Proceedings of the 2006 IASME/WSEAS International Conference on Continuum Mechanics, Chalkida, Greece, May 11-13, 2006 (pp166-169)

Escape Solutions of Two-Degree of Freedom Dynamical System of the Coupled Non-Linear Double Oscillator with Third Order Potential

EVANGELOS P. VALARIS* and MARIA ALEX. LEFTAKI°
*Department of Engineering Sciences
° Department of Mathematics
University of Patras
26500 Patras
GREECE

Abstract: - This paper presents the results of a remarkable effort to approximate the space of escape solutions of the axi-symmetric, two-degree of freedom dynamical system of the non-linear double oscillator corresponding to a third order potential. The initial conditions of escape solutions constitute a compact two-dimensional space in the (x, C) plane and the solutions generated by them, occupy densely its space of motion. Determination of the part of (x, C), within which ordered (non-chaotic) motions prevail, is better achieved through determination of the space of escape motions.

Key words: - Non-linear double oscillator, escape solutions, ordered motions.

1 Introduction

The problem which we have treated here is that of the coupled non-linear double oscillator. The study of the dynamics of coupled non-linear oscillators has attracted the interest of many researchers in recent years. The reason is that they arise in many branches of science in order to model and to explain different physical, chemical, biological, etc., processes.

The potential V of this problem is given by the expression:

$$V = \frac{1}{2}\left(Ax^2 + By^2\right) - bxy^2\,, \tag{1}$$

with A, B and b being constants.

A potential of this type may represent either the 'effective potential' on the meridian plane of an axi-symmetric galaxy near a given circular orbit, or the so-called 'Barbanis potential' following the work of Barbanis in the chemical literature[1].

The equations governing the motion are the following ones:

$$\frac{d^2x}{dt^2} = -Ax + by^2\,, \quad \frac{d^2y}{dt^2} = -By + 2bxy\,, \tag{2}$$

with the expression:

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + Ax^2 + By^2\right] - bxy^2 = C\,, \tag{3}$$

being the integral of its energy.

It is obvious that in the unperturbed case, where b=0, all the orbits can be resonant periodic Lissajous figures, or not resonant, depending on values taken for A and B. In our problem we treat here, a coupling term (b≠0) is present.

The permissible area of motion and the 0-velocity curves are expressed by the inequality

$$C - \frac{1}{2}\left[Ax^2 + By^2\right] + bxy^2 \geq 0\,, \tag{4}$$

the case of equality defining the boundary of this area.

2 Escape Solutions Areas

Our interest is focused on the study of solutions escaping to infinity. The first of our results having been found for another conservative dynamical problem of two degrees of freedom with escape orbits, reveal their spiral form on the plane (x, y)[2]. Because of their significance many researchers have started studies on escape solutions in various problems, especially, of celestial mechanics [3],[4],[5] and [6].

In the plane of initial conditions (x, C) and within the space of permissible motion we mark the points generating escape solutions in the plane (x, y), of one, two, etc., intersections with the x-axis. In Fig.1 we present areas of escape solutions of one intersection (in red color), of two intersections (in green color), of three (in blue color), of four intersections (in cyan color), of five intersections (in magenta color), of six intersections (in yellow color), of seven intersections (in dark yellow color), of eight intersections (in navy blue color) and of nine intersections (in purple color) with the x-axis. The values used for the constants are A=1, B=2, and b=0.5.

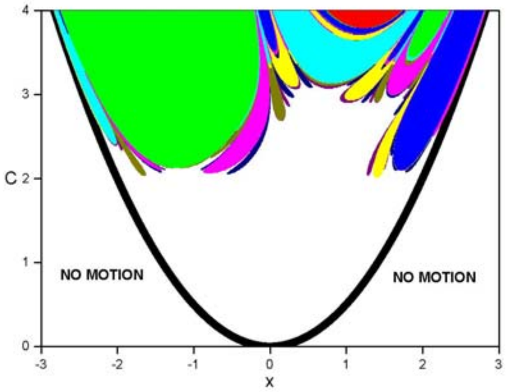

Fig. 1: Escape solutions areas of one up to nine intersections with x-axis.

In Figures 2, 3, 4 and 5 on the (x, y) plane we present four escape solutions of one, two, three and four intersections, respectively, with the x-axis.

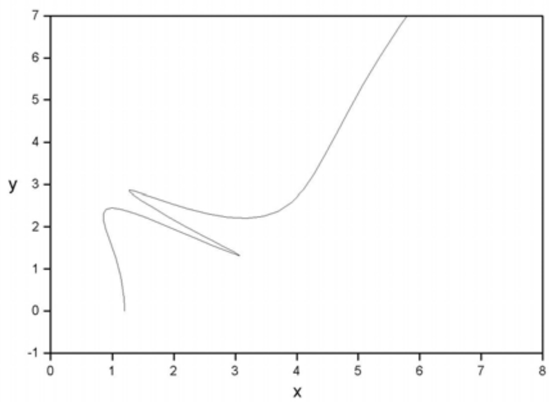

Fig. 2: Escape solution with initial conditions: x=1.2, C=3.8.

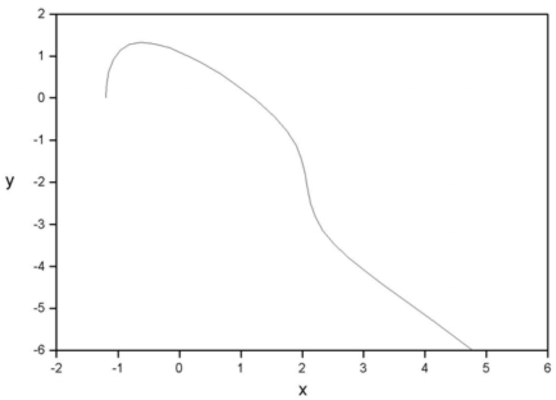

Fig. 3: Escape solution with initial conditions: x=-1.2, C=3.4.

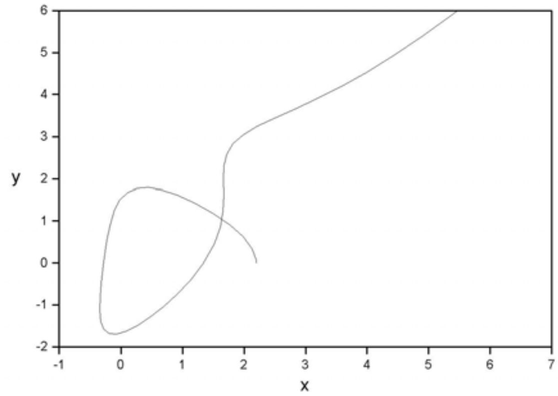

Fig. 4: Escape solution with initial conditions: x=2.2, C=3.4.

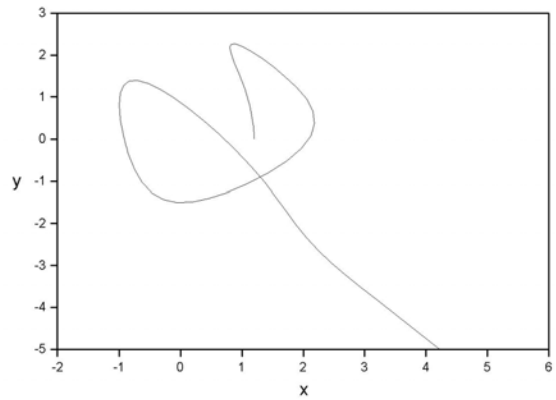

Fig. 5: Escape solution with initial conditions: x=1.2, C=3.4.

For the value, for instance, of C=2.69, we find symmetric periodic solutions of one, two, etc, intersections with x-axis, i.e. solutions which intersect normally the x-axis initially, and at the first, second, etc., respectively, intersection and in this way, simultaneously, we locate the initial points conducting to escape solutions.

