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Properties of a Class of Continuum Damage Models

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Abstract: A class of elasto-plastic models for small deformations coupled with damage effects is considered. The models are derived on the basis of energy equivalence principles and are characterized by a function of damage related to the yield function. Properties of the models are illustrated by uniaxial tensile loadings.

Key–Words: Continuum damage mechanics, Energy equivalence, Isotropy.

1 Introduction

Continuum damage models on the basis of effective strain and effective stress combined with the hypothesis (principle) of energy equivalence have been introduced by Cordebois and Sidoroff [4]. In this work, the principle of energy equivalence was discussed for the case of pure elastic mechanical behavior only. Interesting extensions to elastic-plastic materials were then proposed by Chow and Lu [3] as well as Saanouni, Forster and Hatira [7]. The concept of energy equivalence relies upon the assumption that there exists an undamaged fictitious material, whose response functions serve to establish the corresponding functions for the real damaged material by stetting in relation some energy quantities. Thereby, so-called effective strain and stress variables have to be used for the undamaged fictitious material. Only isotropic hardening is considered in Chow and Lu [3] and an equivalence for the incremental plastic work is postulated. According to the assumptions made, the yield function for the real material is known and the effective accumulated plastic strain is gained by the principle. The latter is used to formulate the isotropic hardening rule for the real material. Both, isotropic and kinematic hardening are assumed to apply in Saanouni, Forster and Hatira [7]. Equivalence is defined for the free energy functions and the dissipation potentials, respectively. This way, the evolution equations governing the hardening response are obtained by making use of the generalized normality rule.

In the present paper, we are concerned with elasto-plasticity coupled with damage. We assume the yield function for the real material to be unknown and postulate an equivalence for the material functions governing the plastic and the hardening powers. As a result, we obtain for the real material the yield function and the evolution equations for the hardening variables. In order to explain here ideas as simple as possible we restrict the presentations only to isotropy and uniaxial loadings. Also, only isotropic hardening and isothermal deformations are assumed to apply. Some of the concepts addressed here have been developed previously in [6], but some arguments there are not formulated clearly, or even they are incorrect, so that the present paper offers a rigorous motivation for these concepts.

2 Underlying Elasto-Plasticity Model

When no damage effects are present, the material response is assumed to be described by the following system of constitutive functions:

$$\varepsilon = \varepsilon_e + \varepsilon_p \quad , \tag{1}$$

$$\psi = \tilde{\psi}(\varepsilon_e, r) = \psi_e + \psi_{is} \quad , \tag{2}$$

$$\psi_e = \tilde{\psi}_e(\varepsilon_e) = \frac{1}{2} E \varepsilon_e^2 \quad , \tag{3}$$

$$\psi_{is} = \tilde{\psi}_{is}(r) = \frac{1}{2} \gamma r^2 \quad , \tag{4}$$

$$\sigma = \frac{\partial \tilde{\psi}_e}{\partial \varepsilon_e} = E \varepsilon_e \quad , \tag{5}$$

$$R := \frac{\partial \tilde{\psi}_{is}}{\partial r} = \gamma r \quad , \tag{6}$$

$$f = \tilde{f}(\sigma, R) = |\sigma| - R \ : \ \text{yield function} \quad , \tag{7}$$

$$\tilde{f}(\sigma, R) = k_0 \quad : \quad \text{yield condition} \quad , \tag{8}$$

$$\dot{\varepsilon}_p = \tilde{p}(\Lambda, \sigma, R) = \Lambda \frac{\partial \tilde{f}}{\partial \sigma} = \Lambda \frac{\sigma}{|\sigma|} \quad , \tag{9}$$

$$\dot{s} := |\dot{\varepsilon}_p| = \Lambda \quad , \quad \Lambda \geq 0 \quad , \tag{10}$$

$$\dot{r} = \tilde{\varrho}(\Lambda, R) = (1 - \beta R)\Lambda \equiv (1 - \beta R)\dot{s} \quad , \tag{11}$$

$$w_p = \tilde{w}_p(\Lambda, \sigma, R) = \sigma \tilde{p}(\Lambda, \sigma, R) = \Lambda \sigma \frac{\partial \tilde{f}}{\partial \sigma} \quad , \tag{12}$$

$$w_{is} = \tilde{w}_{is}(\Lambda, R) = R\tilde{\varrho}(\Lambda, R) = R(1 - \beta R)\Lambda \quad , \tag{13}$$

$$\sigma \dot{\varepsilon}_p - \dot{\psi}_{is} \geq 0 \quad . \tag{14}$$

In these equations, ε is the strain, σ is the stress, ψ denotes the free energy and E, β, γ, k_0 are material parameters. Isotropic hardening is modelled by the internal strain r, which is conjugate to the stress R. Latter reflects isotropic hardening effects in the material behavior. Evolution equations like that in (11) have been intensively investigated by Chaboche (see e.g. [1, 2]). The factor Λ has to be determined from the so-called consistency condition. In Equations (12), (13), w_p and w_{is} denote the plastic power and the rate of energy stored in the material due to isotropic hardening and can be considered to be given by constitutive functions $\tilde{w}_p(\cdot)$ and $\tilde{w}_{is}(\cdot)$, respectively. Finally, Equation (14) represents the so-called dissipation inequality, which can be proved to be satisfied for every admissible process.

3 Elasto-Plasticity Coupled With Damage

Let $D \in [0, 1)$ denote the damage variable. The set of constitutive functions is supposed to be extended by D so that we have:

$$\varepsilon = \varepsilon_e + \varepsilon_p \quad , \tag{15}$$

$$\psi = \bar{\psi}(\varepsilon_e, r, D) = \psi_e + \psi_{is} \quad , \tag{16}$$

$$\psi_e = \bar{\psi}_e(\varepsilon_e, D) \quad , \tag{17}$$

$$\psi_{is} = \bar{\psi}_{is}(r, D) \quad , \tag{18}$$

$$\sigma = \frac{\partial \bar{\psi}_e(\varepsilon_e, D)}{\partial \varepsilon_e} \quad , \tag{19}$$

$$R := \frac{\partial \bar{\psi}_{is}(r, D)}{\partial r} \quad , \tag{20}$$

$$f = \bar{f}(\sigma, R, D) \quad , \tag{21}$$

$$\bar{f}(\sigma, R, D) = k_0 \quad : \quad \text{yield condition} \quad , \tag{22}$$

$$\dot{\varepsilon}_p = \bar{p}(\Lambda, \sigma, R, D) = \Lambda \frac{\partial \bar{f}(\sigma, R, D)}{\partial \sigma} \quad , \tag{23}$$

$$\dot{s} := |\dot{\varepsilon}_p| = \Lambda \left| \frac{\partial \bar{f}}{\partial \sigma} \right| \quad , \tag{24}$$

$$\dot{r} = \bar{\varrho}(\Lambda, R, D) \quad , \tag{25}$$

$$\begin{aligned} w_p = \bar{w}_p(\Lambda, \sigma, R, D) &= \sigma \bar{p}(\Lambda, \sigma, R, D) \\ &= \Lambda \sigma \frac{\partial \bar{f}(\sigma, R, D)}{\partial \sigma} \quad , \end{aligned} \tag{26}$$

$$w_{is} = \bar{w}_{is}(\Lambda, R, D) = R\bar{\varrho}(\Lambda, R, D) \quad , \tag{27}$$

$$\Omega := \frac{\partial \bar{\psi}(\varepsilon_e, r, D)}{\partial D} \quad , \tag{28}$$

$$\mathcal{D} := \sigma \dot{\varepsilon}_p - R\dot{r} - \Omega \dot{D} \geq 0 \quad . \tag{29}$$

A normality rule is assumed to hold in (23), but otherwise the yield function is not specified further. The next step is to determine the constitutive functions $\bar{\psi}_e$, $\bar{\psi}_{is}$, $\bar{f}$ and $\bar{\varrho}$ from those in Section 2 by using an energy equivalence principle.

4 Proposed Energy Equivalence Principle

Following Cordebois and Sidoroff [4], Chow and Lu [4] and Saanouni, Forster and Hatira [7], we introduce effective variables σ^{ef}, ε_e^{ef}, R^{ef}, r^{ef}, Λ^{ef} and determine the constitutive equations for the real material as follows.

