely solvable, and the mappings

\[
X^n : G \to G, \quad X^{-n} : G \to G
\]

are measure preserving \( C^1 \)-diffeomorphisms in \( G \). Moreover, there is a uniquely determined solution

\[
v^{n+1} \in H^3(G) \cap V(G), \quad \nabla v^{n+1} \in H^1(G)
\]

of the equations (7). The velocity field \( v^{n+1} \) satisfies the energy equation

\[
\nu \| \nabla v^{n+1} \|^2 = \langle F, v^{n+1} \rangle.
\]

b) Assume \( v^0 \in H^3(G) \cap V(G) \) and \( F \in H^1(G) \). Let \((v^n)\) denote the sequence of solutions constructed in view of Part a). Then \((v^n)\) is bounded in \( V(G) \) i.e.

\[
\| \nabla v^n \|^2 \leq C_{G,F,\nu} \text{ for all } n \in \mathbb{N}, \text{ where the constant } C_{G,F,\nu} \text{ does not depend on } n.
\]

Moreover, \((v^n)\) has an accumulation point \( v \in V(G) \) satisfying (3), i.e. \( v \) is a weak solution of the Navier-Stokes equations (2).

6 The Nonstationary Navier - Stokes Equations

Let us consider now the motion of a nonstationary viscous incompressible fluid flow in a bounded domain \( G \subset \mathbb{R}^3 \) with a sufficiently smooth boundary \( S \). Without loss of generality, in this section we assume conservative external forces and consider the following Navier-Stokes initial boundary value problem:

Construct a velocity field \( v = v(t, x) \) and some pressure function \( p = p(t, x) \) as a solution of the system

\[
v_t - \nu \Delta v + \nabla p + v \cdot \nabla v = 0 \text{ in } G, \quad t > 0,
\]

\[
\nabla \cdot v = 0 \text{ on } S, \quad t > 0,
\]

\[
v = v_0 \text{ for } t = 0.
\]

Here \( v_0 \) is a suitable prescribed initial velocity distribution.

The existence of a classical solution global in time of this problem without any smallness restriction on the data has not been proved up to now. Hence also a globally stable approximation scheme does not exist for this system. In order to construct classically solvable equations, as in the steady-state case, an approximation of the nonlinear convective term \( v \cdot \nabla v \), which is responsible for the non-global existence of the solution, by means of a Lagrangian difference quotients seems to be reasonable.

In the following we show that the nonstationary Navier-Stokes system \((N)\) can also be approximated by means of Lagrangian differences. The resulting approximate system \((N_\varepsilon)\) is uniquely solvable, its solution exists globally in time, has a high degree of regularity and satisfies the nonstationary energy equation.

7 The Initial Value Problem

Let \( J \) be a compact time interval, and let \( \tilde{v} \in C(J, H^3(G) \cap V(G)) \) be a given velocity field being strongly \( H^3 \)-continuous. Consider the initial value problem

\[
\dot{x}(t) = \tilde{v}(t, x(t)), \quad (s, x_0) \in J \times \overline{G} \quad (A)
\]

concerning the trajectory \( x(t) = X(t, s, x_0) \) of a fluid particle, which at time \( t = s \) is located at \( x_0 \) in \( \overline{G} \).
Due to well-known results on ordinary differential equations, as in the autonomous case, the uniquely determined general solution \( X(t, s, x_0) \) of (A) exists for all times, and the mapping 
\[
X(t, s, \cdot) : \mathcal{G} \rightarrow \mathcal{G}, \quad t, s \in J
\]
is a measure preserving diffeomorphism with inverse function 
\[
X^{-1} = X(s, t, \cdot).
\]

As in the stationary case we now approximate the time dependent nonlinear convective term \( v(t, x) \cdot \nabla v(t, x) \) by a central Lagrangian difference quotient as follows:
\[
v(t, x) \cdot \nabla v(t_0, x) \sim \frac{1}{2\varepsilon} \left( v(t_0, X(t + \varepsilon, t, x)) - v(t_0, X(t, t + \varepsilon, x)) \right).
\]

(8)

Here \( \sim \) means that for a sufficiently regular function \( v \) the right hand side converges to the expression on the left hand side as \( \varepsilon \to 0 \).

The main advantage of the central quotient in (8), which we denote by 
\[
\frac{1}{2\varepsilon} \left( v \circ X - v \circ X^{-1} \right)
\]
for abbreviation, is again the validity of an analogon to the orthogonality relation of Hopf [6]:

Using \( \langle \cdot, \cdot \rangle \) as \( L^2(G) \)-scalar product Hopf obtains the global (in time) existence of weak solutions to the Navier-Stokes system (N) due to the important orthogonality relation 
\[
\langle v \cdot \nabla v, v \rangle = 0, \quad v \in V(G).
\]

Using the measure preserving property of the mapping \( X \), we analogously obtain 
\[
\frac{1}{2\varepsilon} \left( v \circ X - v \circ X^{-1}, v \right) = \frac{1}{2\varepsilon} \left( (v \circ X, v) - (v \circ v \circ X) \right) = 0,
\]
which implies the validity of the energy equation for all sufficiently regular solutions of the approximate system, if central Lagrangian differences instead of one-sided quotients are used.

8 Time Delay and Compatibility at Initial Time

To avoid fixed-point considerations for the solution of the regularized approximate system – the velocity vector \( v \) as well as the mappings \( X \) are unknown – by means of a time delay we replace \( v \cdot \nabla v \) by 
\[
\frac{1}{2\varepsilon} \left( v \circ X - v \circ X^{-1} \right)
\]
with trajectories \( X \) constructed at earlier time points, where the velocity \( v \) is known already.

To do so, on the given time interval \([0, T]\) we define a time grid by 
\[
t_k = k \cdot \varepsilon, \quad k = 0, \ldots, N \in \mathbb{N},
\]
where \( \varepsilon := \frac{T}{N} > 0 \). Setting 
\[
X_k := X(t_k, t_{k-1}, x),
\]
for \( t \in [t_k, t_{k+1}] \) we can use e.g. the approximation 
\[
v(t, x) \cdot \nabla v(t, x) \sim \frac{1}{2\varepsilon} \left( v(t, X_k) - v(t, X_k^{-1}) \right).
\]

(9)

To initiate this procedure we extend the initial value \( v_0 \) continuously to a start function 
\[
v_\varepsilon \in C([-\varepsilon, 0], H^3(G) \cap V(G)).
\]

Then, indeed, on the subintervals \([t_k, t_{k+1}]\) we can successively construct the mappings \( X_k \) from the given velocity field \( v \) and vice versa. Nevertheless, we do not obtain a global on \([0, T]\) existing solution of a problem regularized by (9). This is due to a certain compatibility condition, which always occurs in parabolic problems at the corner of the space time cylinder:

For the unique construction of the mapping \( X_k \), if integer order Sobolev spaces are used, we need a velocity field 
\[
v \in C([t_{k-1}, t_k], H^3(G) \cap V(G)),
\]
i.e. 
\[
v_t \in C([t_{k-1}, t_k], V(G)).
\]

Using
\[
P : L^2(G) \rightarrow H(G) := C_0^\infty(G)^{\mathbb{N}}
\]
as orthogonal projection we obtain in particular the condition 
\[
v_t(t_k) = \mu P \Delta v(t_k) - \frac{1}{2\varepsilon} P \left( (v(t_k, X_k) - v(t_k, X_k^{-1}) \right) \in V(G).
\]

Due to \( v_0 \in H^3(G) \cap V(G) \) we find that the right hand side of (10) is contained in \( H^1(G) \cap H(G) \), only. Hence the condition \( v_t(t_k) \in V(G) \) implies in case of an approximation of the type (9) that we have to impose the condition 
\[
\frac{1}{2\varepsilon} P \left( (v(t_k, X_k) - v(t_k, X_k^{-1}) \right) = 0 \quad \text{on } S.
\]

(11)
9 The Approximate System $(N_\varepsilon)$

Instead of a system regularized by (9) we consider

\[
\begin{align*}
v_t - \mu \Delta v + \nabla p + Z_\varepsilon v &= 0 \quad \text{in } G \text{ for } t > 0, \\
\nabla \cdot v &= 0 \quad \text{on } S, \\
v &= 0 \quad \text{in } G \text{ for } t = 0, \\
v_t &= f \quad \text{in } G \text{ for } t = 0, \\
\end{align*}
\]

where \( f \in V(G) \), and where for \( t \in [t_k, t_{k+1}] \)

\[
Z_\varepsilon v(t, x) := \frac{1}{2\varepsilon} \left( (t-t_k)(v(t, X_k) - v(t, X_k^{-1})) + (t_{k+1} - t)(v(t, X_{k-1}) - v(t, X_{k-1}^{-1})) \right)
\]

is continuously defined on \([0, T]\).

In this case all compatibility conditions are satisfied: The condition for \( t = 0 \) can be fulfilled following a hint of V. A. Solonnikov by prescribing \( v_t(0) = f \in V(G) \) instead of \( v(0) = v_0 \):

For a given function

\[
v_s \in C([-2\varepsilon, -\varepsilon], H^3(G) \cap V(G))
\]

we solve the problem (A) and obtain the mapping \( X_{-1} \). Then we consider the stationary problem

\[
\nu \Delta v_0 - \frac{1}{2\varepsilon} P(v_0 \circ X_{-1} - v_0 \circ X_{-1}^{-1}) = f,
\]

and obtain by well-known existence and regularity results a uniquely determined solution

\[v_0 \in H^3(G) \cap V(G),\]

which, since functions in \( V(G) \) vanish on the boundary \( S \), satisfies the required compatibility condition (11). By linear interpolation between \( v_s(-\varepsilon) \) and \( v_0 \) we then obtain a start function

\[
v_s \in C([-2\varepsilon, 0], H^3(G) \cap V(G)).
\]

Since the compatibility condition in all the following grid points \( t_k \) are automatically satisfied due to the continuity of the function

\[t \rightarrow Z_\varepsilon v(t),\]

we finally obtain, by successively constructing the mappings from the velocity field \( v \) and vice versa, the following result:

**Theorem 3** Let \([0, T]\) be given and let \( f \in V(G) \). Then for every \( \varepsilon > 0 \) exists a uniquely determined function

\[v \in C([0, T], H^3(G) \cap V(G))\]

and a uniquely determined pressure gradient

\[\nabla p \in C([0, T], H^1(G))\]

as the solution of the system \((N_\varepsilon)\). For \( v \) holds on \([0, T]\) the energy equation

\[
\|v(t)\|^2 + 2\nu \int_0^t \|v(s)\|^2 ds = \|v_0\|^2,
\]

and \( H^3 \)-Norm estimates can be constructed uniformly on \([0, T]\) depending on the data, \( T \) and \( \varepsilon \).

References:


