THEORETICAL MODEL OF THE DYNAMIC INTERACTION BETWEEN WAGON TRAIN AND CONTINUOUS RAIL

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Abstract: The paper reveals the vibrations of a wagon with two bogies having elastic links and dampers between the wagon box and the bogie and also between the wheels and the bogie. The entire system has two vertical planes of symmetry therefore the models has 15 d.o.f. The railway is considered elastic solid in both planes horizontal and vertical that induces forced vibrations of the wagon. The vibrations are studied using the Language’s equations and the Hamilton’s principle. By this way it was found for the first time a mathematical approach for the wagon box forced vibrations in connection with the non-linear vibrations of the rail. In the mean time the authors found, by using the Hamilton’s principle and the Dirac function, two partial derivatives equations which describe the forced non-linear vibrations at the rail in two planes namely: vertical and horizontal.

Key words: wagon vibrations, continuous rail

1 Introduction

With the continuous increase at the speed for the railway transport it is necessary to increase the comfort and the safy of the railway traffic. Because of these demands it was developed the research in the area of forced vibrations of the wagon train. These vibrations are due to the geometrical imperfections of the rails and produce, in the case of constant speed of the wagon, the following vibrating movements: vertical, rolling, pitching and gyration.

Also the rail is a deformable solid in both vertical and horizontal plane, having a distributed elasticity and damping in these planes [5]. In the mean time it was taking into account, for the vibrations of the rail, the effects of rotary inertia and shear stress.

The wagon is with one suspended stage, its box being elastic mounted into the bogies. Also the suspension and the anti-gyrations elastic elements have viscous dampers. The paper reveals the forced damped vibrations of the system wagon-rail in interaction.

2 Basic Model and Hypothesis

The wagon model (see figure 1) is composed of two rigid bogies 1 and 2, having the mass \( m_1=m_2 \), the wagon box 3 of mass \( m_3 \) and the wheels “left”, “right” identical 4,5,6 and 7.

The bogies and the box at the wagon have two planes of symmetrie and the rails are simetric in geometrical rotations, being deformable in both horizontal and vertical planes. The rails have distributed damping \( c_y \) and \( c_z \) in the vertical and horizontal planes, the model of the rails being illustrated in figure 2. The model consists into clamped beams at the ends. The links between the bogies and the wagon box, together with the links between the wheels axes and the bogies are identical in vertical and horizontal planes, their mechanical characteristics being:
- stiffness: \( k, k' \) for the system bogies-wagon and \( k, k' \) for the system wheel axes-bogie;
- viscous damping coefficients: \( c, c' \) for the system bogies-wagon and \( c, c' \) for the system wheel axes-bogie;
- mass moments of inertia: \( J_1=J_2 \) (axes \( c_1, c_2 \)), \( J'_1=J'_2 \) (axes \( c_{10}, c_{20} \)), \( J'_3=J'_4 \) (axes \( c_{30}, c_{31} \)) for the bogies; \( J_3 \) (axe \( c_{32} \)), \( J'_3 \) (axe \( c_{33} \)) for the wagon box, where \( c_1, c_2, c_3 \) and \( c_1' \) are the mass centers.

The model has 15 degree of freedom (d.o.f.) as it follows:
- the coordinates \( y_1, y_2, y_3 \) are the vertical displacements of the mass centers;
- \( \varphi, \varphi_2, \varphi_3 \) are the rotation due to the pitching movement;
- \( \theta_1, \theta_2, \theta_3 \) are the rotation due to the rolling movement;
- \( z_1, z_2, z_3 \) are the coordinates of the mass center for the gyration movement;
- \( \psi_1, \psi_2, \psi_3 \) are the rotation due to the gyration movement;

The rail has: the mass moments of inertia \( J_{zo}, J'_{zo}, J_{z1} \) is the geometric moment of inertia; \( E \) is the Young’s modulus and the distance between two clamped ends is \( L_0=19.2 \text{m} \).

The adapted model is a mechanical system with holonomic and scleronomic links, on which is acting elastic forces, viscous damping forces and perturbing forces generated by the imperfections and the deformations of the rail. It is considered that all the wheels are in permanent contact with the rails only inside the length \( L_0 \).

3 The Equations of Vibrating Movement of The Wagon

Using the Lagrange’s equations, the compact matrix equation for the small vibrations is

\[
[M]\{\ddot{q}\} + [D]\{\dot{q}\} + [K]\{q\} = \{F\},
\]

by denoting:
- the vector of generalized coordinates
  \[
  \{q\}^T = \{y_1, y_2, y_3, z_1, z_2, z_3, \theta_1, \theta_2, \theta_3, \psi_1, \psi_2, \psi_3, \varphi_1, \varphi_2, \varphi_3\}
  \] (2)
- the inertia matrix
  \[
  [M] = \begin{bmatrix}
  [m] & [o] & [o] & [o] \\
  [o] & [m] & [o] & [o] \\
  \end{bmatrix}
  \]
  (3)

where

\[
[m] = \begin{bmatrix}
  m_1 & 0 & 0 \\
  0 & m_1 & 0 \\
  0 & 0 & m_3
  \end{bmatrix}, \quad [J] = \begin{bmatrix}
  J_1 & 0 & 0 \\
  0 & J_1 & 0 \\
  0 & 0 & J_3
  \end{bmatrix}
\]

and similar \([J'], [J''']\),
- the damping matrix
Proceedings of the 2006 IASME/WSEAS International Conference on Continuum Mechanics, Chalkida, Greece, May 11-13, 2006 (pp6-10)

\[
[M] = \begin{bmatrix}
D_y & [o] & [o] & [o] & D_{yp} \\
[o] & D_z & [o] & D_{zp} & [o] \\
D_{yp} & [o] & D_o & [o] & [o] \\
\end{bmatrix}
\] \quad (5)

where

\[
D_y = \begin{bmatrix}
4c_j + 2c & 0 & -2c \\
0 & 4c_j + 2c & -2c \\
-2c & -2c & 4c \\
\end{bmatrix}
\]

\[
D_z = \begin{bmatrix}
4c_j' + 2c' & 0 & -2c \\
0 & 4c_j' + 2c' & -2c' \\
-2c' & -2c' & 4c' \\
\end{bmatrix}
\]

\[
D_o = \begin{bmatrix}
c_i I_i' & 0 & 0 \\
0 & c_i I_i' & 0 \\
0 & 0 & c_i I_i' \\
\end{bmatrix}
\]

\[
D_{yp} = \begin{bmatrix}
0 & 0 & -c'I \\
0 & 0 & -c'I \\
0 & 0 & 0 \\
\end{bmatrix}
\]

- the stiffness matrix

\[
[K_y] = \begin{bmatrix}
K_y & [o] & [o] & [o] & K_{yp} \\
[o] & K_z & [o] & K_{zp} & [o] \\
K_{yp} & [o] & K_o & [o] & [o] \\
\end{bmatrix}
\]

\[
[K] = \begin{bmatrix}
K_y & [o] & [o] & [o] & K_{yp} \\
[o] & K_z & [o] & K_{zp} & [o] \\
K_{yp} & [o] & K_o & [o] & [o] \\
\end{bmatrix}
\] \quad (7)

where \( [K_y], [K_z], [K_o], [K_{yp}], [K_{zp}], [K_{yp}] \) are similar with \( [D_y], [D_z], [D_o], [K_{yp}] \).

\[
[D_y], [D_{yp}], [D_{zp}] \quad \text{if is replace } c \text{ by } k, c' \text{ by } k', c_i \text{ by } k_i \text{ and } c_i' \text{ by } k_i';
\]

- the column matrix at the perturbating forces \( \{F\} \) is

\[
\{F\} = \left\{ \sum_{i=4}^{7} \left[ k_i \left( y^L_{si} + y^R_{si} \right) + c_i \left( y^L_{si} + y^R_{si} \right) \right] \right\}^T
\]

\[
\{F_y\} = \left\{ \sum_{i=4}^{7} \left[ k_i \left( y^L_{si} + y^R_{si} \right) + c_i \left( y^L_{si} + y^R_{si} \right) \right] \right\}^T
\]

\[
\{F_z\} = \left\{ \sum_{i=6}^{7} \left[ 2k_i'(z_{si}) + 2c_i'(z_{si}) \right] \right\}^T
\]

\[
\{F_o\} = \left\{ \sum_{i=4}^{7} \left[ k_i I_3 \left( y^R_{si} - y^L_{si} \right) / 2 + c_i I_3 \left( y^R_{si} - y^L_{si} \right) / 2 \right] \right\}^T
\]

\[
\{F_{\psi}\} = \left\{ \sum_{i=4}^{7} \left[ I_i k_i' \left( -1 \right) z_{si} + I_i c_i' \left( -1 \right) \dot{z}_{si} \right] \right\}^T
\]

\[
\{F_o\} = \left\{ \sum_{i=4}^{7} \left[ I_i k_i' \left( -1 \right) \left( y^L_{si} + y^R_{si} \right) + I_i c_i' \left( -1 \right) \left( y^L_{si} + y^R_{si} \right) + 2 \right] \right\}^T
\]

In the relations \( (9) \div (13) \) \( y^L_{si} \), with \( i = 4, 7 \), are the deformations of the rails “Left” and “Right”, in the vertical plane, for the contact points with the wheels. Also, \( z_{si} \) have the same
meaning in the horizontal vibrations of the wagon. The \( y_{si}^{LR} \) and \( z_{si} \) deformations are functions of \( x \) and \( t \) and represent the excitations of the wagon which has a constant speed. To solve the system (1) it is necessary to compute the functions \( y_{si}^{LR}(x,t) \) and \( z_{si}(x,t) \). These functions represent the static and dynamic deformations of the rail in both vertical and horizontal planes.

4 The Forced Vibrations Equations of The Rails

To study the forced vibrations of the rails it was considered that the wagon, throughout a rigid contact, is acting into the rails with dynamic and static forces. These forces represent a system \( \{F_{y_i}^{LR}\}_{i=1}^{7} \), \( \{F_{z_i}^L\}_{i=1}^{7} \) which correspond to the contact with the wheels 4,5,6,7 “Left” and “Right” (see fig.2). The rail is considered to be a beam, containing the axe of the beam and each wheel equally supports the mass force of the wagon. Also, from the Hamilton’s principle, the boundary conditions are

\[
y^{LR}_{y_i}\mid_{x=0} = 0; \quad \frac{\partial y^{LR}_{z_i}}{\partial x}\mid_{x=0} = 0
\]

(15)

In the same manner it was derived the partial derivatives equations of the forced vibrations of the rail into the horizontal plane, obtaining:

\[
EI_{x} \frac{\partial^{4} y_{x}^{LR}}{\partial x^{4}} + \\
+ \left[ m + (J_{y} k_{y} / K) \delta \sum_{i=1}^{4} \delta(x-x_{i}) \right] \frac{\partial^{2} y_{x}^{LR}}{\partial t^{2}} - \\
- (EI_{x} / K) \left[ c_{y} + c_{i} \delta \sum_{i=1}^{4} \delta(x-x_{i}) \right] \frac{\partial^{3} y_{x}^{LR}}{\partial t^{3}} - \\
- J_{y}^{'} + (mEI_{x}) / K \frac{\partial^{4} y_{x}^{LR}}{\partial t^{4}} + \\
+ (mJ_{y}^{'} / K) \frac{\partial^{4} y_{x}^{LR}}{\partial t^{4}} + \\
+ J_{y}^{'} / K \left[ c_{y} + c_{i} \delta \sum_{i=1}^{4} \delta(x-x_{i}) \right] \frac{\partial^{3} y_{x}^{LR}}{\partial t^{3}} + \\
+ \left[ c_{y} + c_{i} \delta \sum_{i=1}^{4} \delta(x-x_{i}) \right] \frac{\partial y_{x}^{LR}}{\partial t} - \\
- k_{y} (EI_{x} / K) \frac{\partial^{2} y_{x}^{LR}}{\partial x^{2}} \sum_{i=1}^{4} \delta(x-x_{i}) + \\
+ k_{y} \frac{\partial^{2} y_{x}^{LR}}{\partial x^{2}} \sum_{i=1}^{4} \delta(x-x_{i}) = 0
\]

(16)

where \( J_{y} = J_{yo} / L_{o} \), \( \delta \) is the Dirac’s function, \( k = (1.4 \pm 2.8) \cdot A \cdot G \) [6], \( A \) is the cross-section area of the beam, \( G \) is the transversal modulus and \( G_{i} \) is the entire mass force of the wagon. The boundary conditions are

\[
z_{x}\mid_{x=0} = 0; \quad \frac{\partial z_{x}}{\partial x}\mid_{x=0} = 0
\]

(17)

In equations (14) and (16) \( x_{i} \) represent the coordinates of the contact points between the wheels and the rail, their expressions being:

\[
x_{1} = v t; \quad x_{2} = x_{1} - I_{1}; \quad x_{3} = x_{2} - I_{2}; \quad x_{4} = x_{3} - I_{3}
\]

(18)

The key of solving the system (1) is therefore the computation of the solutions of equations (14) and (16).

5 Conclusions

By taking into account that the rails are deformable bodies represents the main contribution of the paper. The equations (14) and (16) enable us
the computation of the deformations of the rail which represent the excitations for the wagon.

Another contribution of the paper is that vibrations of the wagon are considered in a complex form of a mechanical system having 15 d.o.f. By this way it is possible to check in the early stage of design the comfort and the safety conditions for the railway traffic.

The system (1) could be solved by numerical methods such as Runge-Kutta’s method, while the equations (14) and (16) could be solved by a mixed Kautorovich’s method coupled with a perturbation method.

Therefore, the paper reveals a complex and useful algorithm for the CAD stages.

References