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Unsteady flow of Oldroyd-B fluids in an uniform rectilinear pipe using 1D models

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Abstract: Using the one-dimensional approach based on the director theory which reduces the exact three-dimensional equations to a system depending only on time and on a single spatial variable, we analyze the axisymmetric unsteady flow of an incompressible viscoelastic fluid of Oldroyd type in an uniform rectilinear pipe with circular cross-section. From this system we obtain the relationship between average pressure gradient and volume flow rate over a finite section of the pipe and the corresponding equation for the wall shear stress. Attention is focused on the steady flow case with rigid and impermeable walls.

Key–Words: Cosserat theory, Oldroyd-B fluid, steady solution, axisymmetric motion, volume flow rate, average pressure gradient.

1 Introduction

In this paper we introduce a 1D model for viscoelastic non-Newtonian Oldrody-B flows in an axisymmetric pipe with circular cross-section, based on the director approach (Cosserat theory) with nine directors developed by Caulk and Naghdi [5]. The theoretical basis of this approach (see Cosserat [6], [7], based on the work of Duhen [8]) is to consider an additional structure of deformable vectors (called directors) assigned to each point on a space curve (the Cosserat curve). With this approach and integrating the axial component of linear momentum for the flow field over the pipe cross-section, the 3D system of equations is replaced by a system of partial differential equations which, apart from the dependence on time, depends only on a single spatial variable. Using this one-dimensional Cosserat theory we can predict some of the main properties of the three-dimensional problem. For additional background information, we refer that the Cosserat theory has been used in studies of rods, plates and shells, see e.g. Ericksen and Truesdell [9], Truesdell and Toupin [18], Green et al. [14], [13] and Naghdi [16]. Later, this theory has been developed by Caulk and Naghdi [5], Green and Naghdi [15], and Green et al. [12] in studies of unsteady and steady flows, related to fluid dynamics. Recently, the nine-director approach has been applied to blood flow in the arterial system by Robertson and Sequeira [17] and also by Carapau and Sequeira [2], [3], [4], considering Newtonian and non-Newtonian flows, respectively.

In this paper we are interested in studying the initial boundary value problem of an incompressible homogeneous Oldroyd-B fluid model in a straight circular rigid and impermeable pipe with constant radius where the fluid velocity field, given by the director theory, can be approximated by the following finite series[1]:

$$\boldsymbol{v}^* = \boldsymbol{v} + \sum_{N=1}^{k} x_{\alpha_1} \dots x_{\alpha_N} \boldsymbol{W}_{\alpha_1 \dots \alpha_N}, \qquad (1)$$

with

$$\boldsymbol{v} = v_i(z,t)\boldsymbol{e}_i, \quad \boldsymbol{W}_{\alpha_1 \dots \alpha_N} = W^i_{\alpha_1 \dots \alpha_N}(z,t)\boldsymbol{e}_i. \quad (2)$$

Here, $\boldsymbol{v}$ represents the velocity along the axis of symmetry z at time t, $x_{\alpha_1} \dots x_{\alpha_N}$ are the polynomial weighting functions with order k (the number k identifies the order of hierarchical theory and is related to the number of directors), the vectors $\boldsymbol{W}_{\alpha_1 \dots \alpha_N}$ are the director velocities which are completely symmetric with respect to their indices and $\boldsymbol{e}_i$ are the associated unit basis vectors. From this velocity field approach, we obtain

[1]Latin indices subscript take the values $1, 2, 3$, Greek indices subscript $1, 2$. Summation convention is employed over a repeated index.

the unsteady relationship between average pressure gradient and volume flow rate, and the correspondent equation for the wall shear stress.

The goal of this paper is to develop a nine-director theory ($k = 3$ in (1)) for the steady flow of an Oldroyd-B fluid in a straight pipe with constant radius, to compare the average pressure gradient for different values of both Reynolds and Weissenberg numbers.

2 Model Problem

Let us consider a homogeneous fluid inside a circular straight and impermeable pipe, the domain $\Omega \subset \mathbb{R}^3$, with boundary $\partial\Omega$ composed by the proximal cross-section Γ_1, the distal cross-section Γ_2 and the lateral wall Γ_w, see Fig.1.

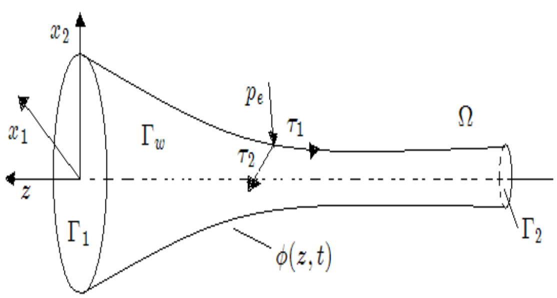

Figure 1: Fluid domain Ω with the components of the surface traction vector τ_1, τ_2 and p_e. Γ_w is the lateral wall of the pipe with equation $\phi(z,t)$, and Γ_1, Γ_2 are the upstream part and downstream districts of the pipe, respectively.

Let x_i $(i = 1,2,3)$ be the rectangular Cartesian coordinates and for convenience set $x_3 = z$. Consider the axisymmetric motion of an incompressible fluid without body forces, inside a surface of revolution, about the z axis and let $\phi(z,t)$ denote the instantaneous radius of that surface at z and time t. The components of the three-dimensional equations governing an Oldroyd type fluid motion are given in $\Omega' = \Omega \times (0,T)$ by[2]

$$\begin{cases} \rho\Big(\dfrac{\partial \boldsymbol{v}^*}{\partial t} + v^*_{,i}\boldsymbol{v}^*_i\Big) = t_{i,i}, \\ v^*_{i,i} = 0, \\ t_i = -p^*\boldsymbol{e}_i + \sigma_{ij}\boldsymbol{e}_j,\ \boldsymbol{t} = \vartheta^*_i t_i, \\ \sigma_{e_{ij}} + \lambda_1 \overset{\nabla}{\sigma}_{e_{ij}} = \mu_e\big(v^*_{i,j} + v^*_{j,i}\big), \end{cases} \quad in\ \Omega', \qquad (3)$$

[2]We use the notation $v^*_{i,j} = \partial v^*_i/\partial x_j$ and $v^*_{,i}\boldsymbol{v}^*_i = v^*_i \partial \boldsymbol{v}^*/\partial x_i$ adopted in Naghdi et al. [5], [11].

with the initial condition

$$\boldsymbol{v}^*(x,0) = \boldsymbol{v}_0(x) \ \text{ in } \ \Omega, \qquad (4)$$

and the boundary condition

$$\boldsymbol{v}^*(x,t) = 0 \ \text{ on } \ \Gamma_w \times (0,T), \qquad (5)$$

where $\boldsymbol{v}^* = v^*_i \boldsymbol{e}_i$ is the velocity field and ρ is the constant fluid density. Equation $(3)_1$ represents the balance of linear momentum and $(3)_2$ is the incompressibility condition. In equation $(3)_3$, p^* is the pressure and σ_{ij} are the components of the (symmetric) extra stress tensor given by

$$\sigma_{ij} = \mu_n\big(v^*_{i,j} + v^*_{j,i}\big) + \sigma_{e_{ij}},$$

where $\sigma_{e_{ij}}$ are the components of its viscoelastic part. Here $\boldsymbol{t}$ denotes the stress vector on the surface whose outward unit normal is $\vartheta^* = \vartheta^*_i \boldsymbol{e}_i$, and t_i are the components of $\boldsymbol{t}$. In equation $(3)_4$ the symbol $\overset{\nabla}{\sigma}_{e_{ij}}$ represents the objective Oldroyd derivative of the tensor $\sigma_{e_{ij}}$ given by (see e.g. [19])

$$\begin{aligned} \overset{\nabla}{\sigma}_{e_{ij}} &= \frac{\partial \sigma_{eij}}{\partial t} + v^*_k \frac{\partial \sigma_{eij}}{\partial x_k} + \sigma_{eik}\big(v^*_{k,j} - v^*_{j,k}\big) \\ &- \big(v^*_{i,k} - v^*_{k,i}\big)\sigma_{ekj} - a\Big[\big(v^*_{i,k} + v^*_{k,i}\big)\sigma_{ekj} \\ &+ \sigma_{eik}\big(v^*_{k,j} + v^*_{j,k}\big)\Big], \end{aligned} \qquad (6)$$

where $a \in [-1,1]$ is a real given parameter. The initial velocity field $\boldsymbol{v}_0$ is assumed to be known. Finally, $\mu_n = (\mu\lambda_2)/\lambda_1$ is the Newtonian viscosity and $\mu_e = \mu\big(1 - \lambda_2/\lambda_1\big) = \mu\lambda$ is the elastic viscosity, with $\mu = \mu_n + \mu_e$ denoting the viscosity coefficient, and the constants λ_1 and λ_2 (with $0 \leqslant \lambda_2 < \lambda_1$) being the relaxation and retardation times, respectively. Models with $\lambda_2 = 0$ are called "of Maxwell type" and those with $\lambda_2 > 0$ "of Jeffreys type". Oldroyd-B fluids correspond to Jeffreys type fluids with $a = 1$ (in equation (6)) and Oldroyd-A fluids correspond to $a = -1$, see e.g. [10].

The lateral surface Γ_w of the axisymmetric domain is defined by

$$\phi^2 = x_\alpha x_\alpha, \qquad (7)$$

and the components of the outward unit normal to this surface are

$$\vartheta^*_\alpha = \frac{x_\alpha}{\phi\big(1+\phi_z^2\big)^{1/2}}, \quad \vartheta^*_3 = -\frac{\phi_z}{\big(1+\phi_z^2\big)^{1/2}}, \qquad (8)$$

where a subscript variable denotes partial differentiation. Since equation (7) defines a material

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surface, the velocity field must satisfy the condition

$$\phi\phi_t + \phi\phi_z v_3^* - x_\alpha v_\alpha^* = 0 \tag{9}$$

at the boundary (7).

Let us consider $S(z,t)$ as a generic axial section of the domain at time t defined by the spatial variable z and bounded by the circle defined in (7) and let $A(z,t)$ be the area of this section $S(z,t)$. The volume flow rate Q is defined by

$$Q(z,t) = \int_{S(z,t)} v_3^*(x_1, x_2, z, t)da, \tag{10}$$

and the average pressure $\bar{p}$ is defined by

$$\bar{p}(z,t) = \frac{1}{A(z,t)} \int_{S(z,t)} p^*(x_1, x_2, z, t)da. \tag{11}$$

In what follows, this general framework will be applied to the specific case of the nine-director theory in a rigid pipe, i.e. $\phi = \phi(z)$. Using condition (1), with $k = 3$, it follows from Caulk and Naghdi [5] that the approximation for the three-dimensional velocity field $\boldsymbol{v}^*$ is given by

$$\begin{aligned} \boldsymbol{v}^* &= \Big[x_1\big(1 - \frac{x_1^2 + x_2^2}{\phi^2}\big)\frac{2\phi_z Q}{\pi\phi^3}\Big]\boldsymbol{e}_1 \\ &+ \Big[x_2\big(1 - \frac{x_1^2 + x_2^2}{\phi^2}\big)\frac{2\phi_z Q}{\pi\phi^3}\Big]\boldsymbol{e}_2 \\ &+ \Big[\frac{2Q}{\pi\phi^2}\big(1 - \frac{x_1^2 + x_2^2}{\phi^2}\big)\Big]\boldsymbol{e}_3 \end{aligned} \tag{12}$$

where the volume flow rate $Q(t)$ is

$$Q(t) = \frac{\pi}{2}\phi^2(z)v_3(z,t). \tag{13}$$

We remark that the initial condition (4) is satisfied when $Q(0) = ct$. Also, from Caulk and Naghdi [5] the stress vector on the lateral surface Γ_w is given by

$$\begin{aligned} \boldsymbol{t}_w &= \Big[\frac{1}{\phi(1+\phi_z^2)^{1/2}}\Big(\tau_1 x_1 \phi_z - p_e x_1 \\ &- \tau_2 x_2 (1+\phi_z^2)^{1/2}\Big)\Big]\boldsymbol{e}_1 \\ &+ \Big[\frac{1}{\phi(1+\phi_z^2)^{1/2}}\Big(\tau_1 x_2 \phi_z - p_e x_2 \\ &+ \tau_2 x_1 (1+\phi_z^2)^{1/2}\Big)\Big]\boldsymbol{e}_2 \\ &+ \Big[\frac{1}{(1+\phi_z^2)^{1/2}}\Big(\tau_1 + p_e\phi_z\Big)\Big]\boldsymbol{e}_3, \end{aligned} \tag{14}$$

