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Stabilization properties for a spherical model of gaseous star

BERNARD DUCOMET
CEA/DAM/DIFF/DPTA
BP12
F-91680 Bruyères le Châtel
FRANCE

ALEXANDER ZLOTNIK
Moscow Power Engineering Institute
Department of Mathematical Modelling
Krasnokazarmennaja 14, 111250 Moscow
RUSSIA

Abstract: We consider viscous compressible flows with spherical symmetry under the action of gravitation and a prescribed outer pressure, outside a rigid core, in order to analyze the stability properties of simple models of gaseous stars. For a general (possibly non-monotone) state function $p = p(\rho)$, we present global-in-time bounds for solutions with arbitrarily large data. In the case of a non-decreasing p, the ω-limit set for the density ρ may be identified. In the more standard case of a strictly increasing p, uniqueness and static stability of the stationary solutions may be investigated together with stabilization rate estimates toward the statically stable solutions.

Key–Words: Navier-Stokes, Gravitation, Free boundary, Stability, Stabilization rate.

1 Introduction

Astrophysics tells us [1] [2] that the existence on large time scales of classical gaseous stars is due to a subtle equilibrium between gas properties (gaseous pressure, radiation, nuclear energy production) which lead to an expansion of the gas in the interstellar medium, and selfgravitation which produces a contraction of the system (the so called gravitational collapse).

In some physical situations it makes sense to study the purely mechanical competition between the gaseous pressure and the gravitation, discarding the exact roles of radiation and thermonuclear processes. This is the case in the standard Eddington's model [1]), in which a constant ratio β between the gaseous part p_g and the radiative part p_{rad} of the pressure is assumed at any point in the star. Observations show [2] that this assumption is valid for stars of mass $m \sim 10\odot$ (ten solar masses), with $\beta \sim 0.8$.

Moreover it is also important to consider monotone (polytropic case [1]) as well as non-monotone equations of state (neutron star case [10]).

2 The spherically symmetric free-boundary problem

Let us consider the following compressible Navier-Stokes system

$$\eta_t = (r^2 v)_x, \ \ \eta = 1/\rho,$$

$$v_t = r^2 \left[\mu(\rho)\rho(r^2 v)_x - p(\rho)\right]_x + f[r],$$

$$r_t = v,$$

in the domain $Q := J \times \mathbf{R}^+$ with $J := (0, M)$, where the unknown density ρ, the velocity v and the radius r depend on the lagrangian mass coordinates (x, t).

We supplement the system with boundary and initial conditions

$$v|_{x=0} = 0,$$

$$\left[\mu(\rho)\rho(r^2 v)_x - 4\mu_1(\rho)\frac{v}{r} - p(\rho)\right]\Big|_{x=M} = -p_\Gamma(t),$$

for $t > 0$, and

$$(\rho, v, r)|_{t=0} = \left(\rho^0, v^0, r^0\right)(x),$$

for $x \in J$, where

$$\frac{1}{3}(r^0)^3 = V_0 + \int_0^x \frac{d\xi}{\rho^0(\xi)}, \quad V_0 = \frac{1}{3}r_0^3, \text{ and } r_0 > 0.$$

In this model, the fluid is supposed to be viscous. In fact our assumptions concerning them will be of mathematical nature (see below) as viscosity coefficients in astrophysical fluids are not precisely identified. All that physics can tell us is they are very small [3].

Moreover the motion takes place in a domain Ω surrounding a hard core (supposed to be inert) with radius r_0, and the external surface of Ω is a free boundary. The existence of a hard core is required as, from a physical point of view, matter is very dense in the central region of the star, and quantum and relativistic effects are requested if one desires to obtain a more complete information on the dynamics. Our aim is more modest and we replace this ill-known region by a hard core with Dirichlet condition for the velocity.

The free-boundary condition is more involved to be justified. In fact the star is evolving in the vacuum or at least in a very rarefied medium. As contact of a gas with vacuum is a complex problem from both modelization and mathematical point of view, we model the underdense matter at the boundary by an external time-dependent, but fixed pressure, playing the role of a stellar atmosphere.

So we postulate an external pressure p_Γ of the form

$$p_\Gamma(t) = p_{\Gamma,S} + \Delta p_\Gamma(t),$$

where $p_{\Gamma,S} > 0$ and $\Delta p_\Gamma(t)$ is a perturbation.

The (gravitational) mass force f has the form

$$f(r,x,t) = f_S(r,x) + \Delta f(r,x,t),$$

where $f_S(r,x) = -G\frac{M_0+j_0x}{r^2}$ with $G>0$, $M_0 \geq 0$, $j_0 = 0$ or 1, but $f_S \not\equiv 0$, and $\Delta f(r,x,t)$ is a perturbation. The case $j_0 = 1$ corresponds to a selfgravitating fluid, the simpler case $j_0 = 0$ supposes that selfgravitation is neglected and only the newtonian attraction by an effective central mass M_0 (the rigid core) is taken into account [4].

Finally, we use the notation $f[r](x,t) := f(r(x,t),x,t)$.

Some related mathematical results on close problems may be found in [5]-[9].

3 Global bounds and stabilization

We consider a general state function p continuous on $\overline{\mathbf{R}}^+$, satisfying only $p(0) = 0$, $\lim_{s\to\infty} p(s) = +\infty$, and $p' \in L^\infty_{loc}(\mathbf{R}^+)$.

The viscosity coefficients μ and μ_1, supposed to be continuous on $\mathbf{R}^+$, are such that $\mu', \mu_1' \in L^\infty_{loc}(\mathbf{R}^+)$ and satisfy

$$0 < \underline{\mu} \leq \mu(s),\ -\underline{\mu}_1 \leq \mu_1(s) \leq \overline{\mu}_1 < \frac{4}{3}\underline{\mu},$$

for $s > 0$.

We also suppose that $\Delta f(r,x,t)$ is measurable on $(r_0,\infty)\times Q$, continuous with respect to $(r,x) \in [r_0,\infty)\times\overline{J}$ for almost all $t \geq 0$ and that

$$|\Delta f(r,x,t)| \leq \overline{f}_1(t) + \overline{f}_2(t),$$

with $\overline{f}_1 \geq 0$ and $\overline{f}_2 \geq 0$.

Throughout this paper we also suppose that

$$\|\overline{f}_1\|_{L^1(\mathbf{R}^+)} + \|\overline{f}_2\|_{L^2(\mathbf{R}^+)} + \|\Delta p_\Gamma\|_{L^2(\mathbf{R}^+)} \leq N,$$

for a given parameter $N > 1$.

We study strong solutions for the above problem, satisfying

$$\rho \in C(\overline{Q}_T),\ \rho_x,\ \rho_t \in C([0,T]; L^2(J)),$$

$$\min_{\overline{Q}_T} \rho > 0,\ v \in H^1(Q_T),$$

and

$$v_{xx} \in L^2(Q_T),$$

for any $T > 0$, where $Q_T := J \times (0,T)$.

They do exist under the necessary conditions:

$$\rho^0, v^0 \in H^1(J),\ \min_{\overline{J}} \rho^0 > 0, \text{ and } v^0(0) = 0$$

together with $\overline{f}_1 \in L^2(0,T)$, $p_\Gamma' \in L^1(0,T)$, for any $T > 0$.

Let us introduce the primitive functions

$$P_0(s) := \int_1^s \frac{p(\zeta)}{\zeta^2}\, d\zeta,$$

and

$$F(r,x) := -G\left(\frac{1}{r_0} - \frac{1}{r}\right)(M_0 + j_0x),$$

and recall the energy conservation law

$$\frac{d}{dt}\left(\mathcal{E} + \mathcal{F}[\rho]\right) + \int_J \mu(\rho)\rho(r^2v)_x^2 dx - 4R\mu_1(\rho_M)v_M^2$$

$$= \int_J \Delta f[r]v\, dx - \Delta p_\Gamma R^2 v_M,$$

where

$$\mathcal{E} := \frac{1}{2}\int_J v^2 dx$$

is the kinetic energy, and

$$\mathcal{F}[\rho] := \int_J \left(P_0(\rho) + p_{\Gamma,S}\rho^{-1} - F[r]\right)\, dx$$

is the potential energy.

