Mode Coalescence in Two-Fluid Boundary Layer Stability

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Abstract: The hydrodynamic stability problem of two-fluid boundary layers is studied. Using the linear stability theory, a fourth order Ordinary Differential Equation (the Orr-Sommerfeld equation) is derived for each of the fluid layers starting from the two-dimensional, incompressible Navier-Stokes equations. The equations, boundary and interface conditions result in an Eigenvalue problem that is solved using an efficient shoot-search technique. The Eigenvalues of the problem are the Reynolds number, wave number, and complex wave velocity. Unlike the single fluid problem, which results in a single unstable solution (Tollmien-Schlichting mode), the two fluid problem has multiple unstable solutions (Tollmien-Schlichting and Interfacial modes). The parameters that influence the stability problem are the viscosity and density ratios of the two fluids, thickness of the sheared fluid layer, surface tension and gravity. For certain combinations of viscosity ratio and fluid thickness the Tollmien-Schlichting mode and the Interfacial modes interact resulting in a composite mode, that has the properties of both parent modes. A parametric study of the problem is presented, which brings out the details of the mode coalescence and the parameter combinations at which it occurs.

Key-Words: hydrodynamic stability, interfacial instabilities, stability of thin films, two fluid flow, mode coalescence

1 Introduction

In this article, the instability problem of a two-fluid boundary layer flow is presented. In this flow, a viscous fluid shears another viscous fluid, that is bounded by the first fluid above and the wall below (Fig. 1). The problem in hand has relevance to flows of de-icing fluids over aircraft wings entrained by airflow, jet wiping, and coating applications. The problem is interesting also from a mathematical point of view, as it poses an Eigenvalue problem with multiple interacting solutions.

In the two-fluid stability problem, in addition to the Tollmien-Schlichting (T-S) mode present in the single layer flow, additional solutions or instability modes can be found (Interfacial modes). The presence of these additional modes was first reported by Yih [1]. Under certain circumstances, these modes interact with each other or with the Tollmien-Schlichting mode. This paper aims at revealing these multiple solutions and the conditions for which they interact (coalesce).

The Eigenvalues of the problem are the Reynolds number $Re$, the wavenumber $\alpha$ and the complex wave velocity $c=c_r+ic_i$. The real part of the wave velocity is the phase speed with respect to the interface, while the imaginary part is the amplification factor determining whether a disturbance is stable; $c_i<0$, unstable; $c_i>0$, or neutrally stable; $c_i=0$. The parameters that enter the problem are the viscosity and the density ratios of the two fluids $m$ and $r$, thickness of the lower fluid layer $l$, surface tension parameter $S$ and the Froude number $Fr$. Two-fluid flow instability problem has attracted considerable interest from the researchers since almost six decades. Among the more recent and relevant studies are those of Hooper and Boyd [2], [3], Yih [4], Miesen and Boersma [5], Timoshin [6], Özgen et al [7], Timoshin and Hooper [8]. It has been reported that, in some parameter ranges it is not possible to identify a mode as the T-S mode or the interfacial mode [7, 8]. In such cases, a crossover occurs between the curves representing the Eigenvalue solutions of the two modes. An example to such a behavior is given in Fig. 8 of Özgen et al [7], where the familiar patterns of the neutral stability curves of the Tollmien-Schlichting and interfacial modes change substantially. The problem has been further elaborated on by Timoshin and Hooper [8], who have demonstrated that mode coalescence does not only occur in the unstable part of the spectrum but also in the stable part. The results of references [7] and [8] agree qualitatively well although very different methods have been used for the solutions (shooting method in the former and an asymptotic method in the latter).

2 Problem Formulation

In the flow geometry shown in Fig. 1, subscripts 1 and 2 denote upper and lower fluid layers, respectively. Viscosities of the fluid layers are denoted by $\mu$, whereas the densities are shown by $\rho$. The asterisk denotes dimensional variables. The velocity profile of the upper fluid layer is the Blasius profile, while the lower layer
profile is taken to be linear. Also, \( m = \mu_2^* / \mu_1^* \) is the viscosity ratio and \( r = \rho_2^* / \rho_1^* \) is the density ratio of the two fluids. The length scale of the upper fluid layer is the Blasius length scale defined as \( d_1^* = \sqrt{V_1^* x^* / U_e^*} \), while \( U_e^* \) is the upper fluid layer freestream velocity with respect to the interface. The depth of the lower fluid layer is \( d_2^* \), while \( l = d_2^* / d_1^* \). Finally, \( a_1 \) and \( a_2 \) are the velocity gradients for the upper and lower fluid layers, respectively. All length are normalized by \( d_1^* \), while all velocities are reduced by \( U_e^* \).

\[
\chi = \chi' = 0 \quad \text{at} \quad y = -l, \tag{4}
\]
\[
\phi \to 0, \quad \phi' \to 0 \quad \text{when} \quad y \to \infty. \tag{5}
\]

In addition to these, four interface conditions are needed to account for the continuity of velocity and stress components at the interface. These are:

\[
\phi(0) = \chi(0), \tag{6}
\]
\[
\phi'(0) - \chi'(0) = \frac{\phi(0)}{c - U_o} (a_2 - a_1), \tag{7}
\]
\[
\alpha^2 \phi(0) + \phi''(0) = ml' \chi''(0) + \alpha^2 \chi(0) \tag{8}
\]
\[
- \imath \alpha R \left[ \phi' + a_1 \phi \right] - \left( \phi'' - 3 \alpha^2 \phi' \right) \tag{9}
\]
\[
+ \imath \alpha R \left[ \chi' + a_2 \chi \right] + ml' \chi'' - 3 \alpha^2 \chi' \tag{9}
\]
\[
= \imath \alpha R \left[ r - l \right] F \alpha - \alpha^2 S \phi / c. \tag{9}
\]

Equations (6) and (7) are the normal and tangential velocity matching conditions, whereas equations (8) and (9) are the tangential and normal stress matching conditions. In equation (9), \( Fr = U_e^* / \sqrt{g^* d_1^*} \) is the Froude number, and \( g^* \) the gravitational acceleration. Also, \( S = \sigma^* / \left( \rho_1^* U_e^* d_1^* \right) \) is the surface tension parameter, \( \sigma^* \) the surface tension coefficient.

### 3 Problem Solution

#### 3.1 Stability Equations

The solution of the equations (2) and (3) together with the boundary and interface conditions given in equations (4)-(9) is accomplished by a shooting method. In this method, equation (2) is integrated starting from a sufficiently large distance from the wall, say \( y_w \), such that \( U(y_w) \to 1 \) and \( U''(y_w) \to 0 \) to a sufficient degree of accuracy. Two asymptotic solutions are found for equation (2), which are the initial values of the integral:

\[
\begin{bmatrix}
\phi_1 & \phi_1' & \phi_1'' & \phi_1^* \\
\phi_2 & \phi_2' & \phi_2'' & \phi_2^*
\end{bmatrix} = \begin{bmatrix}1 & -\alpha & \alpha^2 & -\alpha^3 \\
1 & -\beta & \beta^2 & -\beta^3
\end{bmatrix}, \tag{10}
\]
\[
\begin{bmatrix}
\phi_1 & \phi_1' & \phi_1'' & \phi_1^* \\
\phi_2 & \phi_2' & \phi_2'' & \phi_2^*
\end{bmatrix} = \begin{bmatrix}1 & -\alpha & \alpha^2 & -\alpha^3 \\
1 & -\beta & \beta^2 & -\beta^3
\end{bmatrix}, \tag{11}
\]

with \( \beta^2 = \alpha^2 + \imath \alpha R (1 - c) \).

When the interface \((y=0)\) is reached at the end of the integration, the Orr-Sommerfeld equation (equation 3) for the lower fluid layer needs to be solved. For this, the integration starts at the wall \((y=-l)\) and proceeds towards the interface. The general solution of equation (3) is:
\( \chi = B_1 \chi_1 + B_2 \chi_2 + B_3 \chi_3 + B_4 \chi_4. \) \hspace{1cm} (12)

An asymptotic solution cannot be found for equation (3) in the bounded domain \((-1 \leq y \leq 0)\) like in equation (2), therefore four orthonormal solutions \(\chi_{i=1,4}\) are constructed at \(y=0\), with the unit vectors:

\[
\begin{bmatrix}
\chi_1 \\
\chi_2 \\
\chi_3 \\
\chi_4
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\] \hspace{1cm} (13)

Due to the boundary conditions in equation (4), \(B_1 = B_2 = 0\). The two remaining solutions are the initial values for the integration of equation (3). When the interface is reached at the end of the integration, the interface conditions given in equations (6-9) are to be satisfied. These relate the constants \(A_1, A_2\) of the upper layer to the constants \(B_3, B_4\) of the lower layer solution.

The four conditions yield a system as follows:

\[
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & c_{14} \\
c_{21} & c_{22} & c_{23} & c_{24} \\
c_{31} & c_{32} & c_{33} & c_{34} \\
c_{41} & c_{42} & c_{43} & c_{44}
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
B_3 \\
B_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\] \hspace{1cm} (14)

The elements of the matrix are detailed by Özgen [9]. A vanishing determinant yields a non-trivial solution for the above system, which happens for certain combinations of the Eigenvalues. In order to determine these combinations, two of the Eigenvalues are fixed and the remaining two are solved for using Newton iteration in two variables. For the presentations in this article, either \((a, R)\) are fixed to plot \((a, c_i)\) curves, or \((R, c_i)\) are fixed to plot \((R, a)\) curves. For both cases \(c_i\) is a computed parameter. For the Newton iteration to proceed in the specified \(R\) or \(a\) direction, it requires two known points on the curve to start with. These are provided by a function minimization subroutine available in a software library [10]. During integration, the Gram-Schmidt orthonormalization routine is applied at regular intervals, to preserve the linear independency of the solutions, both at upper and lower layers.

3.2 Mean Flow Equations
The mean velocity terms \(U_1\) and \(U_2\), and \(U'_1\) need to be calculated in equations (2) and (3). In this study, the Blasius equation is solved for the upper layer flow:

\[2 f'' + f' = 0, \] \hspace{1cm} (15)

where \(f\) is a dimensionless streamfunction defined as \(f' = U_1\). The boundary conditions for this equation are:

\[f = f' = 0 \quad \text{at} \quad y = 0, \] \hspace{1cm} (16)

\[f' \to 1 \quad \text{when} \quad y \to \infty. \] \hspace{1cm} (17)

The velocity profile for the lower layer is given as:

\[U_2 = a_2 y, \] \hspace{1cm} (18)

with \(a_2 = a/m\). The boundary conditions are:

\[U_2 = -U_o \quad \text{at} \quad y = -l, \] \hspace{1cm} (19)

\[U_2 = 0 \quad \text{at} \quad y = 0, \] \hspace{1cm} (20)

where \(U_o\) is the interfacial velocity defined as \(U_o = (a_1/m)l\). In the solution, equation (15) is integrated simultaneously with equation (2), and accurate values of \(U_o, U_2, \) and \(U'_1\) are provided.

4 Results and Discussion

First, it is appropriate to identify the instability modes for a two-fluid boundary layer flow. As it was mentioned previously, in the two-fluid flow, in addition to the Tollmien-Schlichting mode of instability also found in single layer flow, multiple instability modes exist due to viscosity stratification. Figure 3b, shows these modes for moderate viscosity stratification. In addition to the T-S mode, there exist three more modes, labeled as I-1, I-2 and I-3. For this parameter combination, the T-S, I-1 and I-2 become unstable \((c_i > 0)\). The T-S mode has the highest amplification factor among these. The physical mechanisms bearing these modes are also quite different. The T-S mode is due to shear production at wall, while the interfacial modes are due to viscosity stratification at the interface. This is why the interfacial modes do not exist in the classical hydrodynamic stability theory. The neutral stability curves corresponding to the same parameter combination can be seen Figure 2b.

In order to study the mode coalescence phenomena, the parameter \(l\) is varied. Figures 2 and 3 show the neutral stability curves and amplification factors for \(m=5\), with values of \(l\) varying between 0.25 and 3.0.

Figure 2a depicts the neutral stability curve for \(l=0.25\). For this case, only the T-S mode neutral stability curve is present, meaning that the entire domain is unstable for the interfacial mode. The curves in Figure 3a show the amplification factors for this configuration at \(R=4000\).

