A Nonlinear Evolution System of Partial Differential Equations With p-Laplacian and Negative Nonlineariry

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Abstract

In this paper we prove the existence of weak solutions of a mixed problem associated to the system

$$\begin{align*}
  u'' + Au - \Delta u' - |v|^{\rho+2}|u|^\rho u &= f_1 \\
  v'' + Av - \Delta v' - |u|^{\rho+2}|v|^\rho v &= f_2
\end{align*}$$

(1)

where $\Delta$ is the usual Laplacian operator in $\mathbb{R}^n$ and $A$ is the pseudo-Laplacian operator given by

$$Au = -\sum_{i=1}^{n} \frac{\partial U}{\partial x_i}, \quad \text{with} \quad U = \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i},$$

$p > 2$ and $\rho$ satisfies a technical condition.

Keywords: p-Laplacian, Young Inequality, Compactness.


1 Introduction

In the paper ([2]) the author has studied a similar evolution system with p-Laplacian operator where the nonlinear functional terms $|v|^{\rho+2}|u|^\rho u$ and $|u|^{\rho+2}|v|^\rho v$ are preceded by the plus signal. Now we study the existence of weak solutions to the system

$$\begin{align*}
  u'' + Au - \Delta u' - |v|^{\rho+2}|u|^\rho u &= f_1 \\
  v'' + Av - \Delta v' - |u|^{\rho+2}|v|^\rho v &= f_2
\end{align*}$$

(2)

The crucial difference between the system (2) and that one we have studied in ([1]) is the presence of the minus signal preceding the nonlinear terms $|v|^{\rho+2}|u|^\rho u$ and $|u|^{\rho+2}|v|^\rho v$. It conduct us to a different way of calculating the necessary estimates we need in order to solve the problem (2).
2 Notation and Main Results

2.1: Let $\Omega$ be a bounded regular domain of $\mathbb{R}^n$, $T > 0$ be a real number and $Q = \Omega \times ]0, T[$. We denote by $(\ , \ )$, $\| \ \|$ and $(\ , \ )$, $|\ |$ the inner product and norm in $H^1_0(\Omega)$ and $L^2(\Omega)$, respectively.

If $X$ is a Banach space we denote by $L^p(0, T; X)$, $1 \leq p \leq \infty$, the Banach space of all $X$-valued measurable function $u : ]0, T[ \rightarrow X$, such that $\| u(t) \|_X$ belongs to $L^p(0, T)$.

If $1 \leq p < \infty$, then

$$\| u \|_{L^p(0, T; X)} = \left( \int_0^T \| u(t) \|_X^p \ dt \right)^{\frac{1}{p}}$$

defines the norm on $L^p(0, T; X)$. ([4])

The norm on $L^\infty(0, T; X)$ is defined by

$$\| u \|_{L^\infty(0, T; X)} = \text{ess sup}_{0 \leq t \leq T} \| u(t) \|_X.$$

2.2: Let $n \in \mathbb{N}$, $p \in \mathbb{R}$, $n > p$, $p > 2$.

If

$$-1 < \rho \leq \frac{4(1 - n + p)}{2(n - p - 1) + np}$$

then

$$\rho < \frac{4}{np - 2}.$$

2.3: If $n \in \mathbb{N}$ and

$$4n + 2 \quad 4 + n < p < n \quad \text{then} \quad \frac{4}{np - 2} < \frac{1}{n - p}$$

2.4: If

$$\theta = \frac{2np(\rho + 2)}{(np - 2)(\rho + 2) + 2np(\rho + 1)}$$

and

$$\delta = \frac{2np(\rho + 2)}{(np + 2)(\rho + 2) - 2np(\rho + 1)}$$

where $n$, $p$ and $\rho$ are as before, then:

1. $1 < \theta < \frac{\rho + 2}{\rho + 1}$
2. $1 < \delta < \frac{np}{n - p}$
3. $\frac{1}{\theta} + \frac{1}{\delta} = 1$

2.5: Let

$$\alpha = \frac{\rho + 2}{(\rho + 1)\theta} \quad \text{and} \quad \beta = \frac{\rho + 2}{(\rho + 2) - (\rho + 1)\theta}.$$ 

Then we have:

1. $\alpha > 1$, $\beta > 1$
2. $\theta \beta = \frac{2np}{np - 2}$
3. \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \)

2.6: \( \mathcal{W}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega) \), if \( 1 \leq q \leq \frac{np}{n-p} \), given that \( n > p \) and \( p \geq 1 \).