The notations $F[r](x,t) = F(r(x,t),x)$ and $\Psi_M := \Psi|_{x=M}$ are used. Accordingly, $R = r_M$ is the radius of the free boundary.

Let $K = K(N)$, $K_i = K_i(N)$, $i = 0,1,...,$ be positive non-decreasing functions of N which may possibly depend on p, μ, μ_1, G, M_0, M etc.

We first state uniform bounds for the solutions.

Theorem 1 *1. Under the following condition on the data*

$$\left\|v^0\right\|_{L^2(J)} + \left\|P_0(\rho^0)\right\|_{L^1(J)} \leq N, \tag{1}$$

the uniform energy bound holds

$$\sup_{t\geq 0}\left(\mathcal{E}(t) + \|P_0(\rho)(\cdot,t)\|_{L^1(J)} + V_M(t)\right)$$

$$+ \left\|\sqrt{\mu(\rho)\rho}(r^2v)_x\right\|_{L^2(Q)} \leq K, \tag{2}$$

where

$$V(t) := \frac{1}{3}r^3 = V_0 + I\eta,$$

with $(I\phi)(x) := \int_0^x \phi(\xi)\, d\xi$.

2. Under the conditions (1) and $\rho^0 \le N$, *the uniform upper bound*

$$\sup_{\overline{Q}} \rho \le K,$$

holds. Moreover the energies stabilize

$$\mathcal{E}(t) \to 0 \;\text{ and }\; \mathcal{F}[\rho](t) \to \mathcal{F}^{(S)} \;\text{ as } t \to \infty. \qquad (3)$$

3. Under the conditions (1) and $N^{-1} \le \rho^0$, *one has the uniform lower bound*

$$K^{-1} \le \inf_{\overline{Q}} \rho.$$

4. Under the conditions $p \in C^1(\mathbf{R}^+)$, $p' > 0$, $N^{-1} \le \rho^0$ *and* $\|v^0\|_{L^2(J)} + \|\rho^0\|_{H^1(J)} \le N$, *one has the uniform* H^1*-bound*

$$\sup_{t\ge 0} \|\rho(\cdot,t)\|_{H^1(J)} \le K.$$

5. Under the conditions of Claim 4 together with

$$\|v_x^0\|_{L^2(J)} \le N, \quad \overline{f}_1 = 0, \quad \|p_\Gamma'\|_{L^2(\mathbf{R}^+)} \le N, \quad (4)$$

the uniform H^1*-bound*

$$\sup_{t\ge 0} \|v(\cdot,t)\|_{H^1(J)} \le K,$$

holds as well, which implies that

$$\|v(\cdot,t)\|_{C(\overline{J})} \to 0 \;\text{ as } t \to \infty.$$

Notice that Claims 1 and 2 imply the uniform bounds $\sup_{t\ge 0} R(t) \le K$ and $r_0 + K^{-1} \le \inf_{t\ge 0} R(t)$ for the “free” radius.

We introduce now the static problem

$$p_x(\rho_S) = \frac{f_S[r_S]}{r_S^2}, \text{ on } J, \qquad (5)$$

$$p(\rho_S)|_{x=M} = p_{\Gamma,S}, \qquad (6)$$

with $V_S := \frac{1}{3}r_S^3 = V_0 + I\eta_S$, and $\eta_S = \frac{1}{\rho_S}$.