In Fig. 6 we present symmetric and periodic solutions in ordered or chaotic areas in black color, while the points which generate escape solutions in red color. The periodic solutions give a picture of order in the middle of this diagram. We see that the ordered area is surrounding by a chaotic region at its borders, whereas the escape regions have gaps containing non-escaping orbits. We also note that the escape regions are shifting toward the middle of the figure as the number of intersections is increasing (from 1 to 100). This configuration seems to "converge", as the number of intersections increases, to some apparently "final" form. The values used for the constants are A=1, B=2, and b=0.5.

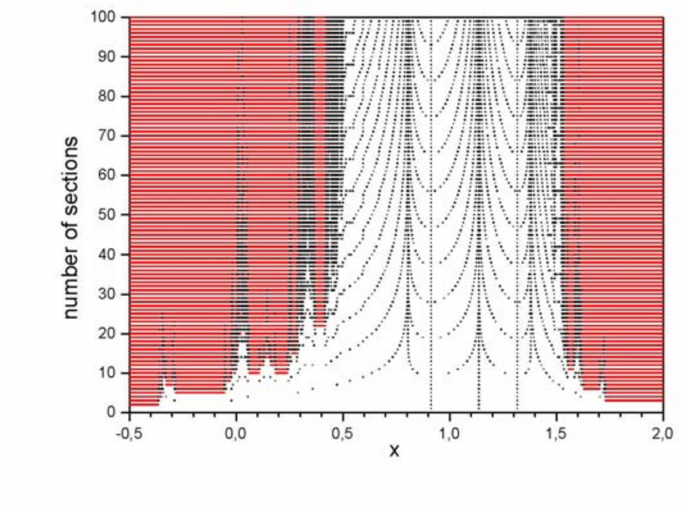

Fig. 6: Periodic solutions and escape solutions for C=2.69.

During the search of symmetric periodic solutions in the (x, C) plane for a large number of intersections with the x-axis, e.g. one hundred intersections within half time of their period, i.e. solutions intersecting normally this axis at the hundredth intersection (the starting point not including), we locate the points conducting to escape solutions. The escape solutions have initial conditions shown as points in red color, in Figure 7 appearing below.

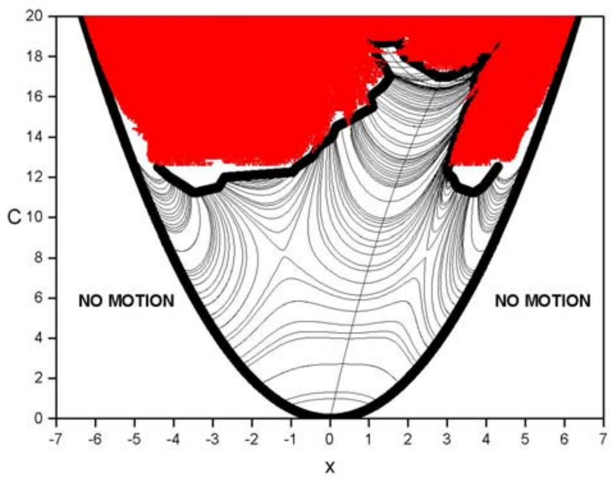

Fig. 7: Area of escape solutions and area of order.

The initial conditions appearing in the Fig. 7 correspond to values of C starting from a certain value and above and extend as far as the space of order (non chaotic). This space is covered densely by stable arcs of families of periodic solutions and it constitutes the space of order in the invariant curves [7]. The values used for the constants are A=1, B=2, and b=0.2.

The curves of families of periodic solutions are continuous in the analytical sense with their periods varying continuously along the same family curve and from 1 to infinite oscillations from start to re-entrance, from curve to curve. Next, and as close as we select, to each curve we can compute another periodic family curve and thus fill the (x, C) space with such curves. The periodic family curves are countably dense and thus as close as we like to any randomly selected point and hence corresponding to a non-periodic solution with probability1, a point resting on a periodic family curve can be found [8].

The invariant curves on a Poincaré surface of section $(x, \dot{x})$ in the Hamiltonian (3) for A=1, B=2, b=0.5 and C=2.69 are presenting in Fig. 8. We notice that the interior of the space of order is occupied by closed and uniformly varying sets of invariant

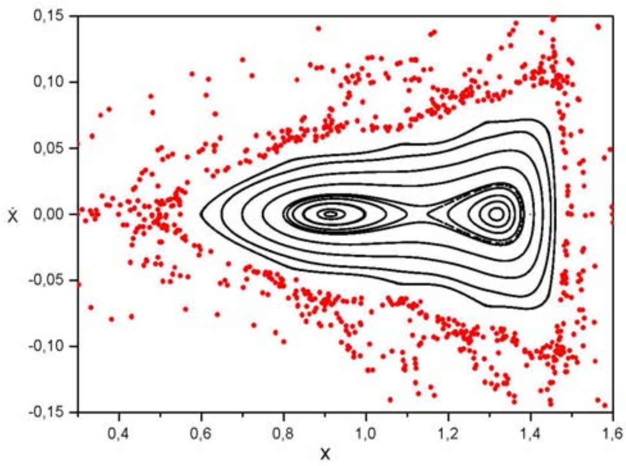

Fig. 8: Invariant curves for C=2.69.

curves, whereas the external region is revealing a picture of chaos which is indicated by points in red color signifying escape solutions. We note that there are no Poincaré surfaces of section when the orbits escape, i.e. there is no return. In this way, the escaping orbits meet the $(x, \dot{x})$ plane at a finite, always, number of points on it. We see that the infinity ''attracts'' them. In this case the theorem of Liouville still holds, but the areas on a surface of section are not conserved.

It is worthwhile mentioning that in the picture of Fig. 8 two simple symmetric and periodic solutions appear. The stable one corresponds to the centers of the small islands. The unstable one (which it was stable for a constant of energy C less than a critical value of C<2.69) in-between the two islands corresponds to the point (1.13, 0).

All numerical integrations were performed by use of the 8th order Runge-Kutta, variable step-size, algorithm that secured variation of the energy constant to less than 10^{-10} for all solutions.

3 Conclusion

The determination of the space of order in the (x, C) plane, either by computing families of periodic solutions and the study of the stability of their members, or by a systematic study of the phase configuration on the surface of section (y=0) of the Hamiltonian (3) for various values of the energy C, is a relatively cumbersome procedure. The same objective is achieved much more easily through determination of the initial conditions, in the (x, C) plane, that generate escape solutions of large number of intersections with the x-axis.

References:

[1] B. Barbanis, *Astron. J.*,Vol. 71, 1966, p. 415

[2] E. P. Valaris, and M. A. Leftaki, Spiral Orbits of a Charged Particle Moving in the Field of Two Revolving Parallel Magnetic Dipoles, *4th International Workshop on "Positional Astronomy and Celestial Mechanics"*, A. Lopez Garcia et al. (eds.) Valencia, 1998, pp.157-160.

[3] G. Contopoulos, *"The Few-Body Problem"*, Kluwer, M. J. Valtonen (ed.), Dordrecht, 1988, p.265.

[4] L. Benet, D. Traumann, and T. H. Seligmann, *Cel. Mech. Dyn. Astr.*,Vol.66,1997, p.203.

[5] L. Benet, T. H. Seligmann, and D. Traumann, *Cel. Mech. Dyn. Astr.*, Vol.71,1999, p.167.

[6]J. P. Anosova, *Astrophysics and Space Sci.*, Vol.124,1986,p.217.

[7] E. Valaris, and M. Leftaki, Approximate General Solution of 2-Degree of Freedom Dynamical System of the Coupled Non-Linear Double Oscillator with Third Order Potential, *1st International Conference "From Scientific Computing to Computational Engineering"*, Athens,2004.

[8] C.L.Goudas, K.Papadakis, and E.Valaris, Approximate General Solution of 2-Degree of Freedom Dynamical System of the Coupled Non-Linear Double Oscillator with 5th Order Potential, *4th GRACM Congress on Computational Mechanics*, Patras,2002.