Determination of Failure Frequency Indices from State Space Decomposition

JOYDEEP MITRA
New Mexico State University
Department of Electrical & Computer Engineering
Las Cruces, New Mexico 88003
USA
jmitra@nmsu.edu

CHANAN SINGH
Texas A&M University
Department of Electrical & Computer Engineering
College Station, Texas 77843
USA
singh@ece.tamu.edu

Abstract: This paper describes a technique for the determination of frequency and duration indices, in the steady state, of discrete capacity systems, using the method of state space decomposition. For the first time direct, closed form expressions are developed and presented, and an efficient implementation is outlined. The approach is illustrated by means of a numerical example and validated using an enumeration-based method.

Key-Words: reliability indices, failure indices, frequency indices, duration indices, state space decomposition.

1 Introduction

There has been considerable research on the problem of reliability analysis of discrete capacity systems. The existing body of knowledge includes analytical methods, Monte Carlo simulation methods, and hybrid methods. In all of these methods, the underlying approach consists of first modeling the system behavior as a stochastic process, and then quantifying the system reliability in terms of the probability or frequency of encountering the failure states, or the period of time the system spends in these states. We thus have probability, frequency and duration indices to describe the reliability of a given system. These three indices are related in that the failure probability equals the product of the failure frequency and the failure duration. Of the three indices, probability indices are the most easily determined, but frequency and duration (F&D) indices convey a clearer and more complete understanding of the system behavior in terms of its reliability. For example, a system may on the average fail once a year and take an average of 88 hours to repair, while another system may fail once in 10 years and take 877 hours to repair; both systems, however, have failure probabilities of 0.01.

There are several analytical techniques for determining steady state reliability indices of systems. All of these techniques are based on identification (by enumeration or classification) of the failure states of the system. While in small systems it is often possible to exhaustively enumerate the failure states, large systems usually have such large numbers of states as to make enumeration impossible. In such cases it becomes necessary to deal with sets of states rather than individual states, or to abandon the analytical approach altogether and use Monte Carlo simulation. Some hybrid methods (e.g. [5]) use combinations of analytical and simulation techniques.

The method of state space decomposition is an analytical technique which classifies the set of all system states into disjoint sets of failure states and acceptable states. It is a recursive method, and is explained in detail in [1, 2, 3, 4]. The method is generally applicable to most systems that can be modeled as flow networks. This includes pipeline networks, transportation networks, communication networks, computer networks, mechanical systems, and electrical networks such as power systems and electronic circuits. Most of these networks are discrete capacity systems; the rest are often modeled as discrete capacity systems to facilitate analysis.

In the past, the method of decomposition has been applied to discrete capacity systems, mostly for determination of probability indices. There are relatively few instances in the literature where F&D indices have been determined from methods dealing with sets of states, because it has been a difficult problem, and prior attempts [5, 6, 7] have used recursive techniques that tend to be cumbersome. In this paper we develop, for the first time, direct closed form expressions that simplify the determination of F&D indices to the point where it requires little additional computing effort beyond the determination of the probability indices.

In the following sections we develop the theory, illustrate the technique by applying it to a sample system, and validate the results using the method of enumeration.

Notations:

\[ A \text{-state} \]: functional (acceptable) state
\[ L \text{-state} \]: loss of functionality (failed) state
\[ A \text{-sets} \]: disjoint sets of A-states

---

\(^1\)Reference [1] is written in English.
2 Theory

We will proceed to develop the theory in three stages. First we will revisit the relevant features of the method of state space decomposition without actually describing the mechanics. Next, we will show how to determine the frequencies of encountering sets of states, and extend this to unions of sets. Finally, we will describe a means of setting up the data in such a manner as to simplify the computation of the indices.

2.1 State Space Representation

A state of a given system is described by the states of all the individual components that constitute the system. If each component can assume several capacity levels, then a list of the prevailing capacity levels of the components would describe the prevailing state of the system. For a system of $N_C$ components, if one were to define an $N_C$-dimensional space, each axis representing the different capacities a corresponding component can assume, then each point in this space would uniquely describe a state of the system. The space would constitute the complete set of states the system can assume, and would be known as the state space of the system. Such an $N_C$-dimensional Cartesian representation has been found to be convenient, and is in widespread use.

In general, this representation is valid for both continuous and discrete capacity systems. Engineering systems are modeled as discrete capacity systems, and the methodology presented in this paper will analyze discrete capacity systems.

For most systems of realistic size, due to the multitude of states (e.g., the state space of a system of $m$ components with $n$ states each consists of $n^m$ states), it is impractical to store information (capacity, probability, frequency) about individual states separately; in such cases, states are dealt with in groups or sets. Sets of states are represented as hypercuboids in the $N_C$-dimensional space, and are denoted, for simplicity, by boundary vectors. (This will become evident in the sections that follow.) However, sets of states that have certain common characteristics (such as failure states) do not necessarily conform to this shape, and must be represented as unions of several adjacent hypercuboids.

2.2 State Space Decomposition

The method of decomposition consists of classifying all points in the state space into disjoint sets of acceptable ($A$), unclassified ($U$) and failure ($L$) states. Each of these sets can be defined by a maximum state and a minimum state such that all states between and including these two constitute the given set. Each $U$-set is further decomposed into an $A$-set, $U$-sets and $L$-sets. This is continued till only $A$-sets and $L$-sets remain. The reliability indices are then determined from the $L$-sets. The classification is achieved by identifying boundary vectors that partition the set being decomposed. The method of identification of boundary vectors is determined by the network model, and is specific to the kind of process or system being analyzed. References [1, 2, 3, 4] provide descriptions of the decomposition method. Now except in very rare cases, the bounding surface of the total set of failure states is very segmented. For instance, the $L$-set in a three dimensional state space may appear as in Fig. 1.

Here, the total $L$-set is given by

$$L = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5$$

(1)

The subsets $L_1, \ldots, L_5$ are obtained in the form of cuboids (hypercuboids in a multidimensional space) because the boundary vectors identify corner points of the sets. This is why the method turns out to be

![Figure 1: Failure sets for a hypothetical system with three components.](image-url)
recursive. Another characteristic of these sets is that every bounding surface shown in Fig. 1 is closed on one side and open on the other, so that all the sets shown are really disjoint.

The reliability indices have to be determined from the union of these sets. This is explained in the next section. It is now appropriate to state the conditions under which the state space can be subjected to decomposition. The state space is required to be coherent, i.e., the failure or degradation of a component cannot improve system performance, and, likewise, the improvement or restoration of a component cannot deteriorate system performance. If this condition is satisfied the following analysis may be applied.

2.3 Determination of Indices

For the purpose of the forthcoming analysis, we will use the following assumptions:

(a) The system is frequency-balanced [8], i.e., the frequency between any pair of states is equal in both directions. This automatically holds true if every component can be represented by a two-state Markov model. For multi-state components we assume that forced frequency balance is imposed as described in [8]. In the next section we will describe a method that automatically ensures frequency balance.

(b) The system can transit in one step from one state to another only by failure or restoration of a single component.

A transition to a higher capacity state is said to be an upward transition; similarly, a transition to a lower capacity state is said to be a downward transition. Assume that decomposition has resulted in \( N_L \) failure sets; then the aggregate failure set is given by

\[
L = \bigcup_{i=1}^{N_L} L_i
\]  

(2)

Obviously since the sets are all disjoint, the failure probability is

\[
P(L) = \sum_{i=1}^{N_L} P(L_i)
\]  

(3)

The following analysis is a generalization of the results developed in [8]. Consider the sums described by equations (4) and (5) given below.

\[
F^+ \{L_i\} = \sum_{x \in L_i} \left[ P(x) \sum_{j=1}^{N_C} \lambda^+_{xj} \right]
\]  

(4)

\[
F^- \{L_i\} = \sum_{x \in L_i} \left[ P(x) \sum_{j=1}^{N_C} \lambda^-_{xj} \right]
\]  

(5)

where \( \lambda^+_{xj} \): transition rate of component \( j \) from its state in system state \( x \) to higher capacity states;

\( \lambda^-_{xj} \): transition rate of component \( j \) from its state in system state \( x \) to lower capacity states.

\( F^+ \{L_i\} \) is the sum of frequencies of all upward transitions from all the states in the subset \( L_i \). Similarly \( F^- \{L_i\} \) is the sum of frequencies of all downward transitions from all the states in the subset \( L_i \). We will use this notation for the remainder of the analysis.