We consider stationary solutions $\rho_S \in L^\infty(J)$, with ess $\inf_J \rho_S > 0$ and $p(\rho_S) \in C^1(\overline{J})$. Note that $p(\rho_S)$ decreases and satisfies $p_{\Gamma,S} \le p(\rho_S) \le \overline{p}_S := p_{\Gamma,S} + G\left(M_0 + \frac{1}{2}j_0 M\right)\frac{M}{r_0^4}$ on $\overline{J}$. Let $R_S := r_S(M)$ be the radius of the static free boundary.

Let us state stabilization properties of ρ in the case of general non-decreasing state function p.

Theorem 2 *Suppose that* $p'(s) \ge 0$ *with* $p(s) > 0$ *for* $s > 0$, *and that conditions* $N^{-1} \le \rho^0 \le N$ *and* $\|v^0\|_{L^2(J)} \le N$ *are valid. Then for any sequence* $t_n \to +\infty$, *there exists a subsequence* θ_n *such that*

$$\eta(\cdot,\theta_n) \to \eta_*(\cdot) \text{ weakly star in } L^\infty(J) \qquad (7)$$

with some $\eta_* \in L^\infty(J)$. *Moreover, for any sequence* $\theta_n \to \infty$, $\theta_n \ge 0$ *such that (7) holds, in fact* $\rho_S := \frac{1}{\eta_*}$ *is a stationary solution the potential energy of which satisfies the equality*

$$\mathcal{F}_S[\rho_S] :=$$

$$\int_J P_0(\rho_S) + p_{\Gamma,S}\rho_S^{-1} - F[r_S]\, dx = \mathcal{F}^{(S)}, \qquad (8)$$

where $\mathcal{F}^{(S)}$ *is given by (3),* $K^{-1} \le \rho_S \le K$, r_S *is such that* $\frac{1}{3}r_S^3 = V_0 + I(1/\rho_S)$, *and the limit relation holds*

$$\rho(\cdot,\theta_n) \to \rho_S(\cdot) \;\text{ in } L^\lambda(J),\ \forall\, 1 \le \lambda < \infty. \qquad (9)$$

Note that (9) implies that $p(\rho(\cdot,\theta_n)) \to p(\rho_S(\cdot))$ in $L^\lambda(\Omega_S)$, $\forall\, 1 \le \lambda < \infty$, and $R(\theta_n) \to R_S$. Accordingly, a by-product of the theorem is an existence result for the static problem.

Let us define the $\omega-$limit set for the density $\mathcal{O}_\rho$ as the set of functions $\rho_* := \frac{1}{\eta_*} \in L^\infty(J)$, where η_* satisfies the limit relation (7), for some subsequence $\theta_n \to \infty$, $\theta_n \ge 0$, and study its properties.

Theorem 3 *Let the hypotheses of theorem 2 be valid. Then the* $\omega-$*limit set* $\mathcal{O}_\rho$ *has the following properties.*

1. *If* $\rho_* \in \mathcal{O}_\rho$, *then* $\rho_* = \rho_S$ *is a stationary solution with fixed potential energy (8), moreover the limit relation (9) holds for the same sequence* θ_n *as in the property* $\rho_* \in \mathcal{O}_\rho$.

2. $\mathcal{O}_\rho$ *is a compact, connected and attracting set in* $L^\lambda(J)$, *for any* $1 \le \lambda < \infty$.

   Here, the attracting property means that

   $$\inf_{\rho_*\in\mathcal{O}_\rho} \|\rho(\cdot,t) - \rho_*(\cdot)\|_{L^\lambda(J)} \to 0 \;\text{ as } t \to \infty.$$

Corollary 1 *Let the hypotheses of Theorem 3 be valid. If, for any real* a, *there exists at most a countable set of stationary solutions satisfying* $\mathcal{F}_S[\rho_S] = a$, *then, for some of the solutions satisfying* $\mathcal{F}_S[\rho_S] = \mathcal{F}^{(S)}$, *the stabilization of* ρ *in a standard sense holds*

$$\rho(\cdot,t) \to \rho_S(\cdot) \text{ in } L^\lambda(J),\ \forall\, 1 \le \lambda < \infty. \qquad (10)$$

