The Invariant Planck Energy Distribution Law and its Connection to the Maxwell-Boltzmann Distribution Function

SIAVASH H. SOHRAB
Robert McCormick School of Engineering and Applied Science
Department of Mechanical Engineering
Northwestern University, Evanston, Illinois 60208
UNITED STATES OF AMERICA

Abstract: - A scale-invariant model of statistical mechanics is described and applied to derive the invariant Planck law of energy distribution from the invariant Boltzmann distribution function. Also, the invariant Schrödinger equation is derived from the invariant Bernoulli equation for potential incompressible flow. It is shown that a homogeneous isotropic turbulent fluid is composed of a spectrum of eddies (energy levels) each composed of a spectrum of molecular clusters with energy spectra governed by the Planck distribution law and harmonious with the Kolmogorov $\kappa^{-5/3}$ law. The stability of clusters, de Broglie wave packets, is shown to be due to a potential that acts as Poincaré stress. Atomic transitions between different size clusters (energy levels) are shown to result in emission/absorption of energy in harmony with Bohr's theory of atomic spectra.

Key-Words: - Planck energy distribution. Statistical theory of turbulence. Schrödinger equation. TOE

1 Introduction

Similarities between stochastic quantum fields [1-16] and classical hydrodynamic fields [17-26] resulted in recent introduction of a scale-invariant model of statistical mechanics [27], and its application to the field of statistical thermodynamics [28].

In the present study, further implications of the model to the foundation of statistical mechanics and quantum mechanics will be examined. It is shown that a stationary body of fluid may be viewed as a statistical field of equilibrium cluster dynamics the existence of which is evidenced by the phenomena of Brownian motions. The invariant form of the Planck law of energy distribution will be derived from the invariant Boltzmann distribution function. Also, the invariant Schrödinger equation will be derived from the invariant Bernoulli equation for potential incompressible flow.

The connection between the problem of turbulence and quantum mechanics is also addressed by identifying the field of homogeneous isotropic turbulence as a spectrum of eddies (energy levels) each composed of a spectrum of molecular clusters the stochastically stationary size of which are governed by the invariant Maxwell-Boltzmann distribution function. Also, the energy spectrum of eddies will be shown to be governed by the invariant Planck energy distribution law. The energy spectrum of eddies is also shown to be harmonious with Kolmogorov’s $\kappa^{-5/3}$ law.

2 A Scale Invariant Model of Statistical Mechanics

Following the classical methods [29-33] the invariant definition of density $\rho_\beta$, and velocity of element $v_\beta$, atom $u_\beta$, and system $w_\beta$ at the scale $\beta$ are [28]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta, \quad u_\beta = v_{\beta-1} \quad (1)$$

$$v_\beta = \rho_\beta^{-1} m_\beta \int u_\beta f_\beta du_\beta, \quad w_\beta = v_{\beta+1} \quad (2)$$

Similarly, the invariant definition of the peculiar and diffusion velocities are introduced as

$$v'_\beta = u_\beta - v_\beta, \quad v_\beta = v_\beta - w_\beta \quad (3)$$

such that

$$v'_\beta = v'_{\beta+1} \quad (4)$$
3 Stochastic Natures of the Planck and the Boltzmann Constants

Because at thermodynamic equilibrium the mean velocity of each particle (oscillator) vanishes \( \beta = 0 \), the energy of the particle can be expressed as

\[
\varepsilon_p = m_p \langle u_p^2 \rangle = \langle p_p \rangle (\lambda_p^2 \langle v_p^2 \rangle)^{1/2} = h_p \omega_p
\]

(5)

where \( m_p \langle u_p^2 \rangle = \langle p_p \rangle \) is the root-mean-square momentum of particle. The result (5) can be expressed in terms of either frequency or wavelength

\[
\varepsilon_p = m_p \langle u_p^2 \rangle = \langle p_p \rangle (\lambda_p^2 \langle v_p^2 \rangle)^{1/2} = h_p \omega_p
\]

(6)

\[
\varepsilon_p = m_p \langle u_p^2 \rangle = \langle p_p \rangle (\lambda_p^2 \langle v_p^2 \rangle)^{1/2} = k_p \lambda_p
\]

(7)

when the definition of stochastic Planck and Boltzmann factors are introduced as

\[
h_p = \langle p_p \rangle (\lambda_p^2 \langle v_p^2 \rangle)^{1/2}
\]

(8)

\[
k_p = \langle p_p \rangle (\lambda_p^2 \langle v_p^2 \rangle)^{1/2}
\]

(9)

At the important scale of EKD vacuum, corresponding to photon gas one obtains the universal constants of Planck [34, 35] and Boltzmann [28] that in view of (8)-(9) become

\[
h = h_k = m_k c (\lambda_k^2 \langle v_k^2 \rangle)^{1/2} = 6.626 \times 10^{-34} \text{ J-s}
\]

(10)

\[
k = k_k = m_k c (\lambda_k^2 \langle v_k^2 \rangle)^{1/2} = 1.381 \times 10^{-23} \text{ J/K}
\]

(11)

Following de Broglie hypothesis for the wavelength of matter waves [2]

\[
\lambda_p = h / p_p
\]

(12)

the frequency of matter waves was introduced as [28]

\[
\nu_p = k / p_p
\]

(13)

Therefore, under the condition of thermodynamic equilibrium between matter and radiation field (12)-(13) can be express as

\[
h_p = h_k = h , \quad k_p = k_k = k
\]

(14)

The definitions (10) and (11) result in the gravitational mass of photon [28]

\[
m_k = (h c / k)^{1/2} = 1.84278 \times 10^{-41} \text{ kg}
\]

(15)

that is much larger than the reported value of \( 4 \times 10^{-51} \text{ kg} \) [36]. The finite gravitational mass of photons was anticipated by Newton [37] and is in accordance with the Einstein-de Broglie theory of light [38-40].

The Avogadro-Loschmidt number is predicted as [28]

\[
N^o = 1/(m_k c^2) = 6.0376 \times 10^{23}
\]

(16)

leading to the universal gas constant

\[
R^o = N^o k = 8.338 \text{ m}^{-1}
\]

(17)

that is related to de Pretto number 8338 [41, 51].

4 Brownian Motions as Evidence for the Existence of Statistical Field of Equilibrium Cluster Dynamics

The scale invariant model of statistical mechanics for the limited range of equilibrium molecular-dynamics EMD, cluster-dynamics ECD, and eddy-dynamics EED and the corresponding non-equilibrium laminar flow fields are shown in Fig.2. Each non-equilibrium statistical field is described in terms of three velocities \( (u_p, v_p, w_p) \) corresponding to the (atom, element, system) of the scale \( \beta \).
According to Fig.2, the statistical field of equilibrium eddy-dynamics EED at the scale $\beta = e$ is a homogenous isotropic turbulent field where the system is a fluid element $l$ which is defined as an ensemble of a spectrum of fluid eddies

$$\text{System}_{\text{EED}} = \text{Fluid element} = f_i = \sum_k c_{ki} \quad (18)$$

Next, the fluid eddy $k$ is defined as an ensemble of a spectrum of molecular clusters of type $jk$

$$\text{Element}_{\text{EED}} = \text{Eddy} = c_k = \sum_j c_{jk} \quad (19)$$

Each eddy of type $k$ will be also identified as the energy level $k$. Also, each cluster $jk$ is identified as the quantum state $jk$ or cells $jk$ of the energy level $k$. Finally, each cluster $jk$ that is the “atoms” of the EED field (Fig.2) is defined as an ensemble of a spectrum of molecules of type $ijk$

$$\text{Atom}_{\text{EED}} = \text{Cluster} = c_{jk} = \sum_i m_{ijk} \quad (20)$$

Molecule of different type $ijk$ does not refer to different species of molecules but rather to their different energy.

In a parallel fashion, at the next lower scale of equilibrium cluster dynamics ECD $\beta = c$ (Fig.2) the system and element are eddy and cluster already defined in (19) and (20) and the latter is also the energy level $jk$ of ECD field. Finally, each molecule $ijk$ of cluster $j$ in eddy $k$ is defined as an ensemble of a spectrum of atoms of type $aijk$

$$\text{Atom}_{\text{ECD}} = \text{Molecule} = m_{ijk} = \sum_a a_{aijk} \quad (21)$$

The above procedure could then be extended to higher and lower scales within the hierarchy shown in Fig.1. The system, element, and “atom” of the adjacent statistical fields will be related as

$$\text{System}_{\beta} = \text{Element}_{\beta+1} \quad (22a)$$

$$\text{Element}_{\beta} = \text{Atom}_{\beta+1} = \text{System}_{\beta-1} \quad (22b)$$

$$\text{Atom}_{\beta} = \text{Element}_{\beta-1} \quad (22c)$$

The evidence for the existence of the statistical field of equilibrium cluster-dynamics ECD (Fig.2) is the phenomena of Brownian motions [23, 42-48]. Modern theory of Brownian motion starts with the Langevin equation [23]

$$\frac{du_p}{dt} = -\beta u_p + A(t) \quad (23)$$

where $u_p$ is the particle velocity. The drastic nature of the assumptions inherent in the division of forces in (23) was emphasized by Chandrasekhar [23].

To account for the stationary nature of Brownian motions fluid fluctuations at scales much larger than molecular scales are needed as noted by Gouy [48]. Observations have shown that as the size of the particles decrease their movement become faster [48]. According to classical arguments Brownian motions are induced by multiple collisions of a large number of molecules with individual suspended particle. However, since the typical size of particle is about 100 times larger than that of individual molecules, such collisions preferentially from one side of the particle could not occur in view of the assumed Maxwell-Boltzmann distribution of molecular motions. On the other hand, if one assumes that Brownian motions are induced by collisions of particles with groups, i.e. clusters, of molecules then in view of the stationary nature of Brownian motions, the motions of such clusters themselves must also be governed by the Maxwell-Boltzmann distribution. But this would mean the existence of the statistical field of equilibrium cluster dynamics.

In experimental measurement of Maxwell-Boltzmann velocity distribution [49] only the fraction of total number of atoms with the velocity $v$ was related to the intensity of ionic flux induced by collision of neutral atoms. Let us consider a heated oven containing a total of $N_{nc}$ atoms under random motions, and let $N_{anj}$ be the number of atoms in atomic-cluster (molecule) $j$ with velocity $u_j$. If one now adjusts the phase and the rotation velocity of the two rotating disks in the experiment [49] for the velocity $u_j$, one will expect the relative number of
atoms $N_{\text{amj}} / N_{\text{ac}}$ to vary like the measured relative intensity of ionic flux $I/I_o$. Therefore, one can argue that the measured number of atoms at any given atomic-cluster (molecular) velocity $u_{\text{amj}}$ denotes the fraction of “molecules” of particular size containing $N_{\text{amj}}$ atoms with the velocity $u_{\text{amj}} = \langle u_{\text{amj}} \rangle = u_{\text{am}}$. In view of the above considerations, the Maxwell-Boltzmann statistical field could be viewed as spectral distribution of different atomic-cluster sizes, with different energy and hence velocities as schematically shown in Fig.3 for the temperature $T' = 300$ K and for arbitrary vertical scale [50].

![Fig.3 Maxwell-Boltzmann velocity distribution viewed as stationary spectra of cluster, molecular, and atomic cluster sizes for EED, ECD, and EMD scales at 300 K.](image)

The modified definition of temperature $T$ and the classical one $T'$ are related by $T' = 2T$ due to the inclusion of two degrees of freedom $(\chi^+, \chi^-)$ for the translational harmonic motion of oscillators [28, 51]. The average stochastically stationary molecular cluster defines the thermodynamic temperature $3kT'_m = 6kT_m = m_m \langle u_m^2 \rangle = 2m_m \langle u_m^2 \rangle$. Therefore, for atmospheric air with $T'_m = 2T_m = 300$ K and $m_m = 28.9 \times 1.656 \times 10^{-27}$ kg by (16), the mean molecular speed will be about $v_m = u_c = \bar{u}_{m+} = 360$ m/s (Fig.2) that is in close agreement with the measured speed of sound in standard atmosphere as initially anticipated by Newton. The association of two degrees of freedom $2(kT / 2) = kT = h\nu$ with the harmonic oscillations of each Planck oscillator [51] also resolves the problem of quantum mechanics that, as opposed to the integral numbers of Planck energy unit $h\nu$, involves integral multiples of Planck energy level $h\nu / 2$ as discussed by Schrödinger [52].

### 5 Derivation of the Invariant Planck Law of Energy Distribution from the Invariant Boltzmann Distribution Function

In this section, the model of statistical mechanics introduced in the previous section will be used to derive an invariant form of the Planck energy distribution law. The analysis is best illustrated for the EED scale $\beta = e$ when (system, element, “atom”) are (fluid element, eddy, cluster). Following classical methods of Boltzmann [53, 54], the number of clusters of type $j$ in the energy level (eddy) $k$ is written as

$$ N_{\text{ck}} = g_{\text{ck}} e^{-\alpha c} e^{-\varepsilon_{\text{ck}} / kT} $$

The term $e^{-\alpha c}$ arises from the Lagrange multiplier associated with the constraint of total number of particles (clusters) being constant [53]. However, in the present model the fluid element is composed of a very large number of eddies each of which are composed of a very large number of clusters. As a result, the number of clusters in the system will be considered as infinite leading to $\alpha_c = 0$ such that

$$ e^{-\alpha c} = 1 $$

(25)

Assuming that the degeneracy of all clusters $j$ of the level $k$ are identical to a constant average value $g_{\text{ck}} = \overline{g}_{\text{ck}}$, after substitution from (25) the total number of clusters in the eddy $k$ by (24) becomes

$$ N_{\text{ck}} = \sum_j g_{\text{ck}} e^{-\alpha c} e^{-\varepsilon_{\text{ck}} / kT} = g_{\text{ck}} \sum_j e^{-\varepsilon_{\text{ck}} / kT} $$

$$ = g_{\text{ck}} \sum_j e^{-\eta / kT} = g_{\text{ck}} \sum_j \left( e^{-\varepsilon_{\text{ck}} / kT} \right)^{b_k / \eta} $$

$$ = \frac{g_{\text{ck}}}{1 - e^{-\varepsilon_{\text{ck}} / kT}} $$

(26)

In the derivation of (26) the relation

$$ \varepsilon_{\text{ck}} = \sum_j \varepsilon_{\text{ckj}} = n_j \varepsilon_{\text{ckj}} $$

(27)

for the energy of the cluster $j$ has been employed.

At the lower scale of ECD, again following the classical methods of Boltzmann [53], one expresses the number of “atoms” (molecules) of type $ij$ in the cluster (energy level) $jk$ as

$$ N_{\text{cjk}} = g_{\text{cjk}} e^{-\alpha c} e^{-\varepsilon_{\text{ckj}} / kT} $$

(28)

The average degeneracy of the quantum state (energy level) is taken as $g_{\text{cm}} = \overline{g}_{\text{cjm}}$. Also, since
the ECD system (eddy) k is composed of a large number of energy levels (clusters) \( jk \) each of which contains a large number of molecules ("atoms") of type \( \eta_{jk} \), the number of atoms will be practically infinite such that the constraint of constant \( N_{mk} \) is removed and hence \( \alpha_m = 0 \). With the values of parameters \( (g_{mj}, \alpha_m) = (g_{me}, 0) \), (28) reduces to

\[
N_{mjk} = g_{me} e^{-\epsilon_{mk}/kT} \tag{29}
\]

Since for \( \alpha_m = 0 \) \[53\]

\[
e^{-\alpha} = N_m / Z_m = 1 \tag{30}
\]

the result (29) can also be expressed as

\[
N_{mj} / N_m = g_{me} e^{-\epsilon_{mj}/kT} / Z_m \tag{31}
\]

when the molecular partition function is defined as

\[
Z_m = \sum_{mj} g_{me} e^{-\epsilon_{mj}/kT} \tag{32}
\]

It is now possible to address the central aim of statistical mechanics that is to determine the distribution of particles (Planck oscillators) among various energy levels with degeneracy under the constraint of a constant total energy \( E = N_h \nu \). In view of (26) and (29), the number of "atoms" (molecules) in the energy level (eddy) \( k \) can be expressed as

\[
N_{mk} = N_{ck} N_{mjk} = \frac{g_{mk}}{e^{\epsilon_{mk}/kT} - 1} \tag{33}
\]

that is the Bose-Einstein distribution and the total degeneracy is \( g_{mk} = g_{ck} g_{nc} \). With the Rayleigh-Jeans expression of degeneracy \[34, 35, 56\]

\[
g_{mk} = \frac{8\pi V}{u_k^3} v_k^2 dV_k \tag{34}
\]

denoting the number of oscillators in system with volume \( V \) and the frequency interval \( v_k \) to \( v_k + dv_k \), (33) gives the Planck distribution function \[34, 35\]

\[
\frac{\epsilon_{j} dN_{mk}}{V} = \frac{8\pi h v_k^3}{u_k^3} e^{h v_k/kT} \tag{35}
\]

when the energy of each oscillator is \( \epsilon_k = h \nu_k \). Therefore, the invariant Planck energy distribution law can be written as \[50\]

\[
\frac{\epsilon_{j} dN_{mk}}{V} = \frac{8\pi h v_j^3}{u_j^3} e^{h v_j/kT} \tag{36}
\]

At the scale of EKD \( \beta = k \) associated with photon gas, i.e. vacuum or the physical space \[50\], (36) describes the spectrum of photons (tachyon clusters) with the velocity \( u_k = v_1 = c \). The notion of “molecules of light” as clusters of photons is in accordance with the perceptions of de Broglie \[57\].

Derivation of the Bose-Einstein distribution function (33) is usually based on the number of complexions for distributing \( N_k \) oscillators among \( g_k \) quantum states of energy level \( k \) first written by Planck \[34\]

\[
W_k = \frac{(N_k + g_k - 1)!}{N_k!(g_k - 1)!} \tag{37}
\]

The most probable distribution (33) is obtained by maximization of the total number of complexions \[53\]

\[
W = \prod_k \frac{(N_k + g_k - 1)!}{N_k!(g_k - 1)!} \tag{38}
\]

subject to the energy constraint \( E = N_h \nu \). However, the physical basis for the probabilistic interpretation of (37) has not been fully understood.

It is suggested that the scale-invariant model of statistical mechanics (Fig.1) provides a new perspective on the probabilistic nature of the Planck formula (37). As was discussed above, each energy level is composed of \( g_{ck} \) clusters. However, the smallest cluster that contains only a single "atom" must be considered as full, since no other particle can be added to this smallest cluster. Because an empty cluster has no physical significance, the total number of available clusters in the energy level will be \( (g_k - 1) \). Therefore, the Planck formula (37) is the exact probability associated with the distribution of \( N_k \) indistinguishable oscillators amongst \( (g_k - 1) \) distinguishable available clusters.

While the number of clusters \( N_k \) and the energy of individual clusters \( \epsilon_{jk} \) within the spectrum of clusters in eddy \( k \) are considered to be both variables, their product that is the energy of the level \( k \) is considered to be identical for all levels (eddies)

\[
E_k = \sum_j \epsilon_{ck} = N_{ck} \epsilon_{ck} = \text{const} \tag{39}
\]

As a result, the total constant energy of system will be

\[
E = \sum_k E_k = N_k E_k \tag{40}
\]

The results (39) and (40) suggest that the size of the system can be arbitrarily enlarged by adding more energy levels \( N_k \) and only limited by the total available energy \( E \).

Amongst the spectrum of energy levels are the largest eddy that is the entire system, i.e. a fluid element, and the smallest eddy, which is composed of a single cluster. However, single clusters are
“atoms” of the EED field and hence their energy defines the thermodynamic temperature of the system such that

$$E_i = e_{i\ell} = m_{i\ell} < u_{i\ell}^2 > = 3kT_c$$ (41)

that with (39) leads to

$$E_k = 3kT_c$$ (42)

Therefore, a single parameter namely the absolute temperature $T_c$ defines the energy of the entire system of EED (Fig.3) at thermodynamic equilibrium

$$m_i < u_i^2 > = m_k < u_k^2 > = m_m < u_m^2 > = 3kT_m$$ (43)

Similar reasoning employed above to arrive at (39)-(43) when applied to the lower scale of ECD leads to

$$E_j = \sum_i e_{mj} = N_{mj} e_{mj} = \text{const} \tan t$$ (44)

$$E_k = \sum_j E_{jk} = N_j E_{jk}$$ (45)

$$m_i < u_i^2 > = m_k < u_k^2 > = m_m < u_m^2 > = 3kT_m$$ (46)

Since by (22), the energy of the adjacent statistical fields are related, (39)-(46) lead to the equality of temperature $T$ of the entire hierarchy of scales [27]

$$T_c = T_m = T_a = \ldots = T$$ (47)

Because of validity of (43) and (46) at equilibrium, the classical paradox of Maxwell's demon will no longer be encountered since particles of all size will have exactly the same energy $3kT$, making selection of more energetic particles by the demon impossible. Transfer of an atom from a small (fast oscillating) cluster $j$ to a large (slow oscillating) cluster $i$ is equivalent to the transition from the high energy level $j$ to the low energy level $i$ (Fig.3). The non-commutable nature of such transitions is in harmony with Heisenberg matrix mechanics [58]. Also, by (6) such a transition releases energy in accordance with Bohr's theory of atomic spectra [59]

$$\Delta e_{aji} = e_{aj} - e_{ai} = h(v_{aj} - v_{ai})$$ (48)

Therefore, the reason for the quantum nature of atomic spectra is that only transitions between clusters (energy levels) are allowed that themselves must satisfy the criteria of stationarity imposed by the Maxwell-Boltzmann distribution function (Fig.3) [50].

It is possible to obtain the invariant Maxwell-Boltzmann distribution function directly from the Planck distribution function (35)-(36) that in view of (8) and (14) can be written as

$$dN_{\nu\beta} = \frac{8\pi \sqrt{\nu^2 v_{\beta}^2}}{u_{\beta}^3} e^{\nu^2 / kT} - 1 = \frac{8\pi \sqrt{m_\beta^2 \lambda_\beta^2 v_{\beta}^2}}{u_{\beta}^3 m_\beta^2} e^{\nu^2 / kT} - 1$$

$$= \frac{8\pi \nu m_\beta^2}{kT} u_{\beta}^3 e^{\nu^2 / kT} - 1$$ (49)

Substituting for the partition function

$$Z = N = \sum g_{\nu\beta} e^{-\nu^2 / kT} = g_{\nu\beta} / (1 - e^{-\nu^2 / kT})$$ (from 26),

and the degeneracy

$$g_{\nu\beta} = 2\nu (m_\beta / h)^3 [2\pi kT / m_\beta]^{3/2}$$ (50)

obtained from the normalization condition

$$\int_0^\infty dN_{\nu\beta} / N = 1$$

into (49) results in the invariant Maxwell-Boltzmann velocity (speed) distribution function

$$\frac{dN_{\nu\beta}}{N} = 4\pi \left( \frac{m_\beta}{2\pi kT} \right)^{3/2} u_{\beta}^2 e^{-m_\beta u_{\beta}^2 / 2kT} du_{\beta}$$ (51)

It is thus suggested that the energy spectrum of eddies in isotropic turbulent fields should correspond to the Planck energy distribution law (35). Preliminary examination of the three-dimensional energy spectrum $E(k, t)$ for isotropic turbulence measured by Van Atta and Chen [60] and its model shown in Fig.5.15a and Fig.5.17 of Landahl and Mollo-Christensen [61] appears to support such a correspondence. Indeed, expressing the cluster energy as $e = mu^2$, the eddy energy as $E = \alpha e$, the cluster velocity as $u = \nu\lambda = 2\pi\nu / \kappa$ such that $du = d(2\pi\nu / \kappa) \propto \kappa^{-2} dx$ where $\kappa = 2\pi / \lambda$ is the wave number, one arrives at

$$dE = 2\alpha du \propto \alpha e^{2/3} (u / e^{2/3}) d(v^2 / \kappa) \propto \alpha e^{2/3} (u^{-1/3}) \kappa^{-2} dx \propto \alpha e^{2/3} \kappa^{-5/3} dx$$ (52)

that is Kolmogorov's famous $\kappa^{-5/3}$ law [61].

6 Derivation of Invariant Schrödinger Equation from the Invariant Bernoulli Equation

Following the classical methods [29-33], the scale-invariant forms of mass and momentum conservation equations are given as [51, 62]

$$\frac{\partial \rho_\beta}{\partial t} + \nabla.(\rho_\beta v_\beta) = 0$$ (53)

$$\frac{\partial (\rho_\beta v_\beta)}{\partial t} + \nabla.(\rho_\beta v_\beta v_\beta) = -\nabla P_\beta$$ (54)
where \( P_\beta \) is the partial stress tensor \([29]\)

\[
P_\beta = m_\beta \int (u_\beta - v_\beta)(u_\beta - v_\beta)f_\beta du_\beta \quad (55)
\]

In the derivation of (54) one employs the definition of the peculiar velocity (3) along with the identity

\[
V'_\beta V'_\beta = (u_\beta - v_\beta)(u_\beta - v_\beta) = u_\beta u_\beta - v_\beta v_\beta \quad (56)
\]

For an irrotational \( v_\beta = \nabla \Phi_\beta \) and incompressible flow with the velocity potential \( \Phi_\beta \), (53)-(54) lead to the invariant Bernoulli equation

\[
\frac{\partial (\rho_\beta \Phi_\beta)}{\partial t} + \frac{(V \rho_\beta \Phi_\beta)^2}{2\rho_\beta} + P_\beta = \cos \tan t = 0 \quad (57)
\]

Comparison of (57) with the Hamilton-Jacobi equation of classical mechanics \([2, 63-65]\)

\[
\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + U = 0 \quad (58)
\]

leads to the invariant definition of the modified action \([66]\)

\[
S_\beta(x, t) = \rho_\beta \Phi_\beta \quad (59)
\]

The relation (3) between the velocities \( v_\beta = u_\beta - V'_\beta \) suggests that

\[
\Phi_\beta(x, t) = \Phi_{\text{ref}}(x) - \varepsilon \Phi'_\beta(x, t) \quad , \quad \varepsilon << 1 \quad (60)
\]

because the relation \( \nabla \times v_\beta = \nabla \times u_\beta - \nabla \times V'_\beta = 0 \) leads to \( u_\beta = \nabla \Phi_{\text{ref}} \) and \( V'(x, t)_\beta = \varepsilon \nabla \Phi'_\beta \). Therefore, (59) and (60) give

\[
S_\beta = S_{\text{ref}} - \varepsilon S'_\beta = S_{\text{ref}} - \varepsilon \Psi'_\beta \quad (61)
\]

with the quantum mechanics wave function \( \Psi'_\beta(x, t) \) defined as \([50, 66]\)

\[
\Psi'_\beta(x, t) = S'_\beta(x, t) = \rho_\beta \Phi'_\beta(x, t) \quad (62)
\]

Substituting from (62) into (57) and separating terms of equal powers of \( \varepsilon \) leads to

\[
\frac{\partial S_{\text{ref}}}{\partial t} + \frac{(\nabla S_{\text{ref}})^2}{2\rho_\beta} + \tilde{U}_\beta = 0 \quad (63)
\]

\[
\frac{\partial \Psi'_\beta}{\partial t} + \frac{(\nabla S_{\text{ref}}) \nabla \Psi'_\beta}{\rho_\beta} = 0 \quad (64)
\]

where the volumetric potential energy density is defined as \([28]\)

\[
\tilde{U}_\beta \equiv P_\beta = n_\beta \tilde{U}_\beta \quad (65)
\]

To analyze (63), one introduces the moving coordinate \( z = x - u_\beta t \) such that

\[
\nabla_z S_{\text{ref}} = \nabla_x S_{\text{ref}} \quad , \quad \frac{\partial}{\partial t} S_{\text{ref}} / \partial t = -u_\beta \nabla_z S_{\text{ref}} \quad (66)
\]

Substitutions from (61), (65) and (66) in (63) give the conservation of energy in the form

\[
\tilde{E}_\beta = \tilde{T}_\beta + \tilde{U}_\beta \quad (67)
\]

where

\[
\tilde{E}_\beta = \rho_\beta u_\beta^2 = n_\beta \tilde{E}_\beta \quad , \quad \tilde{T}_\beta = \rho_\beta u_\beta^2 / 2 = n_\beta \tilde{T}_\beta \quad (68)
\]

The result (67) helps to define the stochastically stationary magnitude of the mean molecular velocity. Since (3) when squared and averaged leads to

\[
<u_\beta^2> = <v_\beta^2> + <u_\beta V'_\beta> + <2v_\beta V'_\beta>
\]

\[
<u_\beta^2> = <v_\beta^2> + <V'_\beta^2> \quad (69)
\]

because \( <2v_\beta V'_\beta> = 0 \). In view of (67)-(68), the stationary mean molecular velocity is obtained from (69) as \( <u_\beta^2> = <v_\beta^2> / 2 \).

Next, taking the first time derivative of (64) and substituting for \( \partial \Psi'_\beta / \partial t \) from (64) itself one obtains the wave equation

\[
\frac{\partial^2 \Psi'_\beta}{\partial t^2} = u_\beta^2 \nabla^2 \Psi'_\beta \quad (70)
\]

With the product solution \( \Psi'_\beta(x, t) = \psi_\beta(x) \Lambda_\beta(t) \) in (70) one obtains

\[
\frac{\psi''_\beta}{\psi_\beta} = \frac{\Lambda''_\beta}{\Lambda_\beta u_\beta^2} = -\sigma_\beta^2 \quad (71)
\]

where \( \sigma_\beta \) is the separation constant. The solution of temporal part of (71) is

\[
\Lambda_\beta = \exp(-i \sigma_\beta u_\beta t) = \exp(-i \omega_\beta t) \quad (72)
\]

suggesting that \( \sigma_\beta u_\beta = \omega_\beta = 2\pi \nu_\beta \) is a circular frequency \( \omega_\beta \).

Following Planck \([34]\), one introduces by (6) the invariant expressions

\[
\begin{align*}
\tilde{U}_\beta & \equiv P_\beta = n_\beta \tilde{U}_\beta \\
To analyze (63), one introduces the moving coordinate \( z = x - u_\beta t \) such that
\end{align*}
\]

\[
\nabla_z S_{\text{ref}} = \nabla_x S_{\text{ref}} \quad , \quad \frac{\partial}{\partial t} S_{\text{ref}} / \partial t = -u_\beta \nabla_z S_{\text{ref}} \quad (66)
\]

Substitutions from (61), (65) and (66) in (63) give the conservation of energy in the form

\[
\tilde{E}_\beta = \tilde{T}_\beta + \tilde{U}_\beta \quad (67)
\]

where

\[
\tilde{E}_\beta = \rho_\beta u_\beta^2 = n_\beta \tilde{E}_\beta \quad , \quad \tilde{T}_\beta = \rho_\beta u_\beta^2 / 2 = n_\beta \tilde{T}_\beta \quad (68)
\]

The result (67) helps to define the stochastically stationary magnitude of the mean molecular velocity. Since (3) when squared and averaged leads to

\[
<u_\beta^2> = <v_\beta^2> + <u_\beta V'_\beta> + <2v_\beta V'_\beta>
\]

\[
<u_\beta^2> = <v_\beta^2> + <V'_\beta^2> \quad (69)
\]

because \( <2v_\beta V'_\beta> = 0 \). In view of (67)-(68), the stationary mean molecular velocity is obtained from (69) as \( <u_\beta^2> = <v_\beta^2> / 2 \).

Next, taking the first time derivative of (64) and substituting for \( \partial \Psi'_\beta / \partial t \) from (64) itself one obtains the wave equation

\[
\frac{\partial^2 \Psi'_\beta}{\partial t^2} = u_\beta^2 \nabla^2 \Psi'_\beta \quad (70)
\]

With the product solution \( \Psi'_\beta(x, t) = \psi_\beta(x) \Lambda_\beta(t) \) in (70) one obtains

\[
\frac{\psi''_\beta}{\psi_\beta} = \frac{\Lambda''_\beta}{\Lambda_\beta u_\beta^2} = -\sigma_\beta^2 \quad (71)
\]

where \( \sigma_\beta \) is the separation constant. The solution of temporal part of (71) is

\[
\Lambda_\beta = \exp(-i \sigma_\beta u_\beta t) = \exp(-i \omega_\beta t) \quad (72)
\]

suggesting that \( \sigma_\beta u_\beta = \omega_\beta = 2\pi \nu_\beta \) is a circular frequency \( \omega_\beta \).

Following Planck \([34]\), one introduces by (6) the invariant expressions
\[ \bar{E}_\beta = m_\beta \mathbf{u}_\beta = h_\beta \mathbf{v}_\beta , \quad \bar{E}_\beta = n_\beta h_\beta \mathbf{v}_\beta , \] (73)

when the harmonic atomic velocity is expressed as \( \mathbf{u}_\beta = \frac{E_\beta}{h} \mathbf{v}_\beta \).

It was anticipated by Dirac [67] that amongst the fundamental constants (c, e, \( h \)), the Planck constant \( h \) might be expressible in terms of other fundamental constants. Using the result (73) one obtains

\[ \sigma_\beta = \frac{2\pi E_\beta}{h_\beta} \] (74)

such that (72) becomes

\[ \Lambda_\beta = \exp(-i2\pi E_\beta/h_\beta) \] (75)

By substitution from (68), (74), and (14) into (71), one obtains the invariant time-independent Schrödinger equation [68]

\[ \nabla^2 \psi_\beta + \frac{8\pi^2 m_\beta}{h^2} (\bar{E}_\beta - \bar{U}_\beta) \psi_\beta = 0 \] (76)

and through multiplication by (75), the invariant time-dependent Schrödinger equation

\[ i\hbar \frac{\partial \psi_\beta}{\partial t} + \frac{\hbar^2}{2m_\beta} \nabla^2 \psi_\beta - \bar{U}_\beta \psi_\beta = 0 \] (77)

that govern the dynamics of particles from cosmic to tachyonic scales (Fig.1) [28, 50]. Since (67)-(68) lead to \( \bar{E}_\beta = \bar{T}_\beta + \bar{U}_\beta \), the Schrödinger equation (76) gives the stationary states of particles that are trapped within de Broglie wave-packet under the potential defined in (65) acting as Poincaré stress. In view of the definition (55), anticipation of an external pressure or stress as being the cause of particle stability by Poincaré [69] is a testimony to the true genius of this great mathematician, physicist, and philosopher.

7 Concluding Remarks

A scale-invariant model of statistical mechanics was described and applied to derive the invariant form of Planck energy distribution law from the invariant Boltzmann distribution function. Also, the invariant Schrödinger equation was derived from the invariant Bernoulli equation for potential incompressible flow. It was shown that the statistical field of homogeneous isotropic turbulence is composed of a spectrum of eddies (energy-levels) each composed of a spectrum of molecular clusters the stationary size of which are governed by the Maxwell-Boltzmann distribution function. The stability of clusters, de Broglie wave packets, was shown to be due to a potential acting as Poincaré stress.

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References:
