Comparison between discrete and continuum modeling of granular spreading

ANNE MANGENEY
University California San Diego
Institut For Non Linear Science
9500 Gilman Drive
La Jolla, CA 92093-0402
USA

LYDIE STARON
Center for Mathematical Sciences
DAMTP
Wilberforce Road
Cambridge CB3 0WA
UK

DMITRI VOLFSON, LEV TSIMRING
University California San Diego
Institut For Non Linear Science
9500 Gilman Drive
La Jolla, CA 92093-0402
USA

Abstract: Numerical simulation of the collapse of granular material over an horizontal plane using both continuum and discrete element approach shows that continuum models based on the Long Wave Approximation (LWA) overestimate the driving forces involved during the collapse. This effect increases when the aspect ratio of the granular mass increases. Comparison between discrete and continuum simulations makes it possible to show that the assumption of hydrostatic normal stress is essentially responsible of the limits of the LWA when the aspect ratio increases. The vertical acceleration, neglected in the vertical momentum equation, is shown to play a significant role when the aspect ratio increases. As a result, numerical simulations of real geophysical flows using thin layer models will be significantly improved if a new asymptotic analysis is performed including the effects of the vertical acceleration.

Key–Words: Granular Flows, Shallow Water equations, Continuum and Discrete Numerical Modeling

1 Introduction

Dense granular flows, triggered by large mass destabilization, are active processes that participate in the evolution of the surface of the Earth and other telluric planets. They also represent natural hazards that are today a threat to many populations and industrial infrastructures.

Granular flows have been extensively studied in physics, mechanics and geophysics since the two last decades. Most of the experimental and numerical studies devoted to granular flows have focused on the flow of a granular material along inclines of slope larger than the material repose angle [1, 2]. In this configuration, the flow is driven by the component of gravity along the slope direction and the flow thickness is usually small enough to permit use of hydrostatic models based on the depth-averaged Long Wave Approximation (hereafter called LWA), i. e. the thin layer approximation [3, 4]. Several assumptions are made in these models: hydrostatic normal stress, constant profile of the downslope velocity, the velocity in the direction perpendicular to the topography is neglected compared to the downslope velocity and the whole granular mass from the bottom to the free surface is supposed to flow. Such simple continuum hydrodynamic models initially proposed by [3] have been shown to reproduce basic features of both experimental dense granular flows along inclined planes [5, 6] and geological flows along real topographies [7, 8, 9].

However, in cliff collapse and more generally in the destabiliszation phase of most landslides, the initial aspect ratio of the destabilized mass is not small, i. e. the thickness H of the mass in the direction perpendicular to the underlying topography could be comparable to its extension R in the direction parallell to the plane. Although simple geometrical scaling arguments cannot be used for such mass collapse, it is hard to know a priori if the Long Wave Approximation can or cannot be applied to describe the spreading. Recent numerical simulations and analytical developments have shown that the LWA together with the Coulomb friction law with a constant friction coefficient μ are in good agreement with laboratory experiments of granular collapse over horizontal plane for aspect ratio $a = H_0/R_0 < 1$, where H_0 is the initial thickness of the granular column and R_0 its initial radius (Figure 1) [10, 11]. In particular the scaling observed experimentally, showing that the normalized quantities such as the maximum thickness of the deposit H_f/R_0 and the final runout distance R_f/R_0 are only dependent on the aspect ratio a of the initial mass, is intrinsically contained in the thin layer equations. In fact, the initial geometry is not preserved during the spreading, leading to a lower aspect ratio geometry. Moreover, the flow, at a distance $r > R_0$ from the center of the mass, bears a strong resemblance to thin layer flow and at a distance $r < R_0$ the flowing region is expected to be located near the surface. The question remains as to what are the typical aspects ratios for which the LWA is no more valid and what are the approximations which are mainly responsible of the limits of the LWA when the aspect ratio of the initial mass a increases.

How is it possible to verify the approximations of the thin layer model for granular flows? For viscous flows, the thin layer model can be validated by comparing with the results obtained from a numerical model solving the full set of Navier-Stockes equations. Such a study has been performed by [12] to establish the limits of the shallow ice approximation for an ice sheet flow near an ice divide. The derivation of 3D equations describing granular flow behaviour is still however an open and challenging problem although several attempts have been made recently in this direction. Discrete elements simulations therefore provide a good paradigm to validate continuum approach and in particular thin layer granular flow models.

The limits of the thin layer model is studied here by comparing numerical results with Discrete Elements modeling (hereafter called DE) for 2D and 3D collapse of granular columns over an horizontal plane. A series of numerical experiments have been carried out using both the LWA and DE models by varying the aspect ratio of the initial mass. The motivation has been to reach the limits of the LWA by increasing the aspect ratio of the initial mass. We propose here to investigate which assumption (velocity profile, depth-averaging, static/flowing interface, hydrostatic pressure, flow law, etc.) is essentially responsible for the limits of the LWA approach as the aspect ratio increases.

2 Models

2.1 Thin layer model (LWA)

A long wave approximation is classical for fluid dynamics problems when it is important to separate large-scale motion from motion on smaller time and length scales. It is based on an asymptotic expansion in powers of one or more small parameters, one being typically a length scale below which rapid fluctuations of the velocity field can be smoothed out. The Savage-Hutter empirical model for granular flows was derived using such an approximation. Such a model describes long-time effects of slowly varying bottom topography, and of weak hydrostatic imbalance on the

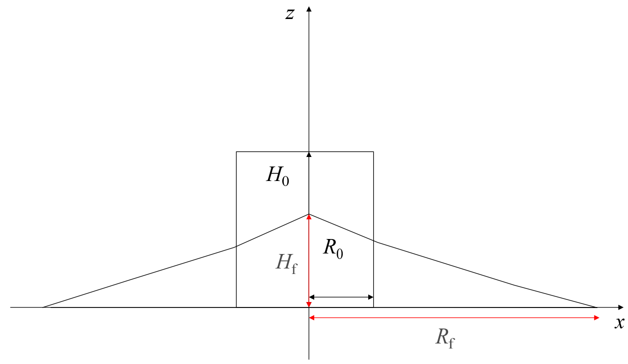

Figure 1: Shematic granular column with initial thickness H_0 and initial radius R_0 released from rest and shematic deposit obtained after the spreading with a final runout radius R_f and a maximum thickness H_f.

vertically-averaged horizontal velocity of an incompressible fluid, with a free surface, that moves under the force of gravity with friction dissipation [1, 4, 6].

Assuming the vertical velocity to be smaller than the characteristic tangential velocity, together with a length scale for the vertical fluctuations of the velocity smaller than that for the horizontal fluctuations of the same order, we shall consider here a minimal model derived from a purely inviscid incompressible fluid together with phenomenological friction dissipation along planes parallel to the bottom topography. The reduced governing equations are then obtained by vertically averaging the equations and by using a leading order approximation neglecting the Lagrangian vertical acceleration. For a flat bottom, the resulting equations are

$$\frac{\partial h}{\partial t} + \mathrm{div}\,(h\mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(hu^2\right) + \frac{\partial}{\partial y}(huv) = -\frac{\partial}{\partial x}\left(g\frac{h^2}{2}\right) + \frac{1}{\rho}T_x \tag{2}$$

$$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}\left(hv^2\right) = -\frac{\partial}{\partial y}\left(g\frac{h^2}{2}\right) + \frac{1}{\rho}T_y \tag{3}$$

where $\mathbf{u} = (u, v)$ denotes the depth-averaged horizontal flow velocity in the horizontal-vertical Cartesian reference frame (x, y, z), h the free upper surface, ρ the mass density and g acceleration due to gravity. These equations model the hydrostatic imbalance in presence of an averaged friction force $\mathbf{T}_t = (T_x, T_y)$, parallel to the horizontal plane, and which is an effective approximation of the friction effects arising both at the bottom and within the bulk due to differential motion between flowing layers parallel to the bottom surface.

The friction force has a direction opposite to the averaged tangential velocity field and when flowing, the amplitude of the friction force is governed by a friction coefficient and the total overall pressure, i.e. $\mu = \|\mathbf{T}_t\|/\rho g h$, where $\mu = \tan\delta$ with δ the friction angle. The transition between static and fluid behavior is simply modeled here using a Coulomb type transition [4], i.e.

$$\begin{aligned} \|\mathbf{T}_t\| \geq \sigma_c \quad &\Rightarrow \quad \mathbf{T}_t = -\mu\rho g h \frac{\mathbf{u}}{\|\mathbf{u}\|}, \\ \|\mathbf{T}_t\| < \sigma_c \quad &\Rightarrow \quad \mathbf{u} = 0, \end{aligned} \tag{4}$$

where $\sigma_c = \mu\rho g h$.

The numerical method used to solve the hyperbolic system (1)-(3) and (4) relies on a finite volume formulation together with the hydrostatic reconstruction scheme developed in [13] for Saint Venant models, and on the apparent topography approach of [14] to deal with friction. The numerical method is described in [5, 14]. This numerical model has recently been successfully applied to the simulation of the collapse of a granular column over a horizontal plane in [10]. The finite volume scheme provides second-order accuracy, in contrast with the first-order method used in [4] and based on a kinetic scheme.

2.2 Discrete element model (DE)

Contrary to the continuum approach, discrete simulations take into account the individual existence of each discrete grain forming the medium. The collection of grains is entirely driven by the standard equations of motion, and the contact laws describing the collisions between the grains. The 2D simulations of the spreading of a rectangular column on an horizontal plane were performed using the contact dynamics algorothim [15, 16] and the 3D simulations of an axisymetric granular column were performed using the soft particle molecular dynamics algorithm [17]. These algorithms assume perfectly rigid grains interacting at contact by mean of a simple Coulomb friction law involving the coefficient of inter-grain friction μ_m. Moreover a Newton coefficient of restitution e controls energy exchanges during binary collisions. Beyond the fact that contact dynamics treats them as strictly non-smooth, the contact laws assumed in both contact dynamics and molecular dynamics methods are essentially similar. The coefficient of restitution was chosen $e = 0.5$. The influence of this coefficient on the numerical results has been studied in [16]. Discrete elements simulations of granular collapse have been able to reproduce scaling laws observed in the experiments [15].

The comparison between continuum and discrete approaches gives similar results in 2D and 3D configurations and essentially 2D results will be shown here for sake of simplicity.

3 A friction law with constant coefficient?

By comparing numerical and experimental results, [10] shows that the normalized runout distance R_f/R_0 calculated using LWA is overestimated for aspect ratio $a > 1$. Question however remains as to how the assumption of constant friction angle is responsible for the disagreement obtained between numerical and experimental results. Discrete simulations shows that the macroscopic effective friction μ_{eff} is constant during the flow for a given value of the microscopic friction coefficient μ_m at the particle scale [15]. The effective friction is shown to be independent on the aspect ratio a. The effective friction coefficient μ_{eff} is defined as the mean friction coefficient applied to the whole granular column from the destabilization to the stopping of the mass. It has been calculated by assuming that the total energy is dissipated by the work of friction forces over the total distance run by the center of mass of the spreading material.

We have investigated the relation between macroscopic (effective) and microscopic (inter-grain) friction in DE and LWA simulations. The effective friction coefficient μ_{eff} depends on the microscopic friction coefficient μ_m describing the friction between two particles. In order to compare this effective friction μ_{eff} with the effective friction coefficient μ used in the LWA model, μ has been fitted for each numerical experiment to recover the runout distance obtained using DE simulations for a given microscopic friction μ_m. Figure 2 shows that the function relating macroscopic to microscopic friction is qualitatively similar in DE and LWA simulations. The saturation observed in DE simulations for increasing microscopic friction is also observed in LWA simulations indicating that for large values of μ, the effective friction seems no longer to depend on the details of the inter-grain friction. The LWA effective friction is however higher than its DE analogue. As a result, the driving forces in LWA models are overestimated. We will investigate which assumption is responsible for this overestimation.

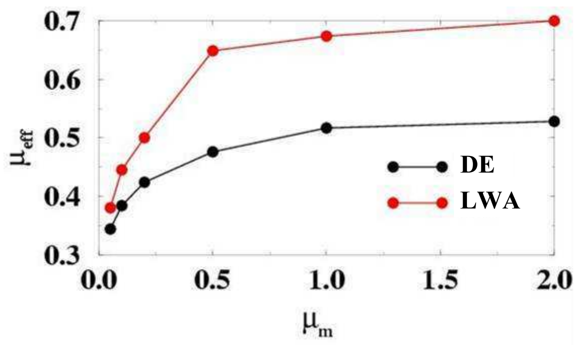

Figure 2: Macroscopic friction coefficient μ_{eff} calculated using the contact dynamics DE model (black circles) and the LWA model (red circles) as a function of the microscopic friction coefficient μ_m during the 2D spreading of a granular column.

4 Thickness profiles

4.1 Calibration of the parameters

In order to compare DE and LWA simulations when varying the aspect ratio of the granular column, it is necessary to fix the friction coefficient in both models whatever the value of a is. For a fixed value of the inter-grain friction coefficient $\mu_m = 0.5$, the friction coefficient μ in the LWA model has been calibrated by reproducing the runout distance calculated by the DE model for the aspect ratio $a = 0.9$. The resulting friction coefficient $\mu = 0.65$ is then used in all the following simulations using the LWA model and $\mu_m = 0.5$ is used in the DE simulations.

4.2 Granular mass changes with time

For small aspect ratio ($a < 0.7$), the dynamics and the deposit calculated using the LWA model are in good agreement with the results of DE simulations for 2D and 3D granular collapse. Similar conclusion was obtained in [10] where LWA was used to simulate experimental results obtained by [18]. However, when increasing the aspect ratio, the dynamics and the shape of the final deposit calculated using the continuum approach differs from those calculated using DE models (figure 3). The mismatch between DE and LWA simulations is essentially observed during the first and last phase of the flow. In fact, during the first phase ($0 < t < t_f$, where t_f is the time where the front stops), the granular mass calculated using the LWA model spreads more rapidly than that calculated using the DE model. During the intermediate phase (350 ms$< t <$600 ms in figure 3), the time changes of the mass thickness calculated with both models are in good agreement. However, once the front stops, the mass calculated using the LWA model stops earlier than the mass calculated using the DE model as was observed in [10]. The stopping phase has been shown to involve shallow surface flows that are not taken into account in the LWA model.

4.3 Deposit morphology

For $a = 0.9$, the deposits obtained with both simulations have similar extension R_f/R_0 but the final maximum thickness of the granular mass H_f/H_0 calculated is smaller when using DE simulations. Interestingly, the runout distance is still well recovered by the LWA approximation when varying the aspect ratio until $a \simeq 1$ without fitting the friction angle (figure 4) although the shape of the deposit is not accurately calculated using LWA simulations. This is an interesting result for geophysical applications because for risk assessment the important parameter is the runout distance more than the shape of the deposit. The question remains however as to what is the hypothesis made in LWA models that fails when increasing the aspect ratio of the granular column.

5 Static/flowing Interface

Experiments and discrete simulations have shown that a layer of flowing grains is moving over a layer of static grains at the base of the flow during the spreading of the mass [18], [15]. This static/flowing transition is not taken into account in the LWA model where all the column is supposed to flow. The question here is to evaluate if the static/flowing transition plays an increasing role when the aspect ratio increases. Indeed, in DE simulations, it is not clear whether the static/flowing effect is more significant when increasing the aspect ratio of the initial released mass [16]. Furthermore LWA model has proven to be quite good capable of reproducing the dynamics and deposit of the spreading of granular columns for small aspect ratio ($a < 0.5$) despite the strong vertical heterogeneity related to the static/flowing transition occurring during the flow.

6 Velocities

6.1 Velocity profi le

When deriving the LWA equations, a given vertical profile of the horizontal velocity $u(z)$ is generally assumed (linear, constant, exponential profile). A constant velocity profile is mostly imposed [1]. The influence of this assumption is still an open question. How-

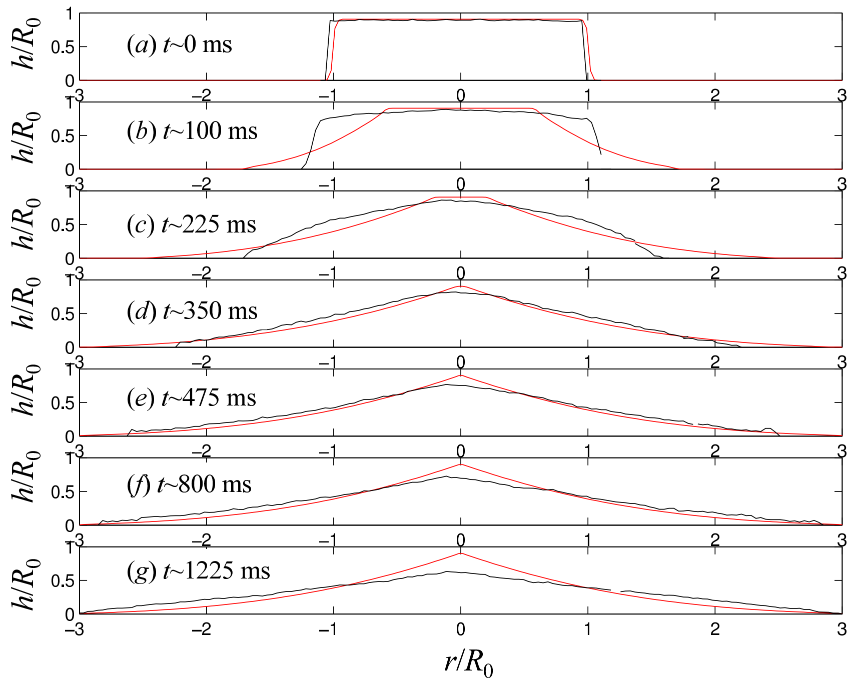

Figure 3: Thickness of the granular mass with aspect ratio $a = 0.9$ at several times calculated using the contact dynamics DE model (black lines) and LWA model (red lines) during the 2D spreading of a granular column.

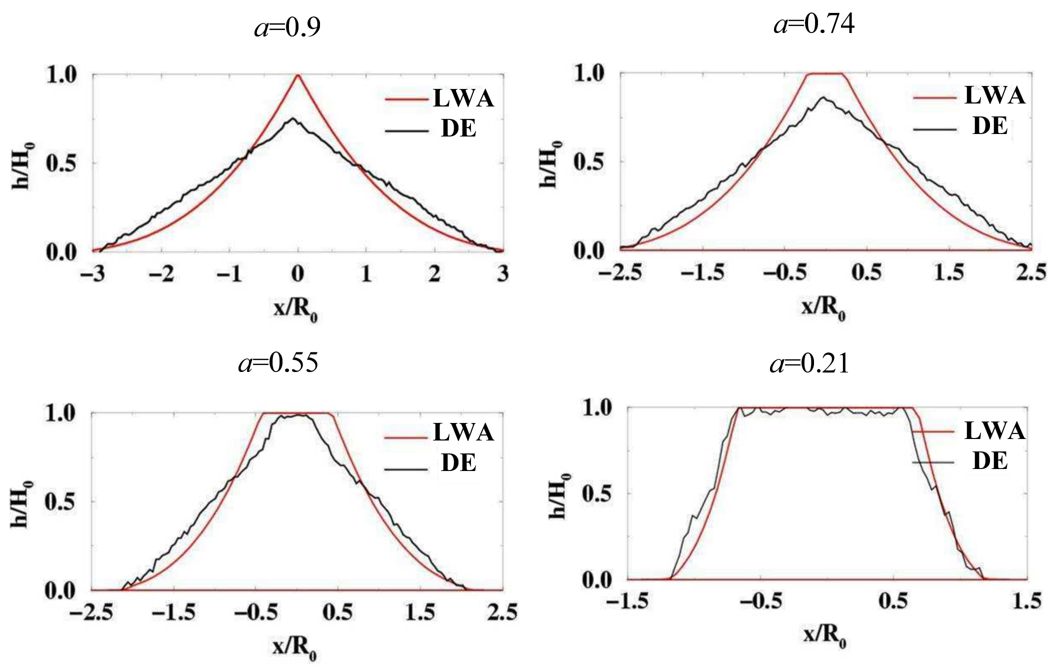

Figure 4: Deposits calculated with the contact dynamics DE model (black lines) and LWA model (red lines) for various aspect ratio.

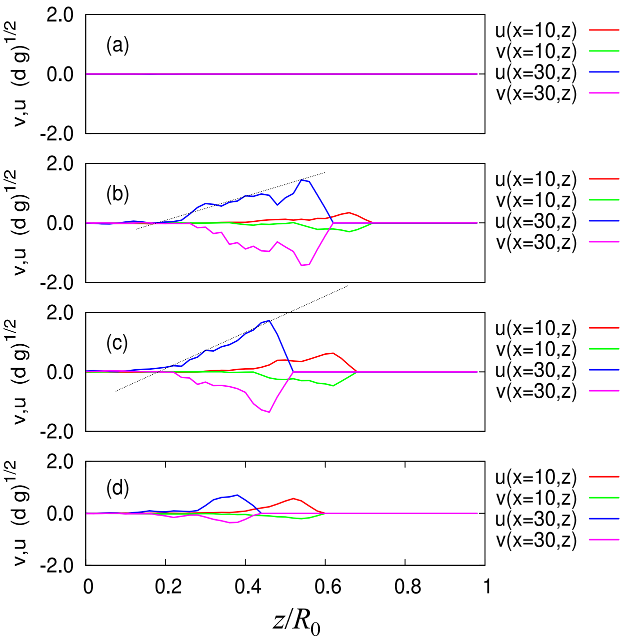

Figure 5: Velocity profiles calculated using a molecular dynamics DE model in the case of the collapse of axisymmetrical granular columns with aspect ratio $a = 0.9$. Figures (b), (c), (d) represent the velocity profiles at several times, (d) is calculated at a time close to the stopping time of the granular mass.

ever [19] and [20] show that the LWA equations can be derived without this assumption by introducing the velocity at half-thickness of the granular layer $u(h/2)$ instead of the depth-averaged velocity. Furthermore, experiments and DE simulations show that the velocity profile changes during the flow (figure 5). In any case, experimental results and DE simulations do not show significant change of the velocity profile when the aspect ratio increases. As a result, the limits of the LWA for increasing aspect ratio are not expected to be due to the assumptions on the velocity profiles.

6.2 Ratio between horizontal and vertical velocities

LWA models only calculate the depth-averaged horizontal velocity u and vertical velocity is neglected v. How the ratio u/v changes with increasing aspect ratio? This has been investigated by calculating the vertical velocity and the horizontal velocity with the DE model. More precisely, the mean values of the horizontal velocity v_x and the mean value of the vertical velocity v_z averaged on all the grains at a given time have been calculated for two aspect ratios $a = 0.21$ and $a = 0.9$ (Figure 6). Figure 6 shows that at $t > T_0$, where T_0 is a characteristic time, the relative magnitude of the vertical velocity compared to the horizontal velocity (v_z/v_x) is smaller for higher aspect ratio whatever the time is. As a result, the presence of vertical velocities is not the reason for the limits of LWA models when the aspect ratio of the granular mass increases.

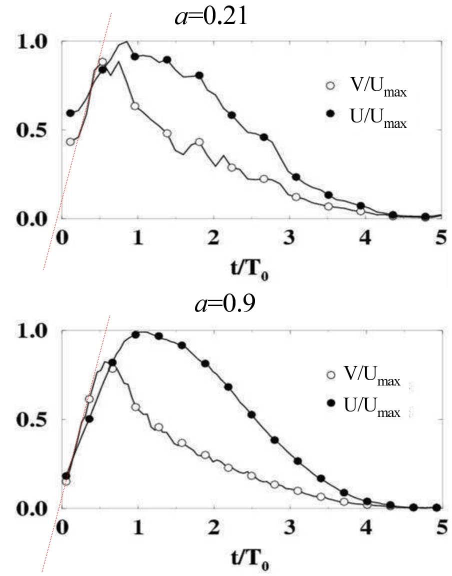

Figure 6: Horizontal v_x and vertical v_y velocities normalized by the maximum velocity obtained for each aspect ratio $a = 0.21$ and $a = 0.9$ as a function of the normalized time during the 2D spreading of a granular column calculated using the contact dynamics DE model.

7 Hydrostatic pressure

In LWA models, the vertical acceleration is neglected compared to the gravity acceleration. The vertical momentum equation then imposes hydrostatic pressure. However, DE simulations show that the vertical acceleration significantly increases with increasing aspect ratio. As an example, for $a = 0.9$, the vertical acceleration is equal to 20% of the gravity acceleration. In LWA, the pressure gradient is then overestimated providing an additional driving force. The hydrostatic approximation seems to be the major assumption that makes the LWA model fail to simulate the spreading of granular masses with high aspect ratio. The new approach proposed by [21] makes it possible to simulate high aspect ratio collapse using LWA type models. In this approach, the higher part of the granular mass is replaced by a given vertical flux deduced from DE simulations. This method cannot be applied to the simulation of real geophysical flows because a general definition of this vertical flux is still lacking. Nevertheless, these results show that extension of the LWA in order to include non-hydrostatic effects is possible and may be applied to granular collapse with high aspect ratio.

8 Conclusion

Numerical simulation of granular collapse over an horizontal plane shows that continuum models based on the Long Wave Approximation overestimate the driving forces involved during the collapse. This effect increases when the aspect ratio of the granular mass increases. Comparison between discrete and continuum simulations makes it possible to show that the assumption of hydrostatic normal stress is essentially responsible of the limits of the LWA when the aspect ratio increases. The vertical acceleration, neglected in the vertical momentum equation, is shown to play a significant role when the aspect ratio increases even for aspect ratio $a \sim 0.8$. The other assumptions as a constant velocity profile, the neglect of vertical velocity and the assumption that all the mass is flowing are actually not verified during granular collapse but do not appear to be less accurate when the aspect ratio of the granular mass increases. For small aspect ratio $a < 0.7$, the Long Wave Approximation model provides numerical results in good agreement with discrete elements modeling, in particular concerning the deposit of the granular mass. Interestingly enough, the runout distance of the deposit calculated using the thin layer model is still in good agreement with discrete element simulations when the aspect ratio increases while the shape of the deposit significantly differs from that calculated using discrete element models.

As a result, numerical simulations of real geophysical flows using thin layer models will be significantly improved if a new asymptotic analysis would be performed including the effects of the vertical acceleration.

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