The results corresponding to \(l=0.5\) are shown in Figures 2b and 3b. In Figure 2b, the lower curve is the neutral stability curve for the T-S mode, while the upper curve is that of the interfacial mode. In Figure 3b, the amplification factor curves for the T-S, I-1 and I-2 modes are evident. In Figure 2b, we observe that the neutral stability curves of the T-S and the I-1 mode approach each other but there still no mode coalescence in the Reynolds number range shown, also inferred from Figure 3b.
All three modes are distinct in this figure, and the lower branch of the neutral stability curve of the interfacial mode corresponds to the I-1 mode, whereas the upper branch corresponds to the I-2 mode, suggesting a mode crossover between the I-1 and I-2 modes. When l is further increased to 0.75, the usual appearances of the neutral stability curves still remain as evidenced by Figure 2c, but Figure 3c shows that a mode coalescence between the T-S and the I-2 modes occur in the stable region of the domain.

**Fig. 2** Neutral stability curves for two-fluid boundary layer stability

\[m=5, \quad r=1, \quad l=0.5, \quad S=0\].
In Figure 2d, one observes that for $l=1.0$, the usual shapes of the neutral stability curves change considerably. Now, mode coalescence between the T-S and the I-1 mode occurs in the unstable region of the domain. The unstable region of the T-S mode becomes smaller, while the stable region of the interfacial mode remains roughly of the same size. In Figure 3d, the mode coalescence of the T-S and the I-1 mode is rendered more obvious. The composite mode exhibits the characteristics of the I-1 mode for very low values of the
wave number, while it demonstrates T-S mode characteristics for higher values. Yet, the characteristics of the interfacial mode are again observed for wave numbers above the neutral stability location. Thus, it is no longer possible to label a mode as T-S or interfacial.

In Figures 2e and 2f for $l=2.0$ and $3.0$, we observe that the composite mode fully dominates the instability, and substantial changes do not occur as $l$ is further increased. This is also evident in the amplification factor curves shown in Figures 3e and 3f. This feature has also been observed for $m=2$ and $m=10$ (not shown).

Another interesting feature of the amplification factors is that their maximum values increase as the value of $l$ increases, especially for $l$ values where the composite mode dominates. There is evidence that the composite mode is more relevant to the interfacial instability mechanism rather than the T-S mechanism. As $l$ increases, the interface moves away from the wall, and since the T-S mode is related to the shear production at the wall, it loses its strength. On the other hand, since the T-S mechanism is weakened, the entire disturbance energy is absorbed by the interfacial mode, which explains the increase in the amplification factors.

In Figure 4, one sees the details of mode coalescence between the T-S mode (thick lines) and the I-1 mode. In this figure, we observe that the amplification factor curves for the T-S and I-1 modes are distinct and separate for $l=0.65$. For $l=0.65$, the curves cross each other at around $\alpha=0.045$ but there is no mode coalescence. However, when $l$ is increased to 0.75, mode coalescence occurs. We again have two distinct stability curves, one plotted with a thick line, the other with thin, to facilitate identification. Here, it is not possible to call the thick lines as the T-S mode and the thin line as the I-1 mode, because each of the curves depicted carry the characteristics of both "parent" modes. For example, the thick curve resembles the T-S mode curve for $\alpha>0.045$ but the same curve shows the characteristics of the I-1 mode for $\alpha<0.045$. The same thing can be said for the thin curve. This time the behavior is Tollmien-Schlichting-like for $\alpha<0.045$ and interfacial-like for $\alpha>0.045$. It is also interesting to note that the thin curve shows a small unstable bandwidth.

**Conclusions**

The stability problem of a two-fluid layer flow is solved using a numerical method. The presence of the interfacial modes has been verified. It has been shown that for sufficiently thick sheared fluid layers, mode coalescence occurs between the Tollmien-Schlichting and the Interfacial modes. When the lower fluid thickness is further increased, it is no longer possible to distinguish the two separate modes, rather a composite mode that shows the characteristics of both parent modes dominate the instability problem. Another interesting feature is the increase in the amplification factors as the lower fluid thickness increases.

For future work, it would be interesting to investigate the effects of surface tension, gravity and density stratification on the composite mode since the effects of these parameters on the interfacial mode and the T-S mode has already been studied [7, 8]. Another interesting question is whether the composite mode can be observed in an experiment.

**References**