where τ_1 represents the wall shear stress in the axial direction of the flow. Instead of satisfying the momentum equation $(3)_1$ pointwise in the fluid, we impose the following integral conditions

$$\int_{S(z,t)} \Big[t_{i,i} - \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v_i^*\Big)\Big]da = 0, \tag{15}$$

$$\int_{S(z,t)} \Big[t_{i,i} - \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v_i^*\Big)\Big]x_{\alpha_1}\ldots x_{\alpha_N}da = 0, \tag{16}$$

where $N = 1, 2, 3$.

Using the divergence theorem and integration by parts, equations (15) – (16) for nine directors, can be reduced to the four vector equations:

$$\frac{\partial \boldsymbol{n}}{\partial z} + \boldsymbol{f} = \boldsymbol{a}, \tag{17}$$

$$\frac{\partial \boldsymbol{m}^{\alpha_1\ldots\alpha_N}}{\partial z} + \boldsymbol{l}^{\alpha_1\ldots\alpha_N} = \boldsymbol{k}^{\alpha_1\ldots\alpha_N} + \boldsymbol{b}^{\alpha_1\ldots\alpha_N}, \tag{18}$$

where $\boldsymbol{n}$, $\boldsymbol{k}^{\alpha_1\ldots\alpha_N}$, $\boldsymbol{m}^{\alpha_1\ldots\alpha_N}$ are resultant forces defined by

$$\boldsymbol{n} = \int_S \boldsymbol{t}_3 da, \quad \boldsymbol{k}^\alpha = \int_S \boldsymbol{t}_\alpha da, \tag{19}$$

$$\boldsymbol{k}^{\alpha\beta} = \int_S \Big(\boldsymbol{t}_\alpha x_\beta + \boldsymbol{t}_\beta x_\alpha\Big)da, \tag{20}$$

$$\boldsymbol{k}^{\alpha\beta\gamma} = \int_S \Big(\boldsymbol{t}_\alpha x_\beta x_\gamma + \boldsymbol{t}_\beta x_\alpha x_\gamma + \boldsymbol{t}_\gamma x_\alpha x_\beta\Big)da, \tag{21}$$

$$\boldsymbol{m}^{\alpha_1\ldots\alpha_N} = \int_S \boldsymbol{t}_3 x_{\alpha_1}\ldots x_{\alpha_N}da. \tag{22}$$

The quantities $\boldsymbol{a}$ and $\boldsymbol{b}^{\alpha_1\ldots\alpha_N}$ are inertia terms defined by

$$\boldsymbol{a} = \int_S \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v_i^*\Big)da, \tag{23}$$

$$\boldsymbol{b}^{\alpha_1\ldots\alpha_N} = \int_S \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v_i^*\Big)x_{\alpha_1}\ldots x_{\alpha_N}da, \tag{24}$$

and $\boldsymbol{f}$, $\boldsymbol{l}^{\alpha_1\ldots\alpha_N}$, which arise due to surface traction on the lateral boundary, are defined by

$$\boldsymbol{f} = \int_{\partial S} \Big(1 + \phi_z^2\Big)^{1/2} \boldsymbol{t}_w ds, \tag{25}$$

$$\boldsymbol{l}^{\alpha_1\ldots\alpha_N} = \int_{\partial S} \Big(1 + \phi_z^2\Big)^{1/2} \boldsymbol{t}_w x_{\alpha_1}\ldots x_{\alpha_N}ds. \tag{26}$$

The equation relating the average pressure gradient with the volume flow rate will be obtained using these quantities (19) – (26).

3 Results and Discussion

Replacing the solutions of equations (19) − (26) into equations (17)−(18) the relationship between average pressure and volume flow rate in a rigid axisymmetric straight pipe with constant radius[3] ϕ, is given by

$$\begin{aligned}\bar{p}_z(z,t) &= -\frac{8\mu_n}{\pi\phi^4}\,Q(t) - \frac{4\rho}{3\pi\phi^2}\,\dot{Q}(t)\\ &+ \frac{2}{\phi^2}(\psi_{33})_z - \frac{4}{\phi^4}(\varpi_{33})_z\\ &- \frac{1}{\phi^2}(\psi_{11})_z + \frac{4}{\phi^4}(\varpi_{11})_z, \qquad (27)\end{aligned}$$

where the functions ψ_{ij} and ϖ_{ij} are defined by[4]

$$\psi_{ij} = \int_S \sigma_{eij}\,da, \quad \varpi_{ij}\delta_\alpha^\beta = \int_S \sigma_{eij}x_\alpha x_\beta\,da,$$

the viscoelastic part of the stress tensor (due to compatibity conditions) takes the particular form

$$\sigma_e = \begin{bmatrix} \sigma_{e11} & 0 & 0\\ 0 & \sigma_{e11} & 0\\ 0 & 0 & \sigma_{e33}\end{bmatrix}, \qquad (28)$$

and, the corresponding wall shear stress τ_1 is given by

$$\begin{aligned}\tau_1 &= \frac{4\mu_n}{\pi\phi^3}Q(t) + \frac{\rho}{6\pi\phi}\dot{Q}(t) - \frac{1}{2\pi\phi}(\psi_{33})_z\\ &+ \frac{2}{\pi\phi^3}(\varpi_{33})_z + \frac{1}{2\pi\phi}(\psi_{11})_z\\ &- \frac{2}{\pi\phi^3}(\varpi_{11})_z. \qquad (29)\end{aligned}$$

Now, let us consider the following dimensionless variables[5]

$$\hat{x} = \frac{x}{\phi}, \quad \hat{\phi} = 1, \quad \hat{t} = \omega_0 t,$$

$$\hat{Q} = \frac{2\rho}{\pi\phi\mu}Q, \quad \hat{\bar{p}} = \frac{\phi^2\rho}{\mu^2}\,\bar{p}, \quad \hat{\sigma_e} = \frac{\phi^2\rho}{\mu^2}\,\sigma_e,$$

where ω_0 is a characteristic frequency for unsteay flow. Substituting these dimensionless variables into equations (27) and $(3)_4$, we obtain, respectively

$$\begin{aligned}\hat{\bar{p}}_{\hat{z}} &= -4\big(1-\lambda\big)\,\hat{Q}(\hat{t}) - \frac{2}{3}\,\mathcal{W}_0^2\,\dot{\hat{Q}}(\hat{t}) + 2\big(\hat{\psi}_{33}\big)_{\hat{z}}\\ &- 4\big(\hat{\varpi}_{33}\big)_{\hat{z}} - \big(\hat{\psi}_{11}\big)_{\hat{z}} + 4\big(\hat{\varpi}_{11}\big)_{\hat{z}}, \qquad (30)\end{aligned}$$

and

$$\hat{\sigma}_{eij} + \mathcal{W}_e\,\overset{\triangledown}{\hat{\sigma}}_{eij} = 2\lambda\hat{D}_{ij} \qquad (31)$$

where $\mathcal{W}_0 = \phi^2\sqrt{(\rho\omega_0)/\mu}$ is the *Womersley number*, which reflect the unsteady flow phenomena, $\mathcal{W}_e = \lambda_1\omega_0$ is the *Weissenberg number*, related with the flow viscoelasticity and

$$D_{ij} = \frac{\mu}{\phi^2\rho}\hat{D}_{ij},$$

where $D_{ij} = \frac{1}{2}\big(v^*_{i,j} + v^*_{j,i}\big)$ is the rate of deformation tensor. Substituting the given dimensionless variables into equation (29), we obtain

$$\begin{aligned}\hat{\tau}_1 &= 2\big(1-\lambda\big)\hat{Q}(\hat{t}) + \mathcal{W}_0^2\frac{1}{12}\dot{\hat{Q}}(\hat{t}) - \frac{1}{2\pi}(\hat{\psi}_{33})_{\hat{z}}\\ &+ \frac{2}{\pi}(\hat{\varpi}_{33})_{\hat{z}} + \frac{1}{2\pi}(\hat{\psi}_{11})_{\hat{z}} - \frac{2}{\pi}(\hat{\varpi}_{11})_{\hat{z}}, \qquad (32)\end{aligned}$$

where

$$\hat{\tau}_1 = \frac{\phi^2\rho}{\mu^2}\tau_1.$$

Integrating condition (30) over the interval $[\hat{z}_1, \hat{z}]$, with $\hat{z}_1$ fixed, we obtain the following relationship between average pressure gradient and volume flow rate

$$\begin{aligned}\hat{\overline{pp}}(\hat{z},t) &= \hat{\bar{p}}(\hat{z},t) - \hat{\bar{p}}(\hat{z}_1,t) \qquad (33)\\ &= 4\big(1-\lambda\big)\,\hat{A}_1(\hat{z})\,\hat{Q}(\hat{t})\\ &+ \frac{2}{3}\,\mathcal{W}_0^2\hat{A}_2(\hat{z})\,\dot{\hat{Q}}(\hat{t})\\ &+ 2\big(\hat{\psi}_{33}(\hat{z},t) - \hat{\psi}_{33}(\hat{z}_1,t)\big)\hat{B}_4(\hat{z})\\ &+ 4\big(\hat{\varpi}_{33}(\hat{z},t) - \hat{\varpi}_{33}(\hat{z}_1,t)\big)\hat{B}_5(\hat{z})\\ &+ \big(\hat{\psi}_{11}(\hat{z},t) - \hat{\psi}_{11}(\hat{z}_1,t)\big)\hat{B}_6(\hat{z})\\ &+ 4\big(\hat{\varpi}_{11}(\hat{z},t) - \hat{\varpi}_{11}(\hat{z}_1,t)\big)\hat{B}_7(\hat{z}),\end{aligned}$$

where $\hat{A}_1(\hat{z}) = \hat{A}_2(\hat{z}) = \hat{B}_5(\hat{z}) = \hat{B}_6(\hat{z}) = \hat{z}_1 - \hat{z}$ and $\hat{B}_4(\hat{z}) = \hat{B}_7(\hat{z}) = \hat{z} - \hat{z}_1$. Now, considering equations (32) and (33) in the steady case and fixing $a = 1$ in (6) we deal with the Oldrody-B fluid model. From, (28) and (31) (dimensionless forms), we obtain

$$\hat{\sigma}_{e11} = \hat{\sigma}_{e22} = 0, \qquad (34)$$

[3]Equation $(3)_4$ introduces some difficulties in handling the general case $\phi = \phi(z)$.

[4]δ_α^β is the two-dimensional Kronecker symbol.

[5]In cases where a steady flow rate is specified, the nondimensional flow rate $\hat{Q}$ is identical to the classical Reynolds number used for flow in pipes, see Robertson and Sequeira [17].

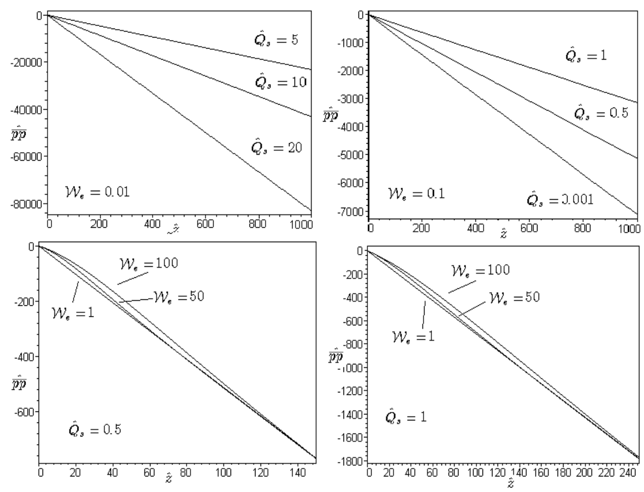

Figure 2: Nondimensional average pressure gradient (33) in the steady case of an Oldroyd-B fluid for different values of the Reynolds number ($\hat{Q}_s = (0.001, 0.5, 1, 5, 10, 20)$) and Weissenberg number ($\mathcal{W}_e = (0.01, 0.1, 1, 50, 100)$).

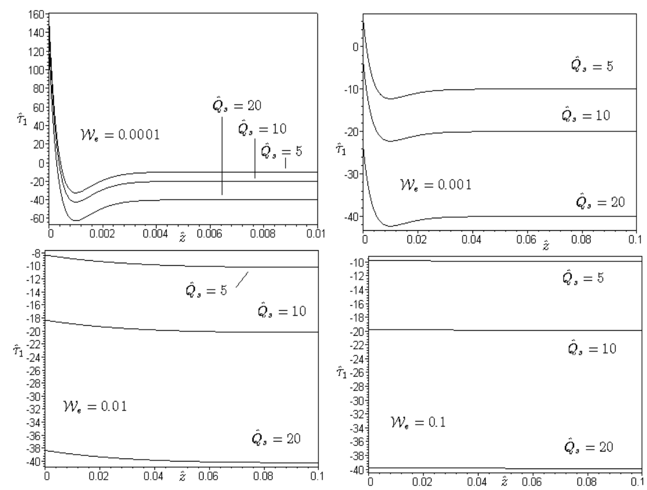

Figure 3: Nondimensional wall shear stress (32) of an Oldroyd-B fluid in the steady case for different values of the Reynolds number ($\hat{Q}_s = (5, 10, 20)$) and Weissenberg number ($\mathcal{W}_e = (0.0001, 0.001, 0.01, 0.1)$).

and

$$\hat{\sigma}_{e33} = exp\Big(\frac{\hat{z}}{\mathcal{W}_e\hat{Q}_s\big(\hat{x}_1^2 + \hat{x}_2^2 - 1\big)}\Big). \quad (35)$$

Using (34), (35) and the approximation

$$exp\Big(\frac{\hat{z}}{\mathcal{W}_e\hat{Q}_s\big(\zeta^2 - 1\big)}\Big) \simeq exp\Big(-\frac{\hat{z}}{\mathcal{W}_e\hat{Q}_s}\Big) - \frac{\zeta^2}{\mathcal{W}_e\hat{Q}_s}\, exp\Big(-\frac{\hat{z}}{\mathcal{W}_e\hat{Q}_s}\Big),$$

we get $\hat{\psi}_{11} = \hat{\varpi}_{11} = 0$,

$$\hat{\psi}_{33} = \pi\, exp\Big(-\frac{\hat{z}}{\mathcal{W}_e\hat{Q}_s}\Big) - \frac{1}{2}\frac{\pi\hat{z}}{\mathcal{W}_e\hat{Q}_s}\, exp\Big(-\frac{\hat{z}}{\mathcal{W}_e\hat{Q}_s}\Big),$$

and

$$\hat{\varpi}_{33} = \frac{1}{4}\pi\, exp\Big(-\frac{\hat{z}}{\mathcal{W}_e\hat{Q}_s}\Big) - \frac{1}{6}\frac{\pi\hat{z}}{\mathcal{W}_e\hat{Q}_s}\, exp\Big(-\frac{\hat{z}}{\mathcal{W}_e\hat{Q}_s}\Big).$$

Again due to compatibility conditions, these results are only valid when $\lambda \simeq 0$, i.e. $\lambda_1 \simeq \lambda_2$. Shown in Fig.2 is the normalized nine-director average pressure gradient steady solution (33) for an Oldroyd-B fluid for different values of Reynolds and Weissenberg numbers in $[0, \hat{z}]$. We conclude that the behavior of the steady solution with fixed Reynolds number does not change when we increase the Weissenberg number. However, with fixed Weissenberg number we can observe a slight change of the steady solution behavior, with increasing Reynolds number. Also, we compare the corresponding wall shear stress (32) for different values of the Reynolds and Weissenberg numbers, see Fig.3, and conclude that it undergoes a small perturbation for $\hat{z}$ close to zero and $\mathcal{W}_e \ll 0.001$. However, for higher values of the Weissenberg number the wall shear stress becomes constant.

4 Conclusion

Contrarily to Newtonian, generalized Newtonian and second order fluids (see e.g. [5], [17], [2], [3], [4], respectively) where the 1D director approach has been applied without restrictions to rectilinear flows, in the case of Oldroyd-B fluids, the 1D theory is only possible when the relaxation and retardation times are close to each other, i.e. $\lambda_1 \simeq \lambda_2$. This is due to compatibility conditions imposed to system (3), as described above. One of the possible extensions of this work is the application of the 1D nine-director approach to other viscoelastic models, including generalized Oldroyd type fluids with shear-dependent viscosity and blood flow models in both rigid and flexible walled straight and curved vessels as well as in vessels with branches or bifurcations. This is the object of ongoing research. More detailed discussion of some of these issues can be found in [1].

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