Corollary 2 *Let the hypotheses of Claim 4 in theorem 1 be valid. Then theorem 2, theorem 3 and corollary 1 may be improved by replacing the space $L^\lambda(J)$ by $C(\overline{J})$. In particular $\rho(\cdot,t) \to \rho_S(\cdot)$ in $C(\overline{J})$ instead of (10).*

Note that, in the case where the equation $p(s) = p(\rho_S(x_0))$ has non-unique solutions for some $x_0 \in J$, the function ρ_S has a jump at $x = x_0$, and thus replacing $L^\lambda(J)$ by $C(\overline{J})$ is impossible in (9) or (10).

4 Dynamical stability and stabilization rates

Let us consider the static problem (5)-(6) supposing that in all the sequel $p \in C^1(\mathbf{R}^+)$, $p' > 0$.

We set $\hat{p}(s) := p(s^{-1})$ and $h(\nu, x) := -G\frac{M_0+j_0x}{(3\nu)^{4/3}}$. Let also $p(\rho_{\Gamma,S}) = p_{\Gamma,S}$.

First, we give some uniqueness conditions.

Proposition 1 *1. Under the conditions*

$$\frac{GM}{r_0 C^{(1)}(\rho_{\Gamma,S})} \le 1, \tag{11}$$

with $C^{(1)}(s_0) := \inf_{s\ge s_0} p'(s) > 0$, the static problem has a unique solution.

2. Under the conditions

$$\frac{2GM^2(2M_0+j_0M)}{r_0^7 C^{(2)}(\rho_{\Gamma,S})} \le q, \tag{12}$$

with $C^{(2)}(s_0) := \inf_{s\ge s_0} s^2 p'(s) > 0$ for some $0 \le q \le 1$, the inequality holds

$$\begin{aligned}\mathcal{B}(\eta_1,\eta_2) &:= \int_J \{(\hat{p}(\eta_2) - \hat{p}(\eta_1))(\eta_1-\eta_2) \\ &\quad + (h[V_2]-h[V_1])(V_1-V_2)\}\ dx \\ &\ge C^{(2)}(\rho_{\Gamma,S})(1-q)\|\eta_1-\eta_2\|^2_{L^2(J)},\end{aligned} \tag{13}$$

for all $\eta_j \in C(\overline{J}),\ \ 0 < \eta_j \le \rho^{-1}_{\Gamma,S},\ j = 1,2$, with $V_j := V_0 + I\eta_j$.

Moreover the inequality is strict provided that $\eta_1(M) = \eta_2(M)$ and $\eta_1 \not\equiv \eta_2$.

Consequently, under condition (12) with $q = 1$, the static problem has a unique solution too.

By eliminating the functions ρ_S and r_S, we reduce the problem to the equivalent boundary value problem for the following quasilinear second order ODE

$$\begin{aligned}&\hat{p}_x((V_S)_x) = h[V_S] \ \text{ on } J, \\ &V_S(0) = 0,\ \ \hat{p}((V_S)_x)(M) = p_{\Gamma,S},\end{aligned} \tag{14}$$

for the function $V_S \in C^1(\overline{J})$, $((V_S)_x)_{min} > 0$; hereafter $\phi_{min} := \min_{\overline{J}} \phi(x)$.

One can linearize the problem near the solution V_S and then pass to the corresponding eigenvalue problem for the second order linear ODE

$$\begin{aligned}&\hat{p}'_x((V_S)_x)W_x + \frac{4h[V_S]}{3V_S}W = \lambda a_0 W \ \text{ on } J, \\ &W(0) = 0,\ \ \left(\hat{p}'((V_S)_x)W_x\right)(M) = 0,\end{aligned} \tag{15}$$

for some $a_0 \in C(\overline{J})$, $(a_0)_{min} > 0$.

Let $\lambda_{min}[\rho_S]$ be the minimal eigenvalue of this problem (with $\rho_S = ((V_S)_x)^{-1}$).

ρ_S is called *statically stable* provided that $\lambda_{min}[\rho_S] > 0$. One checks that this definition is independent of the choice of a_0.

Now one observes that the statically stable solutions are strongly isolated in the following sense.

Proposition 2 *If ρ_S is a statically stable solution, then, for some $\epsilon_0 > 0$ small enough, there exists no stationary solution $\rho_S^{(1)} \ne \rho_S$ such that $\left|V_S(M) - V_S^{(1)}(M)\right| = \left|\int_J (\eta_S - \eta_S^{(1)})\ dx\right| < \epsilon_0$, where $V_S^{(1)}(M) = V_0 + I\eta_S^{(1)}$ and $\eta_S^{(1)} = \frac{1}{\rho_S^{(1)}}$.*

Corollary 3 *The set of the statically stable solutions is at most finite.*

Corollary 4 *Let conditions $N^{-1} \le \rho^0 \le N$ and $\|v^0\|_{L^2(\Omega)} + \|\rho^0\|_{H^1(\Omega)} \le N$ be valid. Suppose also that $\mathcal{O}_\rho$ contains a statically stable solution ρ_S. Then*

$$\mathcal{O}_\rho \backslash \{\rho_S\} = \emptyset,$$

and the stabilization property $\rho(\cdot,t) \to \rho_S(\cdot)$ in $C(\overline{J})$ holds.

Let us discuss now variational aspects of the problem. We can rewrite the static potential energy as

$$\mathcal{F}[\rho_S] = \mathcal{P}[V_S]$$

$$:= \int_J \left(\hat{P}_0(DV_S) + p_{\Gamma,S}(V_S)_x - H[V_S]\right)\ dx,$$

where

$$\hat{P}_0(s) := P_0(s^{-1}) = -\int_1^s \hat{p}(\zeta)\ d\zeta,$$

and

$$H(\nu,x) = -G\left(\frac{1}{(3V_0)^{1/3}} - \frac{1}{(3\nu)^{1/3}}\right)(M_0 + j_0x).$$

Let us introduce the subspace

$$\tilde{C}^1(\overline{J}) := \left\{W \in C^1(\overline{J})\ ;\ W(0) = 0\right\},$$

together with the open set

$$\mathcal{S} := \left\{ V \in C^1(\overline{J}),\ V(0) = V_0 \ ;\ (V_x)_{min} > 0 \right\},$$

on an hyperplane of $C^1(\overline{J})$, and consider the values of $\mathcal{P}[V]$ on $\mathcal{S}$. The first and second variations of $\mathcal{P}$ are given by the formulas

$$\delta\mathcal{P}[V](W) := \left. \frac{d}{d\tau}\mathcal{P}[V+\tau W] \right|_{\tau=0}$$

$$= \int_J \{(p_{\Gamma,S} - \hat{p}(V_x))W_x - h[V]W\}\ dx$$

and

$$\delta^2\mathcal{P}[V](W) := \left. \frac{d^2}{d\tau^2}\mathcal{P}[V+\tau W] \right|_{\tau=0}$$

$$= \int_J \left\{ (-\hat{p}'(V_x)(W_x)^2 + \frac{4h[V]}{3V}W^2 \right\}\ dx,$$

for any $W \in \tilde{C}^1(\overline{J})$. The identity $\delta\mathcal{P}[V_S](W) = 0$ for all $W \in \tilde{C}^1(\overline{J})$ is equivalent to relations (14), i.e. the stationary points of $\mathcal{P}$ are solutions of the static problem (14).

Now, the positivity condition

$$\delta^2\mathcal{P}[V_S](W) > 0, \ \text{ for all } W \in \tilde{C}^1(\overline{J}), \ \ W \not\equiv 0,$$

is equivalent to the condition $\lambda_{min}[\rho_S] > 0$, since $\delta^2\mathcal{P}[V_S]$ is the energy functional of the eigenvalue problem (15), where once again $\rho_S = ((V_S)_x)^{-1}$.

Let us present an extremal characterization of the statically stable solutions.

We say that $V \in \mathcal{S}$ is a point of *local quadratic minimum of* $\mathcal{P}$ if

$$\mathcal{P}[V+W] - \mathcal{P}[V] \geq \delta_0 \|W\|^2_{H^1(J)},$$

$$\forall W \in \tilde{C}^1(\overline{J}), \ \ \|W\|_{\tilde{C}^1(\overline{J})} \leq \epsilon_0,$$

for some $\epsilon_0 > 0$ and $\delta_0 > 0$.

Proposition 3 *$V \in \mathcal{S}$ is a point of local quadratic minimum of $\mathcal{P}$ if and only if V_S is a solution of the static problem (14) such that $\lambda_{min}[\rho_S] > 0$.*

Notice that inequality (13), for some $0 \leq q < 1$, ensures the strong monotonicity of $\delta\mathcal{P}$ on the set

$$\{V \in \mathcal{S} \ ;\ V_x \leq \rho^{-1}_{\Gamma,S}\},$$

since $\mathcal{B}(\eta_1,\eta_2) = (\delta\mathcal{P}[V_1] - \delta\mathcal{P}[V_2])\,(V_1 - V_2)$.

In the following we use also the weaker condition

$$\tilde{\mathcal{B}}(\eta_1,\eta_S) \geq C_{0,\epsilon}\|\eta_1 - \eta_S\|^2_{L^2(J)}, \ \forall \eta_1 \in C(\overline{J}), \quad (16)$$

with $\epsilon \leq \eta_1 \leq \epsilon^{-1}$, for some $\eta_S = (V_S)_x$ with $V_S \in \mathcal{S}$ and any $\epsilon > 0$ small enough, with $C_{0,\epsilon} > 0$.

Under this condition, V_S is the point of *global quadratic minimum of* $\mathcal{P}$ on the sets $\{V_1 \in \mathcal{S} \ ;\ \epsilon \leq (V_1)_x \leq \epsilon^{-1}\}$, for $\epsilon \leq \rho^{-1}_{\Gamma,S}$, since $\mathcal{P}[V_1] - \mathcal{P}[V_S] \geq C_{0,\epsilon}/2\ \|(V_1 - V_S)_x\|^2_{L^2(J)}$.

Let

$$\Gamma_1(s) := \frac{sp'(s)}{p(s)},$$

for $s > 0$, be the so-called first adiabatic exponent of the gas. Obviously $\Gamma_1(s) \equiv \gamma$ in the polytropic case $p(s) = p_1 s^\gamma$.

Theorem 4 *Suppose that*

$$\Gamma_1(s) \geq 4/3,$$

for $\rho_{\Gamma,S} \leq s \leq \overline{\rho}_S$, where $p(\overline{\rho}_S) = \overline{p}_S$. Then

1. *the static problem (5)-(6) has a unique solution,*
2. *this solution is statically stable.*

Now we turn to stabilization rate estimates and to the nonlinear dynamic stability (of exponential type) for the statically stable stationary solutions. We introduce the stabilization errors, for $i = 0, 1$ and $j = 0, 1$

$$\delta_{i,j}(t) := \|\rho(\cdot,t) - \rho_S(\cdot)\|_{H^i(J)} + \|v(\cdot,t)\|_{H^j(J)},$$

where $H^0(J) = L^2(J)$.

Theorem 5 *Let conditions $p \in C^1(\mathbf{R}^+)$, $p' > 0$, $N^{-1} \leq \rho^0 \leq N$ and $\|v^0\|_{L^2(\Omega)} + \|\rho^0\|_{H^1(\Omega)} \leq N$ be valid. Then*

