Efficient Modeling of the Wave Propagation through the Ionosphere using the Finite Element Method

STERGIOS A. ISAAKIDIS, THOMAS D. XENOS, TRAIANOS V. YIOULTSIS, NIKOS J. FARSAIS
Aristotle University of Thessaloniki
Department of Electrical and Computer Engineering
54006 Thessaloniki, Greece

Abstract - In this work, an HF ionospheric wave propagation model, below and above MUF is developed and analyzed using the one-dimensional finite element method. The purpose is to find numerical solutions of the one-dimensional wave equation and study the way that the wave propagates through the ionosphere. The ionospheric medium at heights between 65 to 400 Km, is modeled by its refractive index profile, which is calculated using electron density data from the International Reference Ionosphere model (IRI-95). The frequencies used in this work, in order to illustrate the model are below and above MUF and two slightly different ionospheric profiles have been used in order to evaluate the model response with small variations of the medium characteristics.

1. Introduction

Ionospheric propagation is a complex problem, due to the fact that the ionosphere is an inhomogeneous, anisotropic, turbulent, non-stationary and multipath medium. Ray-tracing models usually give a representation of the behavior of the coherent part, taking into account large and medium scale spatial and temporal variations and anisotropy. On the other hand, many applications require a knowledge of the exact ray path of the radio wave ionospheric propagation. The accuracy of these systems depends not only on the accuracy of the ionospheric electron density model but also on the accuracy of the solutions to the equations used. Due to the fact that although a full numerical solution of the ray equations provides the most accurate result, the computation requires excessive computational resources. Many approximate ray-tracing techniques and models have been developed and are in use nowadays.

In this work, unlike existing analytical studies (Ghermv et al., 1997, 1998), we propose a computational treatment of the wave propagation problem through the ionosphere, based on the Finite Element Method (FEM). The ionospheric medium is modeled by the refractive index profile which, together with the boundary conditions and the wave equation forms the mathematical model which will be dealt with using the one-dimensional FEM. The effect of collisions is incorporated in the model via the collision frequency of the medium, which is calculated using particle density data from the MSIS-E-90 model of NSSDC. Thus, the numerical values for the electric field are obtained at all height levels. First, the mathematical model is presented and the expressions for the refractive index and the differential wave equation are derived. The parameters that have been used for the profile of the medium are also presented and the WKB solution is given as an approximate reference solution. The computational method used for the solution of the problem is, then, described and finally, the results of the method are presented for two different refractive index profiles and compared to the corresponding WKB solutions.

2. The Mathematical Model

Wave expressions

Let an upward propagating plane wave be incident on the ionosphere, at an angle of \( \theta_0 \) with the vertical \( z \)-axis, where we define \( \cos \theta_0 = C \). In free space, the direction of propagation forms the direction cosines \( S_1, S_2, C \), where \( \sin^2 \theta_0 = S_1^2 + S_2^2 \). In general, the electric field component can be expressed as:

\[
E = E_o \exp(j \omega t - j \int kdr) = E_o \exp(j \omega t - jk(S_1x + S_2y + qz)) . \tag{1}
\]

Assuming that \( S_2=0 \), then \( S = \sin \theta_0 \) and neglecting the time harmonic factor "\( j \omega t \)" , the above equation is written:

\[
E = E_o \exp(-jk_o(\sin \theta_0 x + \int q(z)dz)) \tag{2}
\]

where \( k_o \) is the wave factor \( k_o = \omega/c \), with \( \omega = 2\pi f \) and \( c = 3\times10^8 \) the speed of light, while \( q \) corresponds to the vertical component of the refractive index which will be discussed later on.

The factor "\( jk_o \sin \theta_0 x \)" in the exponent is considered constant at all heights. Therefore the above expression can be written in a more simple form, containing only the integral factor which is a function of the height \( z \):

\[
E = E_o \exp(-jk_o \int q(z)dz) . \tag{3}
\]
The phase factor $\Phi = -k_0 \int q(z)dz$ in (3) can be considered an eikonal function and its gradient leads to the eikonal equation (Davies 1990, Budden 1985):

$$k_0q = -\text{grad} \Phi.$$ (4)

**The Refractive Index**

For an isotropic ionosphere the earth's magnetic field is neglected and the medium can be described by its dielectric permittivity which varies only with respect to the height (z-coordinate) and is different from that in free space (Rawer 1993, Budden 1985):

$$\varepsilon(z) = n^2(z) = 1 - X = 1 - \left(f_N / f\right)^2,$$ (5)

where

$$X = \omega^2 / \omega^2 = \text{Ne}^2 / (\varepsilon_o m \omega^2),$$

with $\omega^2 = e^2 N / (\varepsilon_o m)$

In the above equation $f_N$ is the plasma frequency, $f$ is the frequency of the wave, $e=1.602 \times 10^{-19}$ As is the elementary charge, $m=9.109 \times 10^{-31}$ Kg is the electron mass, $\varepsilon_o=8.854 \times 10^{-12}$ As/Vm is the permittivity of vacuum and $N$ is the electron density which is a function of height z. The magnetic permeability of the medium is taken as equal to that of free space ($\mu=\mu_o=1.25 \times 10^{-6}$ Vs/Am).

The vertical component of the refractive index is given by the expression (Budden 1985):

$$q^2 = (n^2 - S^2)^{1/2} = (C^2 - X)^{1/2}.$$ (7)

If collisions among electrons and neutral particles are taken into account, the term "X" has to be replaced by "X/Ut", leading to:

$$q^2 = (C^2 - X / U)^{1/2} = (\text{cos}^2(\theta_o) - X / U)^{1/2},$$ (8)

where:

$$U = 1 - j Z = 1 - j \frac{\psi}{\omega},$$ (9)

Z is a complex quantity inserted to indicate the effect of collisions and is related to the frequency of the wave ($\omega=2\pi f$) and $\psi$ is the collision frequency. The collision model used in this study is (Yeh & Liu, 1972):

$$\psi = 3.3 \times 10^{-16} \sqrt{[N(O_2)+N(N_2)+2 \cdot N(O)]},$$ (10)

where $N(O_2)$, $N(N_2)$ and $N(O)$ are the densities in $m^{-3}$ of $O_2$, $N_2$ and $O$ in the atmosphere. The temperature is in 'K and the collision frequency is given in sec$^{-1}$.

The introduction of the term "$U$" in the refractive index expression (8) leads to a complex wavenumber $k = k_o q(z) = k_o - j k_i = k_o(q, -j q)$ (where the imaginary parts $k_i$, $q_i$ are positive). The substitution of this complex wavenumber in (3) yields:

$$E = E_o \exp(-k_o \int q dz) \cdot \exp(-j k_i \int q_i dz)$$ (11)

which shows that the amplitude of the wave is decreasing exponentially with distance, according to the imaginary part $k_i$ of the wavenumber, representing the attenuation per unit distance and is defined in the direction of $k$ (Davies, 1990).

**Wave Equations**

The use of the Maxwell's equations for propagation in ionosphere leads to two different sets of equations:

$$\frac{\partial E_s}{\partial z} = j k H_x, \quad \frac{\partial H_x}{\partial z} = j k q^2(z) \cdot E_s,$$ (12)

$$\frac{\partial E_s}{\partial z} = -j k q^2(z) H_y,$$ (13)

$$\frac{\partial H_y}{\partial z} = -j k n^2(z) \cdot E_s.$$ (14)

The two sets of equations (12), (13) have been extensively discussed (Budden, 1985). The first one, which is used in this study, depends on $E_s$, $H_x$ and $H_y$ components and it corresponds to a horizontally polarized wave, having only the horizontal $E_s$ component of the electric field. The second set of equations depends on $E_s$, $E_y$ and $H_y$ and corresponds to a vertical polarized wave. Elimination of $H_y$ from the first set (12) gives:

$$\frac{d^2 E_s}{dz^2} + k_o^2 q^2(z) E_s = 0$$ (14)

Equation (14) is a wave equation and can be considered a special form of the general Helmholtz equation:

$$-\frac{d}{dz} \left( \frac{a \cdot dE}{dz} \right) + \beta E = f,$$ (15)

with $a=1$, $f=0$ and $\beta=k_o^2 q^2$. The above equation does not have closed form solutions in terms of known functions, although approximate techniques have been developed such as the WKB technique (Yeh & Liu, 1972). In this study we perform a full wave highly accurate computational treatment using the FEM.

**3. The Medium Profile**

For the study of electromagnetic wave propagation in the ionosphere with the use of equation (14), it is necessary to calculate the refractive index profile as a function of height $z$, using the equations (6), (8), (9) and (10). For the current study, two different electron density profiles, extracted from the International Reference Ionosphere, IRI-95 model, have been used. The input parameters for the profiles are shown in table 1, and the corresponding real and imaginary parts of the refractive index for the frequencies of 24 and 25MHz are shown, respectively, in figures 2 and 3. The lines in figure 3 are shown for the heights of
65-226 Km. Above this height, for the frequency of 24 MHz, the wave cannot pass the ionosphere and the imaginary part of the refractive index is rapidly decreasing, while for the 25 MHz is practically zero. The IRI is an international project sponsored by the Committee on Space Research (COSPAR) and the International Union of Radio Science (URSI) and is based on a wide variety of resources. It provides information about the electron density, electron temperature, ion temperature and ion composition for a wide range of heights and for input parameters like geographical coordinates, the day of year and time.

For the calculation of the collision frequency profile with the use of equation (10), the MSIS-E-90 atmosphere model of NSSDC has been used, with the parameters of table 1. This model describes the neutral temperature, the total density and the densities of He, O, N\(_2\), O\(_2\), Ar, H and N in Earth's atmosphere from ground to thermospheric heights.

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<tr>
<td>Step height (Km)</td>
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</table>

4. The WKB Solutions

When the refractive index changes very slowly with height, approximate solutions of (14) can be obtained:

\[
E_y = E_0 [q_0 / q(z)]^{1/2} \exp(mj \int_{z_{\min}}^{z} q(z)dz) (16)
\]

These two distinct WKB solutions correspond to waves propagating in the positive and negative z direction, respectively. The WKB method, the properties and the validity of the solutions are discussed by Budden (1985) and Yeh & Liu (1972). As the wave propagates in the direction of decreasing refractive index, its electric field increases because of the factor \(q^{-1/2}\) in (16), but its magnetic field decreases by an inverse factor \(q^{1/2}\) (Budden, 1985).

5. The Finite Element Model

For the solution of the wave equation (14) at heights of 65-400Km, the one-dimensional FEM has been used. The advantage of the method is that it can be used with fast varying, or even disturbed media including local inhomogeneities. Also, unlike other methods which demand analytical expressions for the medium profile, FEM can be used with real numerical data taken from any ionspheric model. The problem has been treated with the use of a global mesh of 120000 line elements, while meshes with 150000 and 250000 elements have been investigated with no significant improvement in the accuracy of the solution (figure 8). The refractive index is a function of the electron density profile and the atmospheric densities, which are all functions of height. Therefore a spline interpolation technique has been used for the IRI-95 and MSIS-E-90 data in order to calculate the refractive index at all levels, having each line element assigned to the corresponding refractive index value. The geometry of the model is shown in figure 1. In this figure each element "e" is assigned to its refractive index "\(q_e\)" and the solution is obtained for every node of the mesh.

The boundary conditions that have been applied to this boundary-value problem at the two end-nodes of the geometry at \(z=z_{\min}=65\)Km and \(z=z_{\max}=400\)Km are respectively:

\[
E \bigg|_{z=z_{\min}} = E_0 \exp(-jk_0 q_{z_{\min}} (z-z_{\min})) , (17)
\]

\[
\left[ a \frac{dE}{dz} + \gamma E \right]_{z=z_{\max}} = g , (18)
\]

where \(\alpha=1\), \(g=0\) and \(\gamma=jk_0 q(z)\). The first boundary condition (17) is a Dirichlet condition and is applied at the entrance node, where we assume that no reflections occur at this height. In order to eliminate any possible reflections at the entrance node, a generalized inhomogeneous Neumann boundary condition (Leontovich condition) can be used, but in our case, no significant difference in the results has been observed. Equation (18) is a generalized Neumann or, in other words, a first order Mur absorbing boundary condition, which is used to rescue reflections at the boundary node. In our case, this condition is exact, since the direction of propagation at the exit node is a priori known.

The solution of the problem described above, can be obtained by seeking the stationary point of an appropriate functional \(F(E)\), under the given Dirichlet boundary condition. The functional can, in general, be written (Jin, 1993) as:
6. Computational Results

The following figures of this section show the results of the method described above for the two profiles of section 3. Two different frequencies have been used: $f_1 = 24$ MHz, which is reflected at $h = 227.5$ or $231.5$ km, depending on the profile and $f_2 = 25$ MHz, which, being above MUF, propagates through the ionosphere. The incident angle that is used to derive the refractive index is assumed to be 45°.

Figure 4 shows, for the two profiles, the magnitude of the electric field and the WKB solutions for the corresponding values of $(z,q)$ for the frequency of 25 MHz. The effect of collisions here is neglected, so the refractive index has only real part and no additional attenuation occurs.

In figure 5 the same solutions as in figure 4 are shown, but now the effect of collisions is included, giving rise to an imaginary part of the refractive index as shown in figure 3. This imaginary part gives an additional attenuation to the wave at heights till $227.5$ or $231.5$ km approximately (depending on the profile), where it practically becomes zero. In this case, the difference between the FEM and WKB solutions is clear and it seems that the WKB method probably overestimates the attenuation effect at the heights where the absolute value of the imaginary part of the refractive index is large.

Figure 6 shows the absorption (dB) for the solutions described in figure 5. The absorption in terms of electric field amplitudes $E_r$ (actual received amplitude) and $E_r = 1$ (initial amplitude) is calculated by the equation $L = -10 \log \left( \frac{E_r}{E_0} \right)^2$.

In figure 7 the real part of the solution for the two profiles is shown for the frequency of 24 MHz, which cannot cross the F-layer. It can be seen here that the amplitude becomes extremely large when the wave is reaching the reflection height, taking its maximum value, as it is expected since the real part of the refractive index goes to zero.

In figure 8 the magnitude of the electric field, calculated by the FEM for three different discretizations with 120000, 150000 and 250000 line elements is compared. It can be seen that the discretization with 120000 elements is satisfactory for the specific problem since the solution accuracy is not increasing significantly with the use of larger number of elements.

In figure 9 the gradient of the phase of the wave is shown, together with the factor $k_1q$ for the frequency of 25 MHz and the second profile. This diagram shows that the phase of the wave, calculated by the FEM method, follows the refractive index profile, according to the eikonal equation (4).

The above figures show the behavior of the model for two different ionospheric profiles, for real and complex refractive index. It can be seen that the changes in amplitude and phase can be calculated very efficiently using the one-dimensional FEM with a discretization of 120000 line elements for the whole propagating height from 65 to 400 Km.

7. Conclusions

In this work a transitionospheric wave propagation model has been developed and numerical solutions for the amplitude and phase of the field have been presented by the use of the one dimensional finite elements method. The advantage of this method is that a variety of different ionospheric profiles and wave frequencies can be analyzed. Also, this method can be used with small wavenumbers and at heights where the refractive index is going to zero, where the WKB solutions usually fail. The results have been compared to the corresponding WKB solutions, showing a very good convergence when the effect of collisions is neglected and the refractive index has only real part. When the effect of collisions is taken into account, as it is the case of a realistic ionospheric propagation, and the refractive index is complex the FEM and WKB solutions give quite different results and the FEM solution seems to give a better physical description of the wave propagation.

References

International Reference Ionosphere (IRI-95) model,
http://nssdc.gsfc.nasa.gov/space/model/models/iri.html
NSSDC/MSIS-90 Atmosphere model,
http://nssdc.gsfc.nasa.gov/space/model/models/msis.html

\[ \frac{d^2 E_y}{dz^2} + k^2 q^2 E_y = 0 \]

**Figure 1**
(Illustration of the model’s geometry)

**Figure 2**
(The real part of the refractive index)

**Figure 3**
(The imaginary part of the refractive index)

**Figure 4**
(FEM and WKB solutions for real refractive index)
Figure 5
(FEM and WKB solutions for complex refractive index)

Figure 6
(FEM and WKB absorption in terms of the electric field)

Figure 7
(FEM Solution (E))

Figure 8
(The FEM solution at the cut-off height for $f=24$MHz)

Figure 9
(The gradient of the FEM solution and the factor $k_{oa}$)