First we introduce damage functions $m = m(D)$, $h = h(D)$ and define

$$\sigma^{ef} := \frac{\sigma}{m} \quad , \tag{30}$$

$$\varepsilon_e^{ef} := h\varepsilon_e \quad , \tag{31}$$

$$R^{ef} := \frac{R}{m} \quad , \tag{32}$$

$$r^{ef} := hr \quad . \tag{33}$$

The function m is considered to be given. In particular we assume

$$m = (1 - D)^{q/2} \quad , \tag{34}$$

with q being a nonnegative material parameter. Then, we determine h and $\bar{\psi}_e$, $\bar{\psi}_{is}$ by postulating

$$\bar{\psi}_e(\varepsilon_e, D) = \tilde{\psi}_e(\varepsilon_e) \quad , \tag{35}$$

$$\bar{\psi}_{is}(r, D) = \tilde{\psi}_{is}(r) \quad , \tag{36}$$

$$\sigma^{ef} = \frac{\partial \tilde{\psi}_e(\varepsilon_e^{ef})}{\partial \varepsilon_e^{ef}} \quad , \tag{37}$$

$$R^{ef} = \frac{\partial \tilde{\psi}_{is}(r^{ef})}{\partial r^{ef}} \quad . \tag{38}$$

From these, as well as Equations (19), (20), it follows that

$$h(D) = m(D) = (1 - D)^{q/2} \tag{39}$$

and

$$\bar{\psi}_e(\varepsilon_e, D) = \frac{1}{2}(1-D)^q E \varepsilon_e^2 \quad , \tag{40}$$

$$\bar{\psi}_{is}(r, D) = \frac{1}{2}(1-D)^q \gamma r^2 \quad . \tag{41}$$

In order to obtain the hardening response, we assume the relation

$$\Lambda^{ef} := g\chi\Lambda \quad , \tag{42}$$

and postulate the equivalence for the power functions

$$\bar{w}_p(\Lambda, \sigma, R, D) = \frac{1}{\chi}\tilde{w}_p(\Lambda^{ef}, \sigma^{ef}, R^{ef}) \quad , \tag{43}$$

$$\bar{w}_{is}(\Lambda, \sigma, R, D) = \frac{1}{\chi}\tilde{w}_{is}(\Lambda^{ef}, \sigma^{ef}, R^{ef}) \quad , \tag{44}$$

where g, χ are state functions. The function χ will be relevant when modelling viscoplastic material properties with static recovery terms. However, for plasticity, we deal with here, χ needs not to be specified any further, so that, without loss of generality, we set $\chi \equiv 1$ in what follows. In the present paper we assume g to depend only on the damage variable. Especially, we set

$$g = g(D) = (1-D)^{q/2-n} \quad , \tag{45}$$

where n is a material parameter. From (43), it follows that

$$\Lambda\sigma\frac{\partial \bar{f}(\sigma, R, D)}{\partial \sigma} = \Lambda^{ef}\sigma^{ef}\frac{\partial \tilde{f}(\sigma^{ef}, R^{ef})}{\partial \sigma^{ef}} \tag{46}$$

or

$$\frac{\partial \bar{f}(\sigma, R, D)}{\partial \sigma} = \frac{g}{m}\frac{\partial \tilde{f}(\sigma^{ef}, R^{ef})}{\partial \sigma^{ef}} \quad , \tag{47}$$

which posses the solution

$$\begin{aligned}\bar{f}(\sigma, R, D) &= g(D)\tilde{f}(\sigma^{ef}, R^{ef}) \\ &= g(D)(|\sigma^{ef}| - R^{ef}) \quad .\end{aligned} \tag{48}$$

Similarly, we get from (44)

$$R\bar{\varrho}(\Lambda, R, D) = R^{ef}(1-\beta R^{ef})\Lambda^{ef} \tag{49}$$

or

$$\bar{\varrho}(\Lambda, R, D) = \left(1-\beta\frac{R}{m}\right)\frac{\Lambda^{ef}}{m} \quad , \tag{50}$$

and hence

$$\dot{r} = \bar{\varrho}(\Lambda, R, D) = \left(1-\beta\frac{R}{m}\right)\frac{\Lambda g}{m} = \left(1-\beta\frac{R}{m}\right)\dot{s} \tag{51}$$

with

$$\dot{s} = \frac{\Lambda g}{m} \quad \text{and} \quad \left|\frac{\partial \tilde{f}}{\partial \sigma^{ef}}\right| = 1 \quad , \tag{52}$$

in view of (7). Thus, the yield function and the evolution law for isotropic hardening are established, which completes the system of equations governing the model response.

It remains only to prove the dissipation inequality (29). It is straightforward to show that $\sigma\dot{\varepsilon}_p - R\dot{r} \geq 0$, so that $\mathcal{D} \geq 0$ if $-\Omega\dot{D} \geq 0$. This in turn will be true if $\dot{D} \geq 0$, since

$$-\Omega = \frac{q}{2}(1-D)^{q-1}(E\varepsilon_e^2 + \gamma r^2) \geq 0 \quad . \tag{53}$$

As a simple possibility we chose

$$\dot{D} = \alpha\frac{(-\Omega)^p}{(1-D)^k}\dot{s} \quad , \tag{54}$$

where α, p, k are material parameters. Equation (54) goes back to Lemaitre.