1. Let $\mathcal{O}_\rho$ contain a statically stable solution ρ_S. The following L^2-stabilization rate bound holds

$$\delta_{0,0}(t) \leq K_0 \left(e^{a_0(t_0 - t)}\delta_{0,0}(t_0) \right.$$

$$+ \|e^{a_0(\tau - t)}(\overline{f}_1 + \overline{f}_2)(\tau)\|_{L^1(t_0,t)}$$

$$\left. + \|e^{a_0(\tau - t)}\Delta p_\Gamma(\tau)\|_{L^2(t_0,t)} \right). \quad (17)$$

If in addition $p'' \in L^\infty(\mathbf{R}^+)$, then the following combined bound holds

$$\delta_{1,0}(t) \leq K_1 \left(e^{a_1(t_0 - t)}\delta_{1,0}(t_0) \right.$$

$$+ \|e^{a_1(\tau - t)}(\overline{f}_1 + \overline{f}_2)(\tau)\|_{L^1(t_0,t)}$$

$$\left. + \|e^{a_1(\tau - t)}\Delta p_\Gamma(\tau)\|_{L^2(t_0,t)} \right). \quad (18)$$

Moreover if conditions (4) are valid, then one gets the H^1-bound

$$\delta_{1,1}(t) \leq K_2 \left(e^{a_2(t_0 - t)} \left(\delta_{1,1}(t_0) + |\Delta p_\Gamma(t_0)| \right) \right.$$

$$+\|e^{a_2(\tau-t)}\overline{f}_2(\tau)\|_{L^2(t_0,t)} + \|e^{a_2(\tau-t)}\Delta p_\Gamma(\tau)\|_{L^2(t_0,t)}$$
$$+\|e^{a_2(\tau-t)}p'_\Gamma(\tau)\|_{L^1(t_0,t)}\Big). \tag{19}$$

Here $t \geq t_0$ for sufficiently large t_0, and $a_l := 1/K_l$, for $l = 0, 1, 2$.

2. Let ρ_S be any statically stable stationary solution (contrary to Claim 1). If the data are sufficiently close to the stationary ones, namely

$$\|\rho^0-\rho_S\|_{L^2(J)}+\|v^0\|_{L^2(J)}+\|\overline{f}_1\|_{L^1(\mathbf{R}^+)}+\|\overline{f}_2\|_{L^2(\mathbf{R}^+)}$$
$$+\|\Delta p_\Gamma\|_{L^2(\mathbf{R}^+)} \leq \epsilon_0, \tag{20}$$

for $\epsilon_0 > 0$ small enough, then all the stabilization rate bounds (17)-(19) hold for $t_0 = 0$, and the quantities K_l, a_l are independent of the data.

Moreover this nonlinear dynamical stability result holds even for non-monotone p provided that

$$p'(s) > 0 \ \text{ on some } (s_1, s_2) \supset \rho_S(\overline{J}). \tag{21}$$

3. Let now ρ_S satisfy condition (16), which is more restrictive than the condition on ρ_S in Claim 2.

Then the stabilization rate bounds (17)-(19) hold for $t \geq t_0 = 0$, with K_l independent of the data, without the smallness condition (20).

Notice that bound (17) together with the inequality

$$K^{-1}|R(t) - R_S| \leq \|\rho(\cdot, t) - \rho_S(\cdot)\|_{L^2(J)},$$

for $t \geq t_0$, insure the stabilization rate bound for $R(t) - R_S$ as well.

Condition (21) means that the values of ρ_S belong to an interval where p is stable.

Note that an interesting physical study of related stability problems has been recently given in [11].

The above results follow [12]-[13] where much more general information can be found including the corresponding study in the Eulerian coordinates.

5 Conclusion and comments

Our barotropic spherical model is an oversimplification of the 3D-astrophysical situation (see [3]) where various phenomena can take place (thermonuclear reactions, magnetohydrodynamic effects, radiative transfert...), and which is probably out of reach of present studies.

The dynamical boundary condition for the pressure on the free-boundary is a phenomenological modelling for a complex physical situation. In fact a better approximation would consist in coupling a radiative Navier-Stokes flow for the bulk star to a kinetic description for the surrounding underdense atmosphere, which is, to our knowledge, an open problem in this context.

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