2.7: Let \( u, v \in \mathcal{W}^{1,p}_0(\Omega) \). Then,

1. \( uv \in L^{p+2}(\Omega) \)
2. \( |v|^{p+2}|u|^\rho u = |u|^{p+2}|v|^\rho v \in L^\theta(\Omega) \).

3  Existence Theorem

Let \( n, p, \rho \) be as before and suppose that

\[
\begin{cases}
  f_1, f_2 \in L^2(0,T;L^2(\Omega)) \\
  u_0, v_0 \in \mathcal{W}^{1,p}_0(\Omega) \\
  u_1, v_1 \in L^2(\Omega)
\end{cases}
\]  

(3)

Then there exist functions \( u, v : Q \rightarrow \mathbb{R} \) such that :

\[ u, v \in L^\infty(0,T;\mathcal{W}^{1,p}_0(\Omega)) \]  

(4)

\[ u', v' \in L^\infty(0,T;L^2(\Omega)) \]  

(5)

\[
\begin{align*}
\frac{d}{dt}(u'(t), w) + (Au(t), w) + (u'(t), w) - \\
- \langle |v(t)|^{p+2}|u(t)|^\rho u(t), w \rangle &= (f_1(t), w) \\
\frac{d}{dt}(v'(t), w) + (Av(t), w) + (v'(t), w) - \\
- \langle |u(t)|^{p+2}|v(t)|^\rho v(t), w \rangle &= (f_1(t), w)
\end{align*}
\]  

(6) (7)

para todo \( w \in \mathcal{W}^{1,p}_0(\Omega) \), in the sense of distributions on \( ]0,T[ \). 

Proof. We will essentially use Galerkin’s method, compactness and Young’s Inequality.

Let \( \mathbb{H}_s^0(\Omega), s > 1 + n\left(\frac{1}{2} - \frac{1}{p}\right) \), be a Hilbert space such that \( \mathbb{H}_0^s(\Omega) \hookrightarrow \mathcal{W}^{1,p}_0(\Omega) \). 

We determine a spectral basis \( \{w_j\} \) of \( \mathbb{H}_0^s(\Omega) \) which is an orthonormal complete system in \( L^2(\Omega) \). Let \( V_m = [w_1, w_2, \cdots, w_m] \) be the subspace of \( \mathbb{H}_s^0(\Omega) \) generated by the \( m \) first vectors \( w_1, \ldots, w_m \).

3.1 Approximate Problem

We consider the following approximated problem

\[
(u''_m(t), w) + (Au_m(t), w) + (u'_m(t), w) - \\
\langle |v_m(t)|^{p+2}|u_m(t)|^\rho u_m(t), w \rangle = (f_1(t), w)
\]  

(8)
\[ (v''(t), w) + \langle Av_m(t), v'_m(t) \rangle + (v'_m(t), w) - \langle |u_m(t)|^{p+2} |v_m(t)|^p v_m(t), w \rangle = (f_2(t), w) \] (9)

\[
\begin{aligned}
&u_m(0) = u_{0m} \rightarrow u_0, \quad \text{in } W^p_0(\Omega) \\
u'_m(0) = u_{1m} \rightarrow u_1, \quad \text{in } L^2(\Omega) \\
v_m(0) = v_{0m} \rightarrow v_0, \quad \text{in } W^p_0(\Omega) \\
v'_m(0) = v_{1m} \rightarrow v_1, \quad \text{in } L^2(\Omega)
\end{aligned}
\] (10)

By Carathedory’s Existence Theorem ([5]) the system (8) - (10) has a solution \( \{u_m, v_m\} \) defined on \([0, t_m]\), \( t_m > 0 \). It is possible to extend this solution to the whole interval \([0, T]\). In order to do it some estimates are necessary.