This way, a class of elasto-plasticity models coupled with damage have been got, which are characterized by the damage function $g(D)$.

5 Uniaxial Tensile Loadings

For uniaxial tensile loading we have $\sigma/|\sigma| = 1$, $\dot{s} = \dot{\varepsilon}_p$, and therefore the set of constitutive equations reduces to

$$\varepsilon = \varepsilon_e + \varepsilon_p \quad , \tag{55}$$

$$\sigma = (1-D)^q E\varepsilon_e \quad , \tag{56}$$

$$\sigma = R + (1-D)^n k_0 \quad , \tag{57}$$

$$R = (1-D)^q \gamma r \quad , \tag{58}$$

$$\dot{r} = \left(1-\beta\frac{R}{(1-D)^{q/2}}\right)\dot{\varepsilon}_p \quad , \tag{59}$$

together with Equations (53), (54). ε-σ-responses predicted by these equations are illustrated in Figures 1–8, for the material parameters given in Table 1. From Equations (55)–(59), one can conclude that as $D \to 1$, the elastic strain ε_e remains bounded if and only if $n \geq q$. (For more details see Grammenoudis and Tsakmakis [5].) Consequently, if one assumes that for metallic materials ε_e should remain bounded as $D \to 1$, then $n = q$ has to hold for such materials.

From Figures 1–8 can be recognized that for the range of material parameters considered, and that for $n = q = 1$, the form of the ε-σ-graphs remains concave independent of the evolution law for D. However, if $n = q \neq 1$, then the damage law effects the form of the ε-σ-graphs, which are not more necessarily concave.

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E[MPa]	k_0[MPa]	γ[MPa]	β[MPa]
200000	200	30000	0.01

Table 1: Material parameter used in(55)–(59)

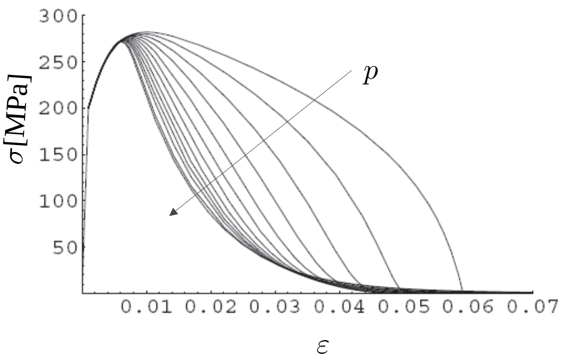

Figure 1: ε-σ-graphs corresponding to the material parameters $n = q = 4$, $\alpha = 1$, $k = 9$, $p \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15\}$.

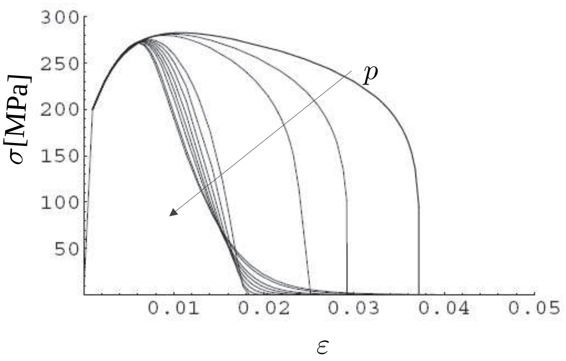

Figure 2: ε-σ-graphs corresponding to the material parameters $n = q = 4$, $\alpha = 1$, $k = 20$, $p \in \{1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.

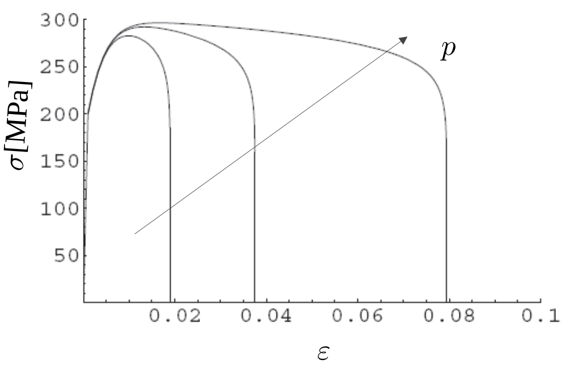

Figure 3: ε-σ-graphs corresponding to the material parameters $n = q = 1$, $\alpha = 50$, $k = 9$, $p \in \{2, 3, 4\}$.

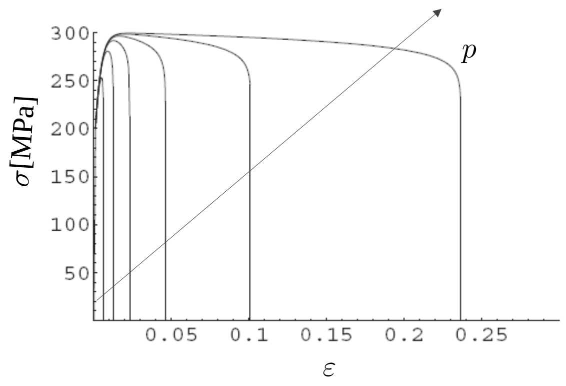

Figure 4: ε-σ-graphs corresponding to the material parameters $n = q = 1$, $\alpha = 50$, $k = 20$, $p \in \{1, 2, 3, 4, 5, 6\}$.

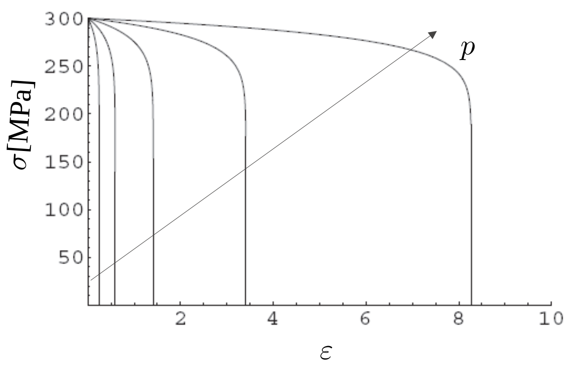

Figure 5: ε-σ-graphs corresponding to the material parameters $n = q = 1$, $\alpha = 1$, $k = 9$, $p \in \{1, 2, 3, 4, 5\}$.

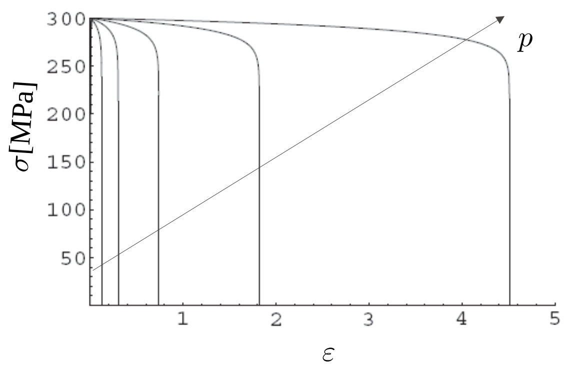

Figure 6: ε-σ-graphs corresponding to the material parameters $n = q = 1$, $\alpha = 1$, $k = 20$, $p \in \{1, 2, 3, 4, 5\}$.

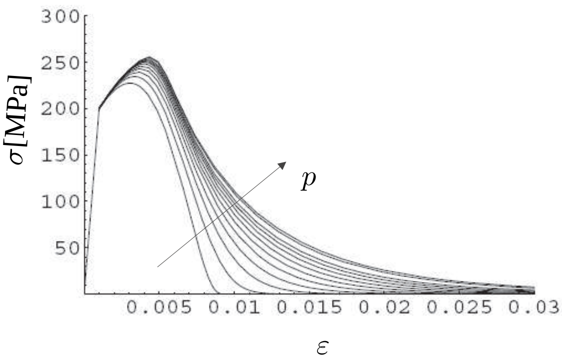

Figure 7: ε-σ-graphs corresponding to the material parameters $n = q = 4$, $\alpha = 50$, $k = 9$, $p \in \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.

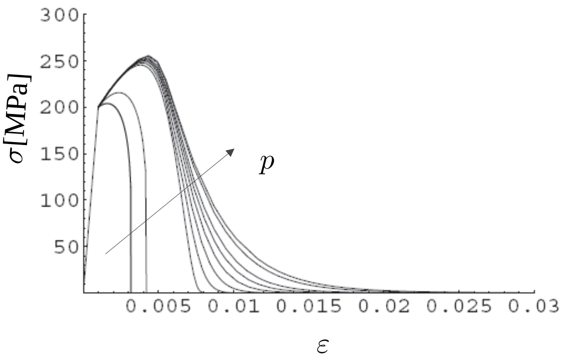

Figure 8: ε-σ-graphs corresponding to the material parameters $n = q = 4$, $\alpha = 50$, $k = 20$, $p \in \{1, 2, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.

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