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On Asymptotic Behavior of Solutions of a Perturbed Non–Steady Stokes Equation in an Exterior Domain

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Abstract: - We formulate sufficient conditions for partial uniform boundedness of the analytic semigroup generated by a perturbed non–steady Stokes operator L in an exterior domain. In spite of the existence of an essential spectrum, we show that the partial (or full) uniform boundedness of the semigroup depends on the behavior of the resolvent operator of L on a finite dimensional space.

Key-Words: - The Stokes equation, Analytic semigroups

1 Introduction

We suppose that Ω is an exterior domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. In this paper, we deal with asymptotic properties of strong solutions of the linear perturbed Stokes problem

$$\begin{aligned} \partial_t \boldsymbol{u} + \zeta\,\partial_1 \boldsymbol{u} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{U} & \\ = -\nabla p + \nu\Delta\boldsymbol{u} \quad & \text{in } \Omega\times(0,+\infty), \quad (1) \\ \operatorname{div}\boldsymbol{u} = 0 \quad & \text{in } \Omega\times(0,+\infty), \quad (2) \\ \boldsymbol{u} = \boldsymbol{0} \quad & \text{on } \partial\Omega\times(0,+\infty) \quad (3) \end{aligned}$$

where $\zeta\in\mathbb{R}$ and

$$\nabla\boldsymbol{U} \in L^3(\Omega)^9 \cap L^{3/2}(\Omega)^9, \tag{4}$$

$$\boldsymbol{U} \in L^s(\Omega)^3 \tag{5}$$

for some $s\in[3,+\infty]$.

We shall use the notation:

- $C^\infty_{0,\sigma}(\Omega)$ is the space of divergence–free vector functions $\boldsymbol{w}\in C^\infty_0(\Omega)^3$.
- $\boldsymbol{H}_0$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^2(\Omega)^3$. The scalar product in $\boldsymbol{H}_0$ is denoted by $(\,.\,,\,.\,)_0$ and the corresponding norm by $\|\,.\,\|_0$. $\boldsymbol{H}_0$ can be characterized as a space of functions from $L^2(\Omega)^3$ whose divergence equals zero in Ω in the sense of distributions and the normal component equals zero on $\partial\Omega$ in the sense of traces.
- Π_σ is the orthogonal projection of $L^2(\Omega)^3$ onto $\boldsymbol{H}_0$.
- $A\boldsymbol{u} = \nu\Pi_\sigma\Delta\boldsymbol{u}$ for $\boldsymbol{u}\in D(A)\equiv \boldsymbol{H}_0\cap W^{1,2}_0(\Omega)^3\cap W^{2,2}(\Omega)^3$.

Operator A is called the Stokes operator. It is a self–adjoint operator in $\boldsymbol{H}_0$ such that its spectrum $\sigma(A)$ coincides with the interval $(-\infty,0]$. Moreover, it is essentially dissipative. It means that $(A\boldsymbol{\phi},\boldsymbol{\phi})_0 \le 0$ for all $\boldsymbol{\phi}\in D(A)$ and $(A\boldsymbol{\phi},\boldsymbol{\phi})_0 = 0 \iff \boldsymbol{\phi}=\boldsymbol{0}$.

We further denote

$$\|\boldsymbol{u}\|_1 \;=\; \left\|(-A)^{1/2}\boldsymbol{u}\right\|_0^{1/2} \;=\; \left(\int_\Omega |\nabla\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}}.$$

- $\boldsymbol{H}_1$ is the completion of $C^\infty_\sigma(\Omega)$ in the norm $\|\,.\,\|_1$.
- If $\boldsymbol{u}\in\boldsymbol{H}_1$ then $\boldsymbol{U}\cdot\nabla\boldsymbol{u}$ and $\boldsymbol{u}\cdot\nabla\boldsymbol{U}$ belong to $L^2(\Omega)^3$. For $\boldsymbol{u}\in\boldsymbol{H}_1$, we put
$$B\boldsymbol{u} \;=\; -\Pi_\sigma(\boldsymbol{U}\cdot\nabla)\boldsymbol{u} - \Pi_\sigma(\boldsymbol{u}\cdot\nabla)\boldsymbol{U}.$$
- The subscript s denotes the symmetric part of an operator or a matrix. Thus, $(\nabla\boldsymbol{U})_s = \frac{1}{2}\,[\nabla\boldsymbol{U} + (\nabla\boldsymbol{U})^T]$ and
$$B_s\boldsymbol{u} \;=\; -\Pi_\sigma[\boldsymbol{u}\cdot(\nabla\boldsymbol{U})_s].$$
- The subscript a denotes the skew–symmetric (= anti–symmetric) part of an operator or a matrix. Thus, $(\nabla\boldsymbol{U})_a = \frac{1}{2}\,[\nabla\boldsymbol{U} - (\nabla\boldsymbol{U})^T]$ and
$$B_a\boldsymbol{u} \;=\; B_a^0\boldsymbol{u} + B_a^1\boldsymbol{u}$$
where
$$\begin{aligned} B_a^0\boldsymbol{u} &= -\zeta\,\Pi_\sigma\,\partial_1\boldsymbol{u}, \\ B_a^1\boldsymbol{u} &= -\Pi_\sigma(\boldsymbol{U}\cdot\nabla)\boldsymbol{u} - \Pi_\sigma[\boldsymbol{u}\cdot(\nabla\boldsymbol{U})_a]. \end{aligned}$$
- We put
$$L\boldsymbol{u} \;=\; A\boldsymbol{u} + B\boldsymbol{u} \;=\; A\boldsymbol{u} + B_s\boldsymbol{u} + B_a\boldsymbol{u}$$
for $\boldsymbol{u}\in D(L)\equiv D(A)$.

Proceedings of the 2006 IASME/WSEAS International Conference on Continuum Mechanics, Chalkida, Greece, May 11-13, 2006 (pp126-131)

- $\sigma(L)$ denotes the spectrum of L and $\rho(L)$ denotes the resolvent set of L.

Obviously, L can be considered to be a perturbed Stokes operator with the disturbance B. Asymptotic properties of operator L play a fundamental role in studies of asymptotic properties of solutions of the Navier–Stokes equation in a neighborhood of a steady solution $\boldsymbol{U}$.

Applying projection Π_σ onto equation (1), we can exclude the term ∇p. Consequently, the problem (1)–(3) can be written in the form of the operator equation

$$\dot{\boldsymbol{u}} = L\boldsymbol{u} \tag{6}$$

in space $\boldsymbol{H}_0$. (The dot denotes the derivative with respect to t.) Under solutions of equation (6) (and similar equations we shall deal with) on a time interval $[0,T)$ (where $0 < T \leq +\infty$), we understand functions $\boldsymbol{u}$ satisfying equation (6) a.e. in $(0,T)$ and such that

$$\boldsymbol{u} \in L^2(J;\, D(A)) \cap L^2(J;\, \boldsymbol{H}_0),$$
$$\dot{\boldsymbol{u}} \in L^2(J;\, \boldsymbol{H}_0)$$

for each bounded interval $J \subset [0,T)$. It can be proved by means of the theory of interpolation spaces that $\boldsymbol{u}$ can be redefined on a set of points from $[0,T)$ of measure zero so that it becomes a continuous mapping from $[0,T)$ to $\boldsymbol{H}_1$.

Stability of the zero solution of the equation of the type (6), possibly also with a nonlinear term, has already been studied for a long time. Let us e.g. mention the works by D. H. Sattinger [15], K. Masuda [9], H. Kielhöfer [5] and [6], P. Maremonti [8], G. P. Galdi and M. Padula [3], W. Borchers and T. Miyakawa [2], G. Mulone [10], S. Lombardo and G. Mulone [7], and J. Neustupa [12], [13]. Some of these works show that if function $\boldsymbol{U}$ is in some sense “sufficiently small” then operator L is negatively definite (possibly in a modified scalar product) and this property can be used to show that the zero solution of equation (6) is stable. Another condition which usually also implies stability of the zero solution of equations of the type (6) is the condition

(i) There exists $\delta > 0$ such that $Re\,\lambda < -\delta$ for all $\lambda \in \sigma(L)$.

In this paper, we do not use a requirement on smallness of $\boldsymbol{U}$ and we also cannot use condition (i) because our considered operator L has an essential spectrum which touches the imaginary axis, independently of the concrete form of function $\boldsymbol{U}$. (See K. I. Babenko [1].) In this paper, we formulate other sufficient conditions for stability (or partial stability) of the zero solution of (6) which are also mostly based on spectral properties of operator L and which replace the role of (i).

2 Auxiliary results

Using mainly the Hölder inequality and inequalities following from theorems on continuous imbedding, we can derive the inequalities

$$\|B_s\boldsymbol{\phi}\|_0^2 \;\leq\; c_1\,\|\boldsymbol{\phi}\|_1^2\,, \tag{7}$$
$$|(B_s\boldsymbol{\phi},\boldsymbol{\psi})_0| \;\leq\; c_2\,\|\boldsymbol{\phi}\|_1\,\|\boldsymbol{\psi}\|_1, \tag{8}$$
$$\|B_a\boldsymbol{\phi}\|_0^2 \;\leq\; \mu\,\|A\boldsymbol{\phi}\|_0^2 + c_3(\mu)\,\|\boldsymbol{\phi}\|_1^2 \tag{9}$$

with $\mu > 0$ arbitrarily small. Moreover, using integrability properties of function $\boldsymbol{U}$, we can show that operators B_s and B_a^1 are A–compact in $\boldsymbol{H}_0$.

Lemma 1 *Let $a \in \mathbb{R}$. Then the operator $A + aB_s$ is selfadjoint. The spectrum of $A + aB_s$ consists of the essential part which coincides with the interval $(-\infty, 0]$ and of at most a finite number of positive eigenvalues, each of whose has a finite algebraic multiplicity. If 0 is an eigenvalue of $A + aB_s$ then it also has a finite algebraic multiplicity.*

Proof. Operator A is selfadjoint. Operator B_s is symmetric. Estimate (7) implies that

$$\|aB_s\boldsymbol{\phi}\|_0^2 \;\leq\; a\,c_1\,(-A\boldsymbol{\phi},\boldsymbol{\phi})_0$$
$$\leq\; \mu\,\|A\boldsymbol{\phi}\|_0 + C(\mu)\,\|\boldsymbol{\phi}\|_0$$

where μ can be chosen arbitrarily small. Thus, the operator $A + aB_s$ is self–adjoint.

The essential spectrum is preserved under a relatively compact perturbation. Since aB_s is such a perturbation of A, the spectrum of $A + aB_s$ contains the essential part which coincides with $(-\infty, 0]$ and at most a countable number of positive eigenvalues which can possibly cluster only at point zero and each of them has a finite algebraic multiplicity. By deeper analysis of the space spanned by all associated eigenfunctions, using the fact that $\|\boldsymbol{\phi}\|_* = (aB_s\boldsymbol{\phi},\,\boldsymbol{\phi})_0^{1/2}$ is a norm in this space and showing that the unit sphere is compact, we prove that this space is finite–dimensional. □

Let $\kappa > 0$ be chosen. Let $E(\lambda)$ denote the resolution of identity corresponding to the selfadjoint operator $A + (1+\kappa)B_s$. Put

$$P' = \int_0^{+\infty} \mathrm{d}E(\lambda), \qquad P'' = I - P'$$

and $\boldsymbol{H}_0' = P'\boldsymbol{H}_0$, $\boldsymbol{H}_0'' = P''\boldsymbol{H}_0$. P', P'' are orthogonal projections in $\boldsymbol{H}_0$ and $\boldsymbol{H}_0'$, $\boldsymbol{H}_0''$ are closed orthogonal subspaces of $\boldsymbol{H}_0$ such that $\boldsymbol{H}_0 = \boldsymbol{H}_0' \oplus \boldsymbol{H}_0''$.

Projections P' and P'' commute with $A + (1+\kappa)B_s$ on $D(A+B_s+\kappa B_s) = D(A)$ and so $P'D(A) \subset$

Proceedings of the 2006 IASME/WSEAS International Conference on Continuum Mechanics, Chalkida, Greece, May 11-13, 2006 (pp126-131)

$D(A)$ and $P''D(A) \subset D(A)$. Moreover, due to the right continuity of the function $E(\lambda)\boldsymbol{\phi}$ (for each $\boldsymbol{\phi} \in \boldsymbol{H}_0$), if 0 is an eigenvalue of $A + (1+\kappa)B_s$ then all corresponding eigenfunctions belong to $\boldsymbol{H}_0''$.

The space $\boldsymbol{H}_0'$ is finite–dimensional due to Lemma 1. Using the expansion of $P'\boldsymbol{\psi}$ with respect to the orthonormal basis of $\boldsymbol{H}_0'$ formed by the eigenvectors of $A + (1+\kappa)B_s$ (for $\boldsymbol{\psi} \in \boldsymbol{H}_0 \cap \boldsymbol{H}_1$), we can show that

$$\|P'\boldsymbol{\psi}\|_0 \;\leq\; c_4\, \|\boldsymbol{\psi}\|_1\,. \tag{10}$$

If $\boldsymbol{\phi} \in \boldsymbol{H}_0'' \cap D(A)$ then $((A + B_s + \kappa B_s)\boldsymbol{\phi}, \boldsymbol{\phi})_0 \leq 0$ and consequently,

$$\begin{aligned} \Big((A + B_s)\boldsymbol{\phi},\, \boldsymbol{\phi}\Big)_0 \;&=\; \frac{\kappa}{1+\kappa}\, (A\boldsymbol{\phi},\, \boldsymbol{\phi})_0 \\ &\quad + \frac{1}{1+\kappa}\, \Big((A + B_s + \kappa B_s)\boldsymbol{\phi},\, \boldsymbol{\phi}\Big)_0 \\ &\leq\; \frac{\kappa}{1+\kappa}\, (A\boldsymbol{\phi},\, \boldsymbol{\phi})_0 \;=\; -c_5\, \|\boldsymbol{\phi}\|_1^2 \end{aligned} \tag{11}$$

where $c_5 = \kappa/(1+\kappa)$. We shall further use this estimate. It cannot be generally excluded that the estimate $((A + B_s)\boldsymbol{\phi}, \boldsymbol{\phi})_0 \leq -c_6\, \|\boldsymbol{\phi}\|_1^2$ holds, with some positive constant c_6, for all $\boldsymbol{\phi} \in \boldsymbol{H}_0'' \cap D(A)$, also in the case when $\kappa = 0$. In such a case, κ can also be chosen to be equal to zero. Otherwise we shall need κ to be positive.

3 Partial uniform boundedness of the semigroup e^{Lt}

The Stokes operator A generates an analytic semigroup in space $\boldsymbol{H}_0$. (See e.g. Y. Giga and H. Sohr [4].) Operator $\zeta\partial_1$ is an A–bounded operator in $\boldsymbol{H}_0$ with a relative A–bound arbitrarily small. (It follows from the estimate $\|\zeta\partial_1\boldsymbol{\phi}\|_0^2 \leq |\zeta|\, \|\boldsymbol{\phi}\|_1^2 = |\zeta|\, (-A\boldsymbol{\phi}, \boldsymbol{\phi})_0$ which holds for all $\boldsymbol{\phi} \in D(A)$.) Furthermore, estimates (7) and (10) imply that the operators B_s and B_a are A–bounded with an arbitrarily small relative A–bound, too. Thus, the operator $L = A + \zeta\partial_1 + B_s + B_a$ also generates an analytic semigroup in $\boldsymbol{H}_0$.

We shall further assume that P is a bounded operator in $\boldsymbol{H}_0$. We shall denote by $R_\lambda(L)$ the resolvent of L. For $\lambda \in \rho(L)$, $R_\lambda(L)$ is a bounded operator in $\boldsymbol{H}_0$ analytically depending on λ. We shall further use, as an important assumption, the condition

(ii) There exists a function $\varphi \in L^2(\mathbb{R}_+)$ such that

$$\|P\,\mathrm{e}^{Lt}\boldsymbol{\phi}\|_0 \;\leq\; \varphi(t)\, \|\boldsymbol{\phi}\|_0 \tag{12}$$

for each $\boldsymbol{\phi} \in \boldsymbol{H}_0'$.

($\mathbb{R}_+$ denotes the interval $(0, +\infty)$.) $\boldsymbol{H}_0'$ is a finite–dimensional subspace of $\boldsymbol{H}_0$. Thus, condition (ii) is the condition on the asymptotic decay of a finite number of functions. We discuss the question of validity of condition (ii) in Section 4.

Lemma 2 *Let $\tau \geq 0$ and $\tau < T \leq +\infty$. Function $\boldsymbol{u}$ is a solution of equation (6) on the time interval $[\tau, T)$ if and only if $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, where $\boldsymbol{v}$, $\boldsymbol{w}$ is a solution of the system*

$$\dot{\boldsymbol{v}} \;=\; A\boldsymbol{v} + (1+\kappa)B_s\boldsymbol{v} + P''(B_a - \kappa B_s)\boldsymbol{v} \tag{13}$$

$$\dot{\boldsymbol{w}} \;=\; (A + B_s + B_a)\boldsymbol{w} + P'(B_a - \kappa B_s)\boldsymbol{v} \tag{14}$$

on $[\tau, T)$, satisfying the initial conditions

$$\boldsymbol{v}(\tau) = P''\boldsymbol{u}(\tau), \qquad \boldsymbol{w}(\tau) = P'\boldsymbol{u}(\tau). \tag{15}$$

Proof. It is obvious that if $\boldsymbol{v}$, $\boldsymbol{w}$ solve (13)–(15) then $\boldsymbol{u}$ is a solution of (6). On the other hand, if $\boldsymbol{u}$ satisfies (6) then we can use properties of operators A, B_s and B_a and prove that the equation (13), together with the first initial condition in (15), has a unique solution $\boldsymbol{v}$. Putting $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ and subtraction equations (6) and (13), we can finally show that $\boldsymbol{w}$ is a solution of equation (14) which satisfies the second initial condition in (15). □

The next theorem presents the main result of this section.

Theorem 3 *Suppose that function $\boldsymbol{U}$ has the properties (4), (5). Suppose that operator L satisfies condition (ii). Then there exist positive constants c_7, c_8 such that if $\tau \geq 0$ then there exists a unique solution $\boldsymbol{v}$, $\boldsymbol{w}$ of the problem (13)–(15) on the time interval $[\tau, +\infty)$, which satisfies*

$$\begin{aligned} &\|\boldsymbol{v}(t)\|_0^2 + \|\boldsymbol{v}(t)\|_1^2 + \|P\boldsymbol{w}(t)\|_0^2 \\ &\quad + c_7 \int_\tau^t \Big(\|\boldsymbol{v}(s)\|_1^2 + \|A\boldsymbol{v}(s)\|_0^2\Big)\, \mathrm{d}s \\ &\quad \leq\; c_8 \Big(\|\boldsymbol{v}(\tau)\|_0^2 + \|\boldsymbol{v}(\tau)\|_1^2 + \|\boldsymbol{w}(\tau)\|_0^2\Big) \end{aligned} \tag{16}$$

(for all $t > \tau$) and

$$\lim_{t \to +\infty} \|\boldsymbol{v}(t)\|_1 \;=\; 0. \tag{17}$$

Proof. Equation (13) is the equation in space $\boldsymbol{H}_0''$. Multiplying it by $\boldsymbol{v}$ and using (8) and (11), we obtain:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\, \frac{1}{2}\, \|\boldsymbol{v}\|_0^2 \;&\leq\; -c_5\, \|\boldsymbol{v}\|_1^2 + \kappa(B_s\boldsymbol{v}, \boldsymbol{v})_0 \\ &\quad - \kappa(P''B_s\boldsymbol{v}, \boldsymbol{v})_0 + (P''B_a\boldsymbol{v}, \boldsymbol{v})_0 \\ &=\; -c_5\, \|\boldsymbol{v}\|_1^2. \end{aligned} \tag{18}$$

Proceedings of the 2006 IASME/WSEAS International Conference on Continuum Mechanics, Chalkida, Greece, May 11-13, 2006 (pp126-131)

Multiplying equation (12) by $(-A\boldsymbol{v})$ and using (7), (10) and (11), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\,\|\boldsymbol{v}\|_1^2 \;&=\; -\|A\boldsymbol{v}\|_0^2 + \kappa(P'B_s\boldsymbol{v}, -A\boldsymbol{v})_0 \\ &\quad + (P''B_a\boldsymbol{v}, -A\boldsymbol{v})_0 \;\leq\; -\|A\boldsymbol{v}\|_0^2 \\ &\quad + \kappa\,\|B_s\boldsymbol{v}\|_0\,\|A\boldsymbol{v}\|_0 + \|B_a\boldsymbol{v}\|_0\,\|A\boldsymbol{v}\|_0 \\ &\leq\; -\frac{3}{4}\,\|A\boldsymbol{v}\|_0^2 + 2\kappa^2\|B_s\boldsymbol{v}\|_0^2 + 2\,\|B_a\boldsymbol{v}\|_0^2 \\ &\leq\; -\frac{3}{4}\,\|A\boldsymbol{v}\|_0^2 + 2\kappa^2 c_1\,\|\boldsymbol{v}\|_1^2 + 2\mu_1\,\|A\boldsymbol{v}\|_0^2 \\ &\quad + 2c_4(\mu_1)\,\|\boldsymbol{v}\|_1^2.\end{aligned}$$

Choosing $\mu_1 = \frac{1}{8}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\,\|\boldsymbol{v}\|_1^2 \;&\leq\; -\frac{1}{2}\,\|A\boldsymbol{v}\|_0^2 \\ &\quad + \left[2\kappa^2 c_1 + 2\,c_4\left(\tfrac{1}{8}\right)\right]\;\|\boldsymbol{v}\|_1^2 \end{aligned} \tag{19}$$

Estimates (18) and (19) imply that if c_9 is chosen so that $c_9\left[2\kappa^2 c_1 + 2c_4\left(\frac{1}{8}\right)\right] = \frac{1}{2}\,c_5$ then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{v}\|_0^2 + c_9\,\|\boldsymbol{v}\|_1^2\right) \\ &\quad + \left(c_5\,\|\boldsymbol{v}\|_1^2 + c_9\,\|A\boldsymbol{v}\|_0^2\right) \;\leq\; 0.\end{aligned}$$

Integrating this inequality from τ to t, we obtain

$$\begin{aligned}&\|\boldsymbol{v}(t)\|_0^2 + c_9\,\|\boldsymbol{v}(t)\|_1^2 \\ &\quad + \int_\tau^t \left(c_5\,\|\boldsymbol{v}(s)\|_1^2 + c_9\,\|A\boldsymbol{v}(s)\|_0^2\right)\,\mathrm{d}s \\ &\leq\; \|\boldsymbol{v}(\tau)\|_0^2 + c_9\,\|\boldsymbol{v}(\tau)\|_1^2 \end{aligned} \tag{20}$$

Denote by $\boldsymbol{\phi}_1,\,\dots,\,\boldsymbol{\phi}_n$ is the orthonormal basis of the space $\boldsymbol{H}_0'$. We shall further use the estimates

$$\begin{aligned}\|P'B_a^1\boldsymbol{v}\|_0 \;&\leq\; \sum_{k=1}^n \left|(B_a^1\boldsymbol{v}, \boldsymbol{\phi}_k)_0\right| \\ &\leq\; \sum_{k=1}^n \left|\int_\Omega \Big[(\boldsymbol{U}\cdot\nabla)\,\boldsymbol{v}\cdot\boldsymbol{\phi}_k + \zeta\partial_1\boldsymbol{v}\cdot\boldsymbol{\phi}_k \right. \\ &\quad \left. + \boldsymbol{v}\cdot(\nabla\boldsymbol{U})_a\cdot\boldsymbol{\phi}_k\Big]\,\mathrm{d}\boldsymbol{x}\right| \\ &\leq\; \sum_{k=1}^n \|\boldsymbol{v}\|_1\,\|\boldsymbol{U}\|_{L^s}\|\boldsymbol{\phi}_k\|_{L^r} + \zeta\,n\,\|\boldsymbol{v}\|_1 \\ &\quad + \sum_{k=1}^n \|\boldsymbol{v}\|_{L^6}\,\|(\nabla\boldsymbol{U})_a\|_{L^3}\,\|\boldsymbol{\phi}_k\|_{L^6} \\ &\leq\; c_{10}\,\|\boldsymbol{v}\|_1 \end{aligned} \tag{21}$$

where $1/r + 1/s = \frac{1}{2}$ and consequently, $2 < r \leq 6$.

Solution $\boldsymbol{w}$ of equation (14) satisfies the integral equation

$$\begin{aligned}\boldsymbol{w}(t) \;&=\; \mathrm{e}^{L(t-\tau)}\boldsymbol{w}(\tau) \\ &\quad + \int_\tau^t \mathrm{e}^{L(t-s)}P'(B_a - \kappa B_s)\boldsymbol{v}(s)\,\mathrm{d}s.\end{aligned}$$

Using (7), (12) and (21), we obtain

$$\begin{aligned}\|P\boldsymbol{w}(t)\|_0 \;&\leq\; \|P\,\mathrm{e}^{L(t-\tau)}\boldsymbol{w}(\tau)\|_0 \\ &\quad + \int_\tau^t \left\|P\,\mathrm{e}^{L(t-s)}P'(B_a - \kappa B_s)\boldsymbol{v}(s)\right\|_0\,\mathrm{d}s \\ &\leq\; \varphi(t-\tau)\,\|\boldsymbol{w}(\tau)\|_0 \\ &\quad + \int_\tau^t \varphi(t-s)\,\left\|P'(B_a - \kappa B_s)\boldsymbol{v}(s)\right\|_0\,\mathrm{d}s \\ &\leq\; \varphi(t-\tau)\,\|\boldsymbol{w}(\tau)\|_0 \\ &\quad + c_{11}\int_\tau^t \varphi(t-s)\,\|\boldsymbol{v}(s)\|_1\,\mathrm{d}s.\end{aligned}$$

Hence

$$\begin{aligned}\|P\boldsymbol{w}(t)\|_0^2 \;&\leq\; 2\,\varphi(t-\tau)^2\,\|\boldsymbol{w}(\tau)\|_0^2 \\ &\quad + 2c_{12}\int_\tau^t \|\boldsymbol{v}(s)\|_1^2\,\mathrm{d}s.\end{aligned}$$

Estimating the integral by means of (20), we get

$$\begin{aligned}\|P\boldsymbol{w}(t)\|_0^2 \;&\leq\; 2\,\varphi(t-\tau)^2\,\|\boldsymbol{w}(\tau)\|_0^2 \\ &\quad + c_{13}\,\|\boldsymbol{v}(\tau)\|_0^2 + c_{14}\,\|\boldsymbol{v}(\tau)\|_1^2. \end{aligned} \tag{22}$$

Now we obtain the inequality (16) if we sum (20) and (22).

Estimate (19) shows that the derivative of $\|\boldsymbol{v}\|_1^2$ with respect to time is upper bounded. This information, together with the integrability of $\|\boldsymbol{v}\|_1^2$ on $(\tau, +\infty)$ (which follows from (16)), implies (17). □

The inequality (16) particularly says that

$$\begin{aligned}&\|\boldsymbol{v}(t)\|_0^2 + \|\boldsymbol{v}(t)\|_1^2 + \|P\boldsymbol{w}(t)\|_0^2 \\ &\quad \leq\; c_8\left(\|\boldsymbol{v}(\tau)\|_0^2 + \|\boldsymbol{v}(\tau)\|_1^2 + \|\boldsymbol{w}(\tau)\|_0^2\right) \end{aligned} \tag{23}$$

By analogy, using only (18) and not (19), we can obtain the estimate

$$\begin{aligned}&\|\boldsymbol{v}(t)\|_0^2 + \|P\boldsymbol{w}(t)\|_0^2 \\ &\quad \leq\; c_8\left(\|\boldsymbol{v}(\tau)\|_0^2 + \|\boldsymbol{w}(\tau)\|_0^2\right) \end{aligned} \tag{24}$$

It would imply the uniform boundedness of the semigroup e^{Lt} in the norm $\|\,.\,\|_0$ if we had $\|\boldsymbol{w}(t)\|_0^2$ and not only $\|P\boldsymbol{w}(t)\|_0^2$ on the left hand side. However, since the left hand side represents only a part of the total norm $\|\boldsymbol{u}\|_0^2$ (where $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$), we call the type of boundedness we have proved the *partial uniform boundedness*.

On the other hand, the freedom in the choice of the bounded operator P offers interesting opportunities. One of them is a projection onto a finite dimensional subspace of $\boldsymbol{H}_0$; Theorem 3 in this case provides the information on asymptotic behavior of solution $\boldsymbol{u}$ of the equation (6), projected into the chosen finite–dimensional subspace. Nevertheless, let us further explain in greater detail another possibility.

Suppose that R is an arbitrarily large positive real number. We shall denote $\Omega_R = \Omega \cap B_R(\mathbf{0})$. Let η be an infinitely differentiable cut–off function defined in Ω such that

$$\begin{array}{ll} \eta \,=\, 1 & \text{on } \Omega_{R+1/4}, \\ 0 \,\leq\, \eta \,\leq\, 1 & \text{on } \Omega_{R+3/4} - \Omega_{R+1/4}, \\ \eta \,=\, 0 & \text{on } \Omega - \Omega_{R+3/4}. \end{array}$$

We denote by $\mathcal{R}$ a linear operator which assigns to a function $g \in L^2(\Omega)$ a function $\mathcal{R}g \in \widehat{W}_0^{1,2}(\Omega)^3$ such that $\mathrm{div}\,(\mathcal{R}g) = g$ a.e. in Ω. ($\widehat{W}_0^{1,2}(\Omega)^3$ is the completion of $C_0^\infty(\Omega)^3$ in the norm identical with $\|\,.\,\|_1$.) We put

$$P\boldsymbol{u} \,=\, \eta\boldsymbol{u} - \mathcal{R}(\nabla\eta\cdot\boldsymbol{u}) \tag{25}$$

for $\boldsymbol{u} \in \boldsymbol{H}_0$. It is well known that $\mathcal{R}(\nabla\eta\cdot\boldsymbol{u})$ can be constructed so that its support is a subset of $\Omega_{R+1} - \Omega_R$. Then we have

$$\begin{aligned} \|P\boldsymbol{u}\|_0^2 \,&=\, \|\eta\boldsymbol{u} - \mathcal{R}(\nabla\eta\cdot\boldsymbol{u})\|_0^2 \\ &\leq\, 2\int_\Omega \Big(|\eta\boldsymbol{u}|^2 + |\mathcal{R}(\nabla\eta\cdot\boldsymbol{u})|^2\Big)\,\mathrm{d}\boldsymbol{x} \\ &\leq\, 2\,\|\boldsymbol{u}\|_0^2 + C\int_\Omega |\nabla\mathcal{R}(\nabla\eta\cdot\boldsymbol{u})|^2\,\mathrm{d}\boldsymbol{x} \\ &\leq\, 2\,\|\boldsymbol{u}\|_0^2 + C\int_\Omega |\nabla\eta\cdot\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} \\ &\leq\, c_{15}\,\|\boldsymbol{u}\|_0^2\,, \end{aligned} \tag{26}$$

(C is a generic constant.) These estimates show that P is a bounded linear operator in $\boldsymbol{H}_0$.

Operator $\mathcal{R}$ is not given uniquely. One of possibilities which satisfies the requirement $\mathrm{supp}\,\mathcal{R}(\nabla\eta\cdot\boldsymbol{u}) \subset (\Omega_{R+1} - \Omega_R)$ is

$$\mathcal{R}(\nabla\eta\cdot\boldsymbol{u}) \,=\, \nabla\eta\times\boldsymbol{\psi} \tag{27}$$

where $\boldsymbol{\psi}$ is a vector potential of $\boldsymbol{u}$, i.e. a function satisfying $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}$ in Ω. Then we have

$$\begin{aligned} P\boldsymbol{u} \,&=\, \eta\boldsymbol{u} - \nabla\eta\times\boldsymbol{\psi} \,=\, \eta\,\mathbf{curl}\,\boldsymbol{\psi} + \boldsymbol{\psi}\times\nabla\eta \\ &=\, \mathbf{curl}\,(\eta\boldsymbol{\psi}). \end{aligned}$$

Thus, if operator P is defined by (25), the inequality (23) provides the boundedness (in the L^2–norm) of functions $\boldsymbol{v}$, $\boldsymbol{w}$ (and consequently also of solution $\boldsymbol{u}$ of the equation (6)) in Ω_R.

4 More on the condition (ii)

In this section, we at first show that condition (ii) follows from another condition

(iii) For each $\boldsymbol{\phi} \in \boldsymbol{H}_0'$, $PR_\lambda(L)\boldsymbol{\phi}$ is a bounded $\boldsymbol{H}_0$–valued function of λ in the right half–plane $\mathbb{C}_+ \equiv \{\lambda\in\mathbb{C};\ Re\,\lambda > 0\}$.

$R_\lambda(L)$ denotes the resolvent operator of L. For $\lambda \in \rho(L)$, $R_\lambda(L)$ is a bounded operator in $\boldsymbol{H}_0$, analytically depending on λ. The spectrum of L has the essential part

$$\begin{aligned} \sigma_{ess}(L) \,&=\, \{\alpha + \mathrm{i}\,\beta\in\mathbb{C};\ -\infty < \beta < +\infty, \\ &\qquad \alpha \leq -\beta^2/\zeta^2\} \end{aligned}$$

(see [1]) and it can also contain at most a countable number of eigenvalues outside $\sigma_{ess}(L)$. Each of them has a finite algebraic multiplicity and they can possibly cluster only at the boundary of $\sigma_{ess}(L)$. Suppose that $\boldsymbol{\phi} \in \boldsymbol{H}_0'$. Then the function $PR_\lambda(L)\boldsymbol{\phi}$ is a bounded holomorphic $\boldsymbol{H}_0$–valued function in $\mathbb{C}_+ - M$ where M is an isolated set in C_+. Each point of M thus represents a removable singularity of $PR_\lambda(L)\boldsymbol{\phi}$ and if we extend the definition of $PR_\lambda(L)\boldsymbol{\phi}$ continuously to M then $PR_\lambda(L)\boldsymbol{\phi}$ becomes a bounded holomorphic function in the whole half–plane $\mathbb{C}_+$.

Let us choose $\lambda_1 \in \mathbb{C}$ such that $Re\,\lambda_1 > 0$. Put $\boldsymbol{\psi} = (L - \lambda_1 I)\boldsymbol{\phi}$ for $\boldsymbol{\phi} \in \boldsymbol{H}_0'$. Then

$$\begin{aligned} PR_\lambda(L)\boldsymbol{\psi} \,&=\, P\,(L-\lambda I)^{-1}\,(L-\lambda_1 I)\boldsymbol{\phi} \\ &=\, P\boldsymbol{\phi} + (\lambda-\lambda_1)\,PR_\lambda(L)\boldsymbol{\phi}. \end{aligned}$$

These identities show that $PR_\lambda(L)\boldsymbol{\psi}$ is bounded and holomorphic function of variable λ in the right half–plane. Since the space $\boldsymbol{H}_0$ is of the Fourier type 2, Theorem 4.3.2, p. 123, in [11] now implies that $P\mathrm{e}^{Lt}(L-\lambda_1)^{-1}\boldsymbol{\psi} \equiv P\mathrm{e}^{Lt}\boldsymbol{\phi}$ belongs to $L^2(\mathbb{R}_+;\ \boldsymbol{H}_0)$. This proves the theorem:

Theorem 4 *Condition (iii) implies condition (ii) and consequently, it also implies the partial uniform boundedness of the semigroup* e^{Lt} *in the sense of inequality (24).*

The question of validity of condition (iii) in the case of concrete types of bounded operator P, discussed at the end of Section 3, is further studied in the paper [14].

5 Conclusion

Obviously, if operator L is negatively–definite then the semigroup e^{Lt} is uniformly bounded. However, we have already explained that we do not assume that operator L is negatively–definite in this paper and on the other hand, we focus on the case when the numerical range of L has a non–empty intersection with $\mathbb{R}_+$. Then, clearly, the zero solution of the equation

$$\dot{\boldsymbol{u}} = A\boldsymbol{u} + B_s\boldsymbol{u} \tag{28}$$

is unstable and the semigroup $\mathrm{e}^{(A+B_s)t}$ is unbounded. If, in spite of this, the semigroup $\mathrm{e}^{Lt} \equiv \mathrm{e}^{(A+B_s+B_a)t}$ is uniformly bounded or at least partially uniformly bounded, it is only due to the stabilizing influence of the skew–symmetric operator B_a in equation (6). Thus, condition (iii) express this stabilizing influence and it has the same effect as the often used assumption (i) (See Section 1.) In our situation, i.e. in the case when L has the essential spectrum which touches the imaginary axis, the condition (i) is useless.

The main contribution of this paper is that it provides a sufficient condition for the partial uniform boundedness of the semigroup e^{Lt} formulated by means of a requirement on boundedness of the operator $PR_\lambda(L)$ (applied to a finite number of functions which form the basis of space $\boldsymbol{H}_0$) in the right complex half–plane.

The question of boundedness or more generally, the asymptotic behavior as $t \to +\infty$, of the semigroup e^{Lt} naturally arises if we study a linearized stability of a steady flow of a viscous incompressible flow in a domain in $\mathbb{R}^3$ which is an exterior of a compact body. In such a case, we are usually interested in behavior of disturbances of the basic flow in just a neighborhood of the body and it is not so important how the disturbances are damped or amplified “far” from the body. The freedom in the choice of a bounded operator P offers the opportunity to decide where and how we measure the size of the disturbances and on the other hand, it also opens a wider field to a possible validity of condition (iii) than the trivial case $P = I$.

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