**Estimate 1.**

We replace \( w \) for \( u'_m(t) \) and \( v'_m(t) \) in equations (8) and (9), respectively, so that we have:

\[
\begin{aligned}
&(u''_m(t), u'_m(t)) + \langle Au_m(t), u'_m(t) \rangle + (u'_m(t), u'_m(t)) - \\
&- \langle |u_m(t)|^{p+2} |u_m(t)|^p u_m(t), u'_m(t) \rangle = (f_1(t), u'_m(t)) \\
&\quad (v''_m(t), v'_m(t)) + \langle Av_m(t), v'_m(t) \rangle + (v'_m(t), v'_m(t)) - \\
&- \langle |u_m(t)|^{p+2} |v_m(t)|^p v_m(t), v'_m(t) \rangle = (f_1(t), v'_m(t))
\end{aligned}
\] (11)

Adding this two expressions it follows that:

\[
\frac{d}{dt} \left[ \frac{1}{2} |u'_m(t)|^2 + \frac{1}{2} |v'_m(t)|^2 \right] + \frac{1}{p} \frac{d}{dt} \| u_m(t) \|_p^p + \frac{1}{p} \| v_m(t) \|_p^p + \\
+ \| u'_m(t) \|^2 + \| v'_m(t) \|^2 = \\
= \frac{1}{\rho + 2} \frac{d}{dt} \| u_m(t) v_m(t) \|_{L^{p+2}(\Omega)}^{p+2} + (f_1(t), u'_m(t)) + \\
+ (f_2(t), v'_m(t)).
\] (13)

Integration in (13) from 0 to \( t < t_m \), implies:

\[
\frac{1}{2} |u'_m(t)|^2 + \frac{1}{2} |v'_m(t)|^2 + \frac{1}{p} \| u_m(t) \|_p^p + \frac{1}{p} \| v_m(t) \|_p^p + \\
+ \int_0^t \| u'_m(s) \|^2 + \| v'_m(s) \|^2) ds \leq \\
\frac{1}{\rho + 2} \| u_m(t) v_m(t) \|_{L^{p+2}(\Omega)}^{p+2} + \frac{1}{\rho + 2} \| u_m(0) v_m(0) \|_{L^{p+2}(\Omega)}^{p+2} + \\
+ \frac{1}{2} |u'_m(t)|_0 + \frac{1}{2} |v'_m(0)|_0 + \frac{1}{p} \| u_m(0) \|_p^p + \\
+ \frac{1}{p} \| u_m(0) \|_p^p + \\
+ \int_0^t |(f_1(s), u'_m(s))| ds + \int_0^t |(f_2(s), v'_m(s))| ds
\] (14)

Now, by Cauchy-Schwarz and Young’s Inequalities, we obtain

\[
\int_0^t |(f_1(s), u'_m(s))| ds \leq \frac{c^2}{2} \int_0^T |f_1(s)|^2 ds + \frac{1}{2} \int_0^t \| u'_m(s) \|^2 ds
\]
and,
\[ \int_0^t |(f_2(s), v'_m(s))| ds \leq \frac{c^2}{2} \int_0^T |f_2(s)|^2 ds + \frac{1}{2} \int_0^t \| v'_m(s) \|^2 ds \]

Taking these estimates into account, the inequality in (14) can be put in the form,
\[ \frac{1}{2} |u'_m(t)|^2 + \frac{1}{2} |v'_m(t)|^2 + \frac{1}{p} \| u_m(t) \|^p + \frac{1}{p} \| v_m(t) \|^p + \frac{1}{\rho + 2} \| u_m(t) v_m(t) \|_{L^{\rho+2}(\Omega)}^p \]
\[ + \frac{1}{\rho + 2} \| u_m(t) v_m(t) \|_{L^{\rho+2}(\Omega)}^p \]
\[ \leq \frac{1}{\rho + 2} \| u_m(t) v_m(t) \|_{L^{\rho+2}(\Omega)}^p \]
\[ + \frac{1}{\rho + 2} \| u_m(t) v_m(t) \|_{L^{\rho+2}(\Omega)}^p + \frac{1}{p} \| u_m(t) \|^p + \frac{1}{\rho + 2} \| u_m(t) v_m(t) \|_{L^{\rho+2}(\Omega)}^p \]
\[ + \frac{1}{p} \| u_m(t) \|^p + \frac{c^2}{2} \int_0^t |f_1(s)|^2 ds + \frac{c^2}{2} \int_0^T |f_2(s)|^2 ds \]  

(15)

**Analysis of** \( \frac{1}{\rho + 2} \| u_m(t) v_m(t) \|_{L^{\rho+2}(\Omega)}^p \).

By Hölder and the trivial inequality \( ab \leq \frac{1}{2} (a^2 + b^2) \), we have,
\[ \frac{1}{\rho + 2} \| u_m(t) v_m(t) \|_{L^{\rho+2}(\Omega)}^p \leq \frac{1}{2(\rho + 2)} \| u_m(t) \|_{L^{2(\rho+2)}(\Omega)}^{2(\rho+2)} + \frac{1}{2(\rho + 2)} \| u_m(t) \|_{L^{2(\rho+2)}(\Omega)}^{2(\rho+2)} \]

Now by Sobolev Imbedding Theorem ([7]) we have \( W_0^{1,p}(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega) \), so that
\[ \| u_m(t) \|_{L^{2(\rho+2)}} \leq C \| u_m(t) \|_0 \]

and
\[ \| v_m(t) \|_{L^{2(\rho+2)}} \leq C \| v_m(t) \|_0 \]

Now taking \( r > 1 \) such that,
\[ 2(\rho + 2)r = p > 4(\rho + 2), \]
and \( s \) such that \( \frac{1}{r} + \frac{1}{s} = 1 \), we obtain,
\[ \frac{C}{2(\rho + 2)} \| u_m(t) \|_0^{2(\rho+2)} \leq \frac{1}{p} \| u_m(t) \|_0^{p/2}.2 + C. \]

Finally, using Young's Inequality once again, we have,
\[ \frac{C}{2(\rho + 2)} \| u_m(t) \|_0^{2(\rho+2)} \leq \frac{1}{2p} \| u_m(t) \|_0^p + 2 + C \]

or yet,
\[ \frac{C}{2(\rho + 2)} \| u_m(t) \|_0^{2(\rho+2)} \leq \frac{1}{2p} \| u_m(t) \|_0^p + C \]  

(16)
and, by similar development,

\[
\frac{C}{2(\rho + 2)} \| v_m(t) \|_0^{2(\rho + 2)} \leq \frac{1}{2p} \| v_m(t) \|_0^p + C \tag{17}
\]

from where we get that,

\[
\frac{1}{\rho + 2} \| u_m(t)v_m(t) \|_{L^{\rho + 2}(\Omega)} \leq \frac{1}{2p} \| u_m(t) \|_0^p + \frac{1}{2p} \| v_m(t) \|_0^p + C.
\]

The constant \( C \) in the analysis we have just made is not the same, evidently, and does not depend on \( m \) and \( t \).

Taking these results into account and remembering the hypothesis in (10), we have from (14) that,

\[
\frac{1}{2} |u_m'(t)|^2 + \frac{1}{2} |v_m'(t)|^2 + \frac{1}{2p} \| u_m(t) \|_0^p + \frac{1}{2p} \| v_m(t) \|_0^p + \int_0^t (\| u_m(s) \|^2 + