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Estimates of optimal accuracy for the Brezzi-Pitkäranta approximation of the Navier-Stokes problem.

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Abstract: This paper deals with a singular perturbation of the stationary Navier-Stokes system. Thereby the term $\varepsilon^2\Delta p$ is added to the continuity equation, where ε is small parameter. For sufficiently regular and small data, existence of a unique solution is proved. This solution converges to the corresponding (unique) solution of the Navier-Stokes problem in $H^{5/2-\delta}$ for the velocity parts and in $H^{3/2-\delta}$ for the pressure parts, respectively.

Key–Words: Stokes problem, Navier- Stokes problem, quasi-compressibility methods, penalty system, singular perturbation.

1 Introduction

In [1] a singular perturbation of the Stokes problem was introduced in order to obtain stable numerical methods. The equation of continuity, $\mathrm{div}\ v = 0$, is substituted by the equation $\mathrm{div}\ v - \varepsilon^2\Delta p = 0$, which leads to a strongly elliptic system of second order for both, the velocity vector v and the pressure p, and thus, a boundary condition has to be added for the pressure, too. Finally this ends up with the problem

$$\left.\begin{array}{rcl} -\Delta v^\varepsilon + \nabla p^\varepsilon &=& f', \\ -\varepsilon^2\Delta p^\varepsilon + \mathrm{div}\ v^\varepsilon &=& f_4 \end{array}\right\} \text{ in } \Omega, \qquad \left.\begin{array}{rcl} v^\varepsilon &=& g', \\ \partial_n p^\varepsilon &=& g_4 \end{array}\right\} \text{ on } \partial\Omega, \tag{S$_\varepsilon$}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with boundary $\partial\Omega$ of class C^2 at least, $f = (f_1, \ldots, f_4)$ and $g = (g_1, \ldots, g_4)$ are given vector fields, while $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon, v_3^\varepsilon)$ and p^ε are the quantities to be found. The Stokes problem (S$_0$) appears formally if we set $\varepsilon = 0$ and omit the Neumann condition for the pressure part, and of course, estimates for the differences $u^\varepsilon - u^0$ are needed, as $\varepsilon \searrow 0$. Contributions to numerical results using this problem were developed further in [9, 8, 11], e.g., while in [3] the problem (S$_\varepsilon$) was used to show the existence of weak solutions to problems with shear dependent viscosities. In the papers cited above $f_4 = 0$ and $g = 0$ was prescribed, then energy methods were used to estimate the differences $v^\varepsilon - v^0$, $p^\varepsilon - p^0$, which leads to convergence in $H^1(\Omega)$ and $L^2(\Omega)$, respectively, as $\varepsilon \searrow 0$.

In [7], the approach of [8, 9] was exploited to obtain estimates in Sobolev spaces $H^l_\varkappa(\Omega, \varepsilon)$ depending on the small parameter ε. The index $\varkappa$ is related to the part of the norms remaining stable as $\varepsilon \searrow 0$. With those estimates it is possible to show that

$$\|v^\varepsilon - v^0; H^{5/2-\delta}(\Omega)\| + \|p^\varepsilon - p^0; H^{3/2-\delta}(\Omega)\| = O(\varepsilon^\delta),$$

as $\varepsilon \searrow 0$,(see Theorem 2 below), where $\delta \in (0, 3/2)$ can be arbitrarily close to the endpoints of the interval. From the problem setting it is clear that this is the optimum with respect to the regularity properties. By constructing boundary layers, it was also shown in [7], that the result is optimal with respect to the order of convergence as $\varepsilon \searrow 0$. Here we dwell upon the nonlinear singular perturbed system

$$\left.\begin{array}{rcl} -\Delta v^\varepsilon + \nabla p^\varepsilon + N(v^\varepsilon, v^\varepsilon) &=& f', \\ -\varepsilon^2\Delta p^\varepsilon + \mathrm{div}\ v^\varepsilon &=& f_4 \end{array}\right\} \text{ in } \Omega, \qquad \left.\begin{array}{rcl} v^\varepsilon &=& g', \\ \partial_n p^\varepsilon &=& g_4 \end{array}\right\} \text{ on } \partial\Omega, \tag{NS$_\varepsilon$}$$

with either $N(v, w) = (v \cdot \nabla)w$ or $N(v, w) = ((v \cdot \nabla)w + 2^{-1}(\mathrm{div}\ v)w$. If $\varepsilon = 0$ and the Neumann condition is cancelled, we obtain the stationary Navier-Stokes problem

$$\left.\begin{array}{rcl} -\Delta v + \nabla p + N(v, v) &=& f', \\ \mathrm{div}\ v &=& f_4 \end{array}\right\} \text{ in } \Omega, \qquad v = g' \quad \text{on } \partial\Omega, \tag{NS$_0$}$$

For $f_4 = 0$ and $g = 0$ and $f' \in L^2(\Omega)$, e.g., it is well known that existence of weak solutions to (NS_0) can be shown by application of the Leray-Schauder principle (see, e.g., [10]), obviously then both variants of the bilinear operator N coincide. Note that in (NS_ε) only the second variant for the bilinear operator enables to use Schauder's fix point theorem to prove existence of weak solutions. Here we obtain estimates of the same accuracy as for the linear problems but for small solutions of the nonlinear problems, and we can use both variants for the nonlinear term, since there is no difference in the arguments.

The decisive point here is to find an ε independent bound for the bicontinuity constant which is related to the nonlinear operator N in $H^l_\varkappa(\Omega,\varepsilon)$-spaces (Lemma 4). For small data, an application of the Banach contraction principle leads to simultaneous existence and uniqueness of small strong solutions both to (NS_ε) and (NS_0) and error estimates for the difference of the same order as for the linear problems.

2 Results for the linear problems

Before we recall the results for the linear problem, we introduce some general notations. As already mentioned, $\Omega \subset \mathbb{R}^3$ is a bounded domain with closure $\overline{\Omega}$, boundary $\partial\Omega$, and for $x \in \partial\Omega$ the exterior unit normal vector is denoted by $n(x)$, if it exists. For any $t \in \mathbb{R}$, we call $[t]$ the integer part of t, i.e. $[t] = \max\{j \in \mathbb{Z} : j \leq t\}$, while the number $t_+ = (t + |t|)/2$ means the positive part of t.

We use the common multi-index terminology: $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1}\frac{\partial^{\alpha_2}}{\partial x_2}\frac{\partial^{\alpha_3}}{\partial x_3}$, with $\alpha \in \mathbb{N}_0^3$. If $k \in \mathbb{N}$, then ∇^k indicates the collection of all partial derivatives of order k. For any function space X, we indicate the norm in X by $\|\cdot; X\|$. Given $l \in \mathbb{N}$, the Sobolev space $H^l(\Omega)$ consists of all $\varphi \in L^2(\Omega)$ with distributional derivatives $\partial^\alpha\varphi \in L^2(\Omega)$, $|\alpha| \leq l$, supplied with the usual norm. If $s > 0$, $s \notin \mathbb{N}$, then for an open set $G \subset \mathbb{R}^n$, (here we need $n = 2$ and $n = 3$) the Sobolev-Slobodetskii space $H^s(G)$ coincides with the set $\{\varphi \in H^{[s]}(G) : \|\varphi; H^s(G)\| < \infty\}$, while

$$\|\varphi; H^s(G)\|^2 = \|v; H^{[s]}(G)\|^2 + \sum_{|\alpha|=[s]} \int_G\int_G \frac{|\partial^\alpha\varphi(x) - \partial^\alpha\varphi(\xi)|^2}{|x-\xi|^{n+2(s-[s])}} d\xi dx.$$

Furthermore, $H^{-s}(G)$ is the dual space of $\overset{\circ}{H}{}^s(G)$, here the supscript $\circ$ indicates the closure of $C_0^\infty(G)$ in $H^s(G)$. Then $H^s(\partial\Omega)$ is defined, by using local coordinates and a partition of unity on $\partial\Omega$, via the definition of $H^s(\mathbb{R}^2)$, while $H^{-s}(\partial\Omega)$ is the dual space of $H^s(\partial\Omega)$ (see [4] for details.)

To obtain asymptotically precise estimates for solutions u^ε to problem (S_ε), the spaces $H^l(\Omega)$ are supplied with equivalent norms, where the small parameter $\varepsilon \in (0, 1]$ is included (see [5, 7]). For $\varkappa \leq l$ and $\varphi \in H^l(\Omega)$ we set

$$\|\varphi; H^l_\varkappa(\Omega;\varepsilon)\|^2 = \|\varphi; H^\varkappa(\Omega)\|^2 + \sum_{k=0}^{l} \varepsilon^{2(k-\varkappa)_+}\|\nabla\varphi; L^2(\Omega)\|^2. \tag{1}$$

Note that the exponents in the powers of ε depend step-like on order k of the derivatives. Then for any differential operator ∂^α with $|\alpha| \leq l$, and k with $\varkappa + k \leq l$, we have

$$\|\partial^\alpha\varphi; H^{l-|\alpha|}_{\varkappa-|\alpha|}(\Omega;\varepsilon)\| \leq C\|\varphi; H^l_\varkappa(\Omega;\varepsilon)\|, \tag{2}$$

$$\|\varepsilon^k\partial^\alpha\varphi; H^{l-|\alpha|}_{\varkappa-|\alpha|+k}(\Omega;\varepsilon)\| \leq C\|\varphi; H^l_\varkappa(\Omega;\varepsilon)\|. \tag{3}$$

We process the trace spaces $H^{l-1/2}(\partial\Omega)$ in the following way. We define $\|v; \mathcal{H}^{l-1/2}_{\varkappa-1/2}(\partial\Omega;\varepsilon)\|$ as $\Big(\|v; H^{\varkappa-1/2}(\partial\Omega)\|^2 + \varepsilon^{2(l-\varkappa)}\|v; H^{l-1/2}(\partial\Omega)\|^2\Big)^{1/2}$ for $\varkappa > 1/2$; and $\varepsilon^{-\varkappa+1/2}\Big(\|v; L^2(\partial\Omega)\|^2 + \varepsilon^{2l-1}\|v; H^{l-1/2}(\partial\Omega)\|^2\Big)^{1/2}$ for $\varkappa < 1/2$. The norms of the trace operators $\partial_n^h : H^l_\varkappa(\Omega;\varepsilon) \to H^{l-h-1/2}_{\varkappa-h-1/2}(\partial\Omega;\varepsilon)$, where $h = 0$ and $h = 1$, can be bounded independent on $\varepsilon \in (0, 1]$, the converse result on extensions is also true. To formulate the complete result we also need $H^l_\varkappa(\Omega,\varepsilon)$ with $\varkappa > l$. In this case we choose $t \in \mathbb{N} = \{1, 2, \dots\}$ such that $t + l \geq \varkappa > l$. We introduce the Helmholtz operator $\mathcal{L}_\varepsilon = 1 - \varepsilon^2\Delta$ with domain $\mathscr{D}(\mathcal{L}_\varepsilon) = H^2(\mathbb{R}^3)$, if $\Omega = \mathbb{R}^3$ and $\mathscr{D}(\mathcal{L}_\varepsilon) = \{\varphi \in H^2(\Omega) : \varphi = 0 \text{ on } \partial\Omega\}$, if Ω is a proper subset of $\mathbb{R}^3$. Since the spectrum of $\mathcal{L}_\varepsilon$ is contained in the interval $[1, \infty)$, there exists $\mathcal{L}_\varepsilon^{-t/2}u \in H^{l+t}(\Omega)$, if $u \in H^l(\Omega)$ and we set

$$\|u; H^l_\varkappa(\Omega,\varepsilon)\| = \|\mathcal{L}_\varepsilon^{-t/2}u; H^{l+t}_\varkappa(\Omega,\varepsilon)\|. \tag{4}$$

Let $\partial\Omega$ be of class C^{l+2} for some $l \in \mathbb{N}$. It is well known that for $f \in H^{l-1}(\Omega) \times H^l(\Omega)$, and

$g' \in H^{l+1/2}(\partial\Omega)^3$, subject to the compatibility condition

$$\int_\Omega f_4 - \int_{\partial\Omega} g' \cdot n do = 0, \tag{5}$$

problem (S_0) possesses a solution $u^0 = (v^0, p^0) \in H^{l+1}(\Omega)^3$ which is unique under the condition that p^0 is mean value free. In [7] the following inequality was proved for this solution:

$$\begin{aligned} &\|v^0; H^{l+1}_{\varkappa+1}(\Omega;\varepsilon)\| + \|p^0; H^l_\varkappa(\Omega;\varepsilon)\| \\ &\le C \Big(\|f'; H^{l-1}_{\varkappa-1}(\Omega;\varepsilon)\| + \|f_4; H^l_\varkappa(\Omega;\varepsilon)\| \\ &\qquad + \|g'; H^{l+1/2}_{\varkappa+1/2}(\partial\Omega;\varepsilon)\| \Big), \end{aligned} \tag{6}$$

For solutions to the problem (S_ε) the following result is valid – we recall it only for the case $g_4 = 0$.

Theorem 1 [7] *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega$ of class C^{l+2}, $l \in \mathbb{N}$, $\varkappa \in [0, 3/2)$, $\varkappa \le l$ and $g_4 = 0$ in* (S_ε)*. Then for any $f \in H^{l-1}(\Omega)^4$ and $g' \in H^{l+1/2}(\partial\Omega)^3$ complying with* (5) *there exists a solution $u^\varepsilon = (v^\varepsilon, p^\varepsilon) \in H^{l+1}(\Omega)$ to problem* (S_ε)*, which is unique under the orthogonality condition $\int_\Omega p^\varepsilon = 0$. This solution satisfies the estimate*

$$\begin{aligned} &\|v^\varepsilon; H^{l+1}_{\varkappa+1}(\Omega;\varepsilon)\| + \|p^\varepsilon; H^{l+1}_\varkappa(\Omega;\varepsilon)_\perp\| \\ &\le C \Big(\|f'; H^{l-1}_{\varkappa-1}(\Omega;\varepsilon)\| + \|f_4; H^{l-1}_\varkappa(\Omega;\varepsilon)\| \\ &\qquad + \|g'; H^{l+1/2}_{\varkappa+1/2}(\partial\Omega;\varepsilon)\| \Big), \end{aligned} \tag{7}$$

where C is a constant depending on $\varkappa, \partial\Omega$ and l, but neither on $\varepsilon \in (0, 1]$ nor on the data (f, g). The subscript $\perp$ indicates the subspace of mean value free functions.

To shorten the notations for the following proofs, we abbreviate the left hand side of (6) to $\|u^0, \mathbf{D}^l_\varkappa(\Omega, \varepsilon)\|$ and the right hand side to $\|(f, g'); \mathbf{R}^l_\varkappa(\Omega, \partial\Omega, \varepsilon)\|$. In a similar manner we denote the left hand side of (7) by $\|u^\varepsilon; \mathscr{D}^l_\varkappa(\Omega, \varepsilon)\|$ and the right hand side by $\|(f, g'); \mathscr{R}^l_\varkappa(\Omega, \partial\Omega, \varepsilon)\|$. The expression $(f, g') \in \mathbf{R}^l_\varkappa(\Omega, \partial\Omega, \varepsilon)_\perp, \mathscr{R}^l_\varkappa(\Omega, \partial\Omega, \varepsilon)_\perp$ means that (5) is fulfilled, while $u = (v, p) \in \mathbf{D}^l_\varkappa(\Omega, \partial\Omega, \varepsilon)_\perp$, $\mathscr{D}^l_\varkappa(\Omega, \partial\Omega, \varepsilon)_\perp$ indicates again that $\int_\Omega p = 0$.

Depending on the smoothness of the data, we may apply the estimates (6) or (7) to the difference $u^\varepsilon - u^0$, which leads to the following result.

Theorem 2 [7] *Let $\varkappa \in [0, 3/2)$, $f \in H^l(\Omega)^4$, $g' \in H^{l+3/2}(\partial\Omega)$ fulfilling (5), moreover we assume*

$$\delta \in [0, \varkappa],\ l \in \mathbb{N}_0 \text{ with } l \ge \max\{\varkappa - \delta, \varkappa - 1\}, \tag{8}$$

The difference $u^\varepsilon - u^0$ satisfies the following inequality with a constant independent of $\varepsilon \in (0, 1]$ and the data:

$$\begin{aligned} &\|u^\varepsilon - u^0; \mathbf{D}^l_{\varkappa-\delta}(\Omega;\varepsilon)\| \le \\ &\quad C\varepsilon^\delta \Big(\|f; H^l(\Omega)^4\| + \|g'; H^{l+3/2}\| \Big). \end{aligned} \tag{9}$$

If, in addition to (8)*, the requirements*

$$l \ge \varkappa, \quad f_4 \in H^{l+1}(\Omega) \tag{10}$$

are met, then the inequality (9) *can be strengthened to*

$$\begin{aligned} &\|u^\varepsilon - u^0; \mathscr{D}^l_\varkappa(\Omega;\varepsilon)\| \le C\varepsilon^{3/2-\varkappa} \Big(\|f'; H^l(\Omega)^3\| \\ &\quad + \|f_4; H^{l+1}(\Omega)\| + \|g'; H^{l+3/2}\| \Big). \end{aligned} \tag{11}$$

3 The Navier-Stokes problem

In order to obtain similar estimates in the case of the nonlinear problems, at least for small data, we start with proving the existence of unique strong solutions (under smallness conditions) to the singular perturbed nonlinear problem (NS_ε). This system as well as the Navier-Stokes system (NS_0) have the structure $\mathbf{S}\mathbf{u} + \mathbf{N}(\mathbf{u}, \mathbf{u}) = \mathbf{f}$, where $\mathbf{S}$ is a linear operator and $\mathbf{N}$ is a bilinear operator acting between certain function spaces. There is a well-known technique to solve such problems, which we recall in the following lemma. The proof is obvious and uses the Banach contraction principle.

Lemma 3 *Let $\mathscr{D}$, $\mathscr{R}$ be Banach spaces, $\mathbf{S} : \mathscr{D} \to \mathscr{R}$ a bounded invertible linear operator with operator norm $\|\mathbf{S}^{-1} : \mathscr{R} \to \mathscr{D}\| = C_S$. Let also $\mathbf{N} : \mathscr{D} \times \mathscr{D} \to \mathscr{R}$ be a bilinear, bicontinuous operator, i.e., $\|\mathbf{N}(\mathbf{u}, \mathbf{w}); \mathscr{R}\| \le C_N \|\mathbf{u}; \mathscr{D}\| \, \|\mathbf{w}; \mathscr{D}\|$. Then, for any $\mathbf{f} \in \mathscr{R}$ with*

$$\|\mathbf{f}, \mathscr{R}\| < 2C_S^2 {C_N}^{-1}, \tag{12}$$

in the set $\{\mathbf{u} : \|\mathbf{u}; \mathscr{D}\| < 2C_S {C_N}^{-1}\}$, there exists a unique solution $\mathbf{u}$ to the nonlinear equation $\mathbf{S}\mathbf{u} + \mathbf{N}(\mathbf{u}, \mathbf{u}) = \mathbf{f}$. The solution $\mathbf{u}$ fulfills the estimate

$$\|\mathbf{u}; \mathscr{D}\| \le 2C_S \|\mathbf{f}; \mathscr{R}\|. \tag{13}$$

□

In order to solve (NS_ε), we apply this lemma with the function spaces $\mathscr{D} = \mathscr{D}^l_\varkappa(\Omega, \varepsilon)$, $\mathscr{R} = \mathscr{R}^l_\varkappa(\Omega, \partial\Omega, \varepsilon)$. Thereby we use the operator $\mathbf{S}u = (S_\varepsilon u, B_\varepsilon u)$, with

$S_\varepsilon u = (-\Delta v + \nabla p, -\varepsilon^2 \Delta p + \mathrm{div}\ v)$ and $B_\varepsilon u = (v|_{\partial\Omega}, \partial_n v|_{\partial\Omega})$, The nonlinear operator is defined by $\mathbf{N}(u,\tilde{u}) = (N(v,\tilde{v}),0,0)$ with $u,\tilde{u} \in \mathscr{D}^l_\varkappa(\Omega,\varepsilon)$ and $N(v,\tilde{v}) = (v\cdot\nabla)\tilde{v}$ or $N(v,\tilde{v}) = (v\cdot\nabla)\tilde{v} + 2^{-1}(\mathrm{div}\ v)\tilde{v}$. Based on embeddings theorems one can prove the following result:

Lemma 4 *If $l \geq \varkappa \geq 0$, $i = 1,2,3$, then*

$$\begin{aligned} &\|v\,\partial_i w; H^{l-1}_{\varkappa-1}(\Omega,\varepsilon)\| \\ &\quad\leq\ C\,\|v; H^{l+1}_{\varkappa+1}(\Omega,\varepsilon)\|\,\|w; H^{l+1}_{\varkappa+1}(\Omega,\varepsilon)\|, \end{aligned} \tag{14}$$

where C depends neither on $\varepsilon \in (0,1]$, nor on v and w.

Corollary 5 *Let $\varkappa \in [0,3/2)$, and $l \geq \varkappa$. There exist constants $\rho > 0, M > 0$, independent of $\varepsilon \in (0,1]$, with the property: For any set of data $(f,g) \in \mathscr{R}^l(\Omega,\partial\Omega)_\perp$ (cf. the notations after Theorem 1), which fulfil the smallness condition*

$$\|(f,g); \mathscr{R}^l_\varkappa(\Omega,\partial\Omega,\varepsilon)\| \leq \rho, \tag{15}$$

there exists a unique solution $u^\varepsilon \in \mathscr{D}^l(\Omega,\varepsilon)_\perp$ to the nonlinear problem (NS$_\varepsilon$), with

$$\|u^\varepsilon; \mathscr{D}^l_\varkappa(\Omega,\varepsilon)\| \leq M\,\|(f,g); \mathscr{R}^l_\varkappa(\Omega,\partial\Omega,\varepsilon)\|. \tag{16}$$

PROOF. This result follows immediately from Lemma 3 if we observe that Theorem 1 gives an estimate for C_S independent of ε, while Lemma 4 does this job for the constant C_N. Moreover, the compatibility condition (5) is not influenced by the nonlinear term, therefore $\mathbf{N}(u,u) \in \mathscr{R}^l(\Omega,\partial\Omega)_\perp$ for any $u \in \mathscr{D}^l H(\Omega)$. □

The same arguments are used to prove the corresponding result for the stationary Navier-Stokes system. Again, we shorten notations to $\mathbf{R}^l(\Omega,\partial\Omega) = H^{l-1}(\Omega)^3 \times H^l(\Omega) \times H^{l+1/2}(\partial\Omega)^3$ and $\mathbf{D}^l(\Omega) = H^{l+1}(\Omega)^3 \times H^l(\Omega)$.

Corollary 6 *Let $l \in \mathbb{N}$, and $(f,g') \in \mathbf{R}^l(\Omega,\partial\Omega)$ be given such that the compatibility condition (5) is met. There exist constants $\rho_0 > 0$ and $M_0 > 0$ with the property: If the data (f,g') fulfill the smallness condition*

$$\|(f,g'); \mathbf{R}^l(\Omega,\partial\Omega)\| \leq \rho_0, \tag{17}$$

then the Navier-Stokes system (NS$_0$) possesses a unique solution $u \in \mathbf{D}^l(\Omega)_\perp$ which fulfills

$$\|u; \mathbf{D}^l(\Omega)_\perp\| \leq M_0\,\|(f,g'); \mathbf{R}^l(\Omega,\partial\Omega)\|. \tag{18}$$

Note that (18) implies

$$\|u; \mathbf{D}^l_\varkappa(\Omega,\varepsilon)_\perp\| \leq M_0\,\|(f,g'); \mathbf{R}^l(\Omega,\partial\Omega)\|. \tag{19}$$

Like for the linear problems, we want to compare the solution to the perturbed problem (NS$_\varepsilon$) with $g_4 = 0$ to those of (NS$_0$). To this end we observe that for $f \in H^l(\Omega)^4$ and $g' \in H^{l+3/2}(\partial\Omega)$,

$$\begin{aligned} &\|(f,g'); \mathbf{R}^l_\varkappa(\Omega,\partial\Omega,\varepsilon)\| \\ &\quad\leq\ \|(f,(g',0)); \mathscr{R}^{l+1}_\varkappa(\Omega,\partial\Omega,\varepsilon)\| \\ &\quad\leq\ \|f; H^l(\Omega)^4\| + \|g'; H^{l+3/2}(\partial\Omega)^3\| \\ &\quad=:\ \mathfrak{F} \end{aligned} \tag{20}$$

holds true with a constant independent of ε and $\varkappa$ (recall that $\varkappa \leq l$). If we assume that (f,g) satisfies (15) at $\varepsilon = 1$, then this condition is valid for all $\varepsilon \in (0,1]$. Thus we can find a $\hat{\rho} \leq \min\{\rho,\rho_0\}$ where the condition (17) for (f,g') , with ρ replaced by $\hat{\rho}$, implies (15) for (f,g). In this case we obtain unique solutions u^ε and u^0 to the nonlinear problems (NS$_\varepsilon$) and (NS$_0$) as well, moreover, from the inclusions $H^{l+1}(\Omega)^4 \subset H^{l+1}(\Omega)^3 \times H^l(\Omega) \subset H^l(\Omega)^3$, and the estimates (16), (19) and (20) we have

$$\|u^\varepsilon; \mathbf{D}^l_\varkappa(\Omega)_\perp\|, \|u^0; \mathbf{D}^l_\varkappa(\Omega)_\perp\| \leq \widetilde{M}\,\mathfrak{F} \tag{21}$$

independent of ε.

Theorem 7 *Let $\varkappa$, l, δ satisfy the conditions of Theorem 2. Let f, g' be fixed under the conditions explained above, furthermore, $u^0 = (v^0,p^0)$ and $u^\varepsilon = (v^\varepsilon,p^\varepsilon)$ denote the solutions to the Navier-Stokes problem (NS$_0$), and the perturbed problem (NS$_\varepsilon$), respectively, while the pressure components p^0 and p^ε have zero mean value in Ω. Then there exists a $\hat{\rho}_0 \leq \hat{\rho}$ such that, under the condition $\mathfrak{F} \leq \hat{\rho}_0$, the difference $u^\varepsilon - u^0$ can be estimated by inequality (9), in particular.*

$$\begin{aligned} \|v^\varepsilon - v^0; H^{\varkappa+1-\delta}(\Omega)^3\| \ &+\ \|p^\varepsilon - p^0; H^{\varkappa-\delta}(\Omega)\| \\ &\leq\ C\,\varepsilon^\delta \mathfrak{F}. \end{aligned} \tag{22}$$

PROOF. The differences $r^\varepsilon = v^\varepsilon - v^0$, $q^\varepsilon = p^\varepsilon - p^0$ satisfy the following perturbed Stokes problem:

$$\begin{aligned} -\Delta r^\varepsilon + N(v^\varepsilon,r^\varepsilon) - N(r^\varepsilon,v^0) + \nabla q^\varepsilon &= 0, \\ \mathrm{div}\ r^\varepsilon = \varepsilon^2 \Delta p^\varepsilon \quad &\text{in } \Omega, \\ r^\varepsilon = 0 \qquad &\text{on } \partial\Omega. \end{aligned}$$

From the results quoted in Section 2, we conclude, that the mapping $u = (v,p) \mapsto (-\Delta v + \nabla p, \mathrm{div}\ v)$

defines an isomorphism

$$\begin{aligned}\mathbf{S} : \mathbf{D}^{l,0}_{\varkappa-\delta} := \{u \in \mathbf{D}^{l}_{\varkappa-\delta}(\Omega, \partial\Omega, \varepsilon) :\\ v = 0 \text{ on } \partial\Omega, \textstyle\int_\Omega p = 0\}\\ \to \mathbf{R}^{l,0}_{\varkappa-\delta} = \{f \in \mathbf{R}^{l}_{\varkappa-\delta}(\Omega) : \textstyle\int_\Omega f_4 = 0\}.\end{aligned}$$

Here the norm of $\mathbf{S}^{-1}$ is bounded independent of $\varepsilon \in (0,1]$ due to (6) while ε independent bounds for $\mathbf{S}$ follow immediately from the definition of the norms. The error system above has the structure $\mathbf{Sr} + \mathbf{Pr} = \mathbf{f}$ with a linear perturbation $\mathbf{P}$, and the operator $\mathbf{S} + \mathbf{P}$ keeps the properties of $\mathbf{S}$, if $\mathbf{P}$ is small. Lemma 4 leads to the estimate

$$\begin{aligned}&\|\mathbf{P}(r^\varepsilon, q^\varepsilon); \mathbf{R}^{l}_{\varkappa-\delta} H(\Omega,\varepsilon)\|\\ &= \ \|N(v^\varepsilon, r^\varepsilon) - N(r^\varepsilon, v^0); H^{l-1}_{\varkappa-\delta-1}(\Omega,\varepsilon)\|\\ &\leq \ C\Big(\|v^\varepsilon; H^{l+1}_{\varkappa-\delta+1}(\Omega,\varepsilon)\|\\ &\quad + \|v^0; H^{l+1}_{\varkappa-\delta+1}(\Omega,\varepsilon)\|\Big)\|r^\varepsilon; H^{l+1}_{\varkappa-\delta+1}(\Omega,\varepsilon)\|\\ &\leq \ C\,\mathfrak{F}\,\|r^\varepsilon; H^{l+1}_{\varkappa-\delta+1}(\Omega,\varepsilon)\|,\end{aligned}$$

for the last inequality we used (21). Thus, we have proved that

$$\|\mathbf{P} : \mathbf{D}^{l,0}_{\varkappa-\delta} \to \mathbf{R}^{l,0}_{\varkappa-\delta}\| \leq C\,\mathfrak{F}$$

independent of $\varepsilon \in (0,1]$, and by a classical perturbation argument, the assertions and arguments of Theorem 2 remain valid for the operator $\mathbf{S} + \mathbf{P}$ if $\mathfrak{F}$ is small enough. □

Conclusions. For small data, the Navier-Stokes problem (NS) as well as as the perturbed problem (NS$_\varepsilon$) possess a unique solution. If the data are smooth enough, the velocity part v^ε converges to v^0 in $H^2(\Omega)^3$, while the pressure p^ε converges to p^0 in $H^1(\Omega)$ as ε tends to 0. The convergence order in $H^1(\Omega)^3$ and $L^2(\omega)$, respectively, is higher than the order obtained by energy methods.

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