Now consider a state \( x \) in set \( L_i \). The sum of frequencies of upward transitions from this state is

\[
F^+ \{L_i(x)\} = P(x) \sum_{j=1}^{N_C} \lambda^+_{xj}
\]  

(6)

and similarly, the downward frequency is

\[
F^- \{L_i(x)\} = P(x) \sum_{j=1}^{N_C} \lambda^-_{xj}
\]  

(7)

Since the system is considered coherent, the \( F^- \{L_i(x)\} \) transitions cannot cross the boundary between \( L \)-sets and \( A \)-sets. The upward transitions can be split into two components,

\[
F^+ \{L_i(x)\} = F^+_0 \{L_i(x)\} + F^+_1 \{L_i(x)\}
\]  

(8)

such that the \( F^+_0 \{L_i(x)\} \) component is the one that crosses the boundary between the \( L \)-sets and \( A \)-sets and \( F^+_1 \{L_i(x)\} \) remains within the \( L \)-set. Since the components are assumed independent and are frequency balanced for any two states \( x_i \) and \( x_j \) [8],

\[
F \{x_i \rightarrow x_j\} = F \{x_j \rightarrow x_i\}
\]  

(9)

Now if we consider

\[
F(L) = \sum_{i=1}^{N_L} \left[ F^+ \{L_i\} - F^- \{L_i\} \right]
\]  

(10)

\( F(L) \) can be expressed as follows:

\[
F(L) = \sum_{i=1}^{N_L} \sum_{x \in L_i} \left[ F^+_0 \{L_i(x)\} + F^+_1 \{L_i(x)\} - F^- \{L_i(x)\} \right]
\]  

(11)

Because of (9), we have

\[
\sum_{i=1}^{N_L} \sum_{x \in L_i} F^+_0 \{L_i(x)\} = \sum_{i=1}^{N_L} \sum_{x \in L_i} F^- \{L_i(x)\}
\]  

(12)

This can be understood as follows. Starting from a failed state, an upward transition that does not cross
the boundary is actually an upward transition from a failed state to another failed state. For every upward transition from a failed state $x_i$ to another failed state $x_j$, there is a downward transition from $x_j$ to $x_i$, of equal frequency. Hence the two sums in (12) are equal. From (11) and (12), we have

$$F_i = \sum_{j=1}^{N_{C}} \sum_{x \in L_i} \left[ F_0^+ \{L_i(x)\} \right]$$  \hspace{1cm} (13)

Equation (13) represents the expected transitions across the boundary of $L$-sets and $A$-sets. It is known that when the state space is divided into two disjoint sets, the steady state frequency is equal in both directions [9]. Therefore the frequency from $L$-sets to $A$-sets is the same as from $A$-sets to $L$-sets. Thus equation (13) represents the failure frequency. Equation (10) is equivalent to equation (13) and is suitable for calculating the frequency of failure of the system, as shown in the next section.

Having obtained the probability and frequency of system failure, the determination of $T \{L\}$, the expected duration of system failure, is trivial:

$$T \{L\} = \frac{P \{L\}}{F \{L\}}$$  \hspace{1cm} (14)

In the rest of this paper, therefore, we will only deal with failure probability and frequency.

### 2.4 Implementation Strategy

The equations (3) and (10) are used to determine the probability and frequency of system failure. We will now show how the data can be constructed so that calculation of $P \{L\}$ and $F \{L\}$ can be achieved with very little effort. This can be done as follows.

Consider a network with $N_{C}$ components where each component has several capacity states, of which every state has a known probability and frequency of occurrence. Construct cumulative probability and frequency distributions for each component, so that $p_{kj}$ and $F_{kj}$ denote, respectively, the cumulative probability and frequency of state $k$ of component $j$. The states are so indexed that a higher index implies a higher capacity. For this system, a state space is constructed, and decomposed into disjoint $A$-sets and $L$-sets, using the method described in [1, 3, 4].

Now consider a failure set $L_i$. Recall that $L_i$ is described by a maximum state and a minimum state. Assume that states $n$ and $m$ of component $j$ define the maximum and minimum states of set $L_i$. For state $k$ of component $j$, where $m \leq k \leq n$, the cumulative probability and frequency can be expressed as follows [6, 8],

$$p_{kj} = p_{(k-1)j} + p_{kj}$$  \hspace{1cm} (15)

and

$$F_{kj} = F_{(k-1)j} + p_{kj} \left( \lambda_{kj}^+ - \lambda_{kj}^- \right)$$  \hspace{1cm} (16)

where $p_{kj}$: probability of state $k$ of component $j$; $\lambda_{kj}^+$, $\lambda_{kj}^-$: transition rates from state $k$ of component $j$ to higher and lower capacity states respectively.

From (15) and (16), we get

$$\lambda_{kj}^+ - \lambda_{kj}^- = \frac{F_{kj} - F_{(k-1)j}}{p_{kj} - p_{(k-1)j}}$$  \hspace{1cm} (17)

If we now represent the states from $m$ to $n$ of component $j$ by an equivalent state $s$, then the equivalent values for this can be expressed in the same manner as (17):

$$\lambda_{sj}^+ - \lambda_{sj}^- = \frac{F_{nj} - F_{(m-1)j}}{p_{nj} - p_{(m-1)j}}$$  \hspace{1cm} (18)

Similarly, the probability of this equivalent state can be written as

$$p_{sj} = p_{nj} - p_{(m-1)j}$$  \hspace{1cm} (19)

In a similar fashion, for every component, the states between the maximum and minimum of set $L_i$ can be represented by an equivalent state. Then the probability of $L_i$ is:

$$P \{L_i\} = \prod_{j=1}^{N_{C}} p_{sj}$$  \hspace{1cm} (20)

and the frequency is:

$$F^+ \{L_i\} - F^- \{L_i\} = P \{L_i\} \sum_{j=1}^{N_{C}} \left( \lambda_{sj}^+ - \lambda_{sj}^- \right)$$  \hspace{1cm} (21)

Hence the probability of system failure is determined by substituting $P \{L_i\}$ from (20) into (3), and the frequency is determined by substituting $F^+ \{L_i\} - F^- \{L_i\}$ from (21) into (10), as shown below.

$$F \{L\} = \sum_{i=1}^{N_{C}} P \{L_i\} \sum_{j=1}^{N_{C}} \left( \lambda_{sj}^+ - \lambda_{sj}^- \right)$$  \hspace{1cm} (22)

Notice that the determination of the failure frequency using the expression given in (21) adds little to the computational effort expended in decomposing the state space and calculating the failure probability using (20).

This approach will now be illustrated by means of an example.
3 Example

Consider a two-node system, as shown in Fig. 2. This represents a hypothetical two-area power system with seven identical generators, each of capacity 100 MW, repair rate $\mu = 0.02$ per hour and failure rate $\lambda = 0.0022$ per hour (or 19.47 per year). This means every generator has availability of 0.9. Three of the generators are in Area 1 and four are in Area 2. The two areas are connected by a tie-line of capacity 100 MW and are assumed to have constant area loads of 200 MW and 300 MW respectively.

The tables of generation data shown in the figure describe the cumulative probability and frequency distribution functions of the generation capacities in the two areas. The probability distribution functions are easy to understand and construct. The frequency distribution functions can be understood as follows. $F\{G_i \leq G_j\}$ represents the probability of encountering states of capacity $G_i$ or less, starting from states of capacity larger than $G_j$. The frequency distribution can be determined using (16) or as described in [6]. The frequencies shown in Fig. 2 are expressed in per hour.

Since in this system there are only two variables ($G_1$ and $G_2$), the state space can be represented (see section 2.1) in two dimensions, as shown in Fig. 3. Fig. 3 also shows the $L$-sets, the boundary between the failure and available states, and the transitions that cross the boundary from available to success states. The actual process of decomposition is not described here, but can be easily understood from [3, 4].

The sets obtained during the process of decomposition are shown in Fig. 4. As described in section 2.2, decomposition is a recursive process, and for this simple system the process completes in just two recursions, shown as Level 1 and Level 2 in Fig. 4. Since this system was quite simple, the decomposition was performed manually. In Fig. 4, the sets are denoted as follows: the bottom row provides the coordinates of the lower limit of the set (bottom left corner in Fig. 3), and the top row denotes the upper limit of the set (top right corner).

Table 1 shows the probabilities and frequencies of the four $L$-sets, and the total probability and frequency of system failure. Transition rates and frequencies are expressed in per hour. In this table, $P\{L_i\}$ is obtained using (20). $\Delta \lambda\{L_i\}$ is determined by

$$\Delta \lambda\{L_i\} = \sum_{j=1}^{N_c} (\lambda_{ij}^+ - \lambda_{ij}^-)$$

where the term in the parentheses is obtained using (18). $F\{L_i\}$ corresponds to the left hand side of (21).

![Figure 2: Two-node system.](image)

<table>
<thead>
<tr>
<th>Level 0</th>
<th>Level 1</th>
<th>Level 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1 = \begin{bmatrix} 300 &amp; 400 \ 200 &amp; 300 \end{bmatrix}$</td>
<td>$L_1 = \begin{bmatrix} 0 &amp; 400 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>$A_2 = \begin{bmatrix} 100 &amp; 400 \ 100 &amp; 400 \end{bmatrix}$</td>
</tr>
<tr>
<td>$U_0 = \begin{bmatrix} 300 &amp; 400 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>$L_2 = \begin{bmatrix} 300 &amp; 100 \ 100 &amp; 0 \end{bmatrix}$</td>
<td>$A_3 = \begin{bmatrix} 100 &amp; 400 \ 100 &amp; 200 \end{bmatrix}$</td>
</tr>
<tr>
<td>$U_1 = \begin{bmatrix} 100 &amp; 400 \ 100 &amp; 200 \end{bmatrix}$</td>
<td>$L_3 = \begin{bmatrix} 100 &amp; 300 \ 100 &amp; 200 \end{bmatrix}$</td>
<td>$A_4 = \begin{bmatrix} 300 &amp; 200 \ 300 &amp; 200 \end{bmatrix}$</td>
</tr>
<tr>
<td>$U_2 = \begin{bmatrix} 300 &amp; 200 \ 200 &amp; 200 \end{bmatrix}$</td>
<td>$L_4 = \begin{bmatrix} 200 &amp; 200 \ 200 &amp; 200 \end{bmatrix}$</td>
<td>$L_4 = \begin{bmatrix} 200 &amp; 200 \ 200 &amp; 200 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

![Figure 4: State space decomposition.](image)
The total probability and frequency of system failure appear on the last row of Table 1.

These results can easily be validated by enumeration. Since the number of $A$-states is less than that of $L$-states, it is easier to calculate the indices using $A$-states. The six $A$-states are marked $a$ through $f$ in Fig. 3.

\[ P(L) = 1 - (p_a + p_b + p_c + p_d + p_e + p_f) \]
\[ F(L) = (p_a + p_b + p_c)5\lambda \]
\[ = 0.124003 \times 0.011111 \]
\[ = 1.3778 \times 10^{-3} \text{ per hour} \]

The $ps$ denote exact state probabilities. The $5\lambda$ term is because each of states $a$, $b$, and $c$ corresponds to a total of 5 generators available, so the total downward transition rate from each of these states is $5\lambda$.

It is important to note that in most practical systems the state space is much larger, and the decomposition needs to be performed using a computer program. References [3, 4] demonstrates the use of decomposition on larger systems for probability computation; while [5] also performs frequency computations, the method it uses is much more involved.

4 Conclusion

This paper has described a method for calculating the frequency of failure of a discrete capacity system, using the method of state space decomposition. The formulation of the general method is then followed by the description of a special technique for easy implementation.

The strength of the method lies in the following features:

(a) the method of decomposition deals with sets of states rather than individual states, and is therefore more efficient than methods involving enumeration;

(b) the expressions used for determination of F&D indices are closed-form, unlike some previously used recursive forms;

(c) the calculation of F&D indices requires only slight additional computational effort beyond the effort required for calculation of probability indices.

While the method of decomposition and calculation of probability indices therefrom is well known, closed form expressions for calculation of F&D indices from decomposition have been presented for the first time in this paper.

Acknowledgements: This research was supported by the National Science Foundation under Grants ECS-0410092 and ECS-9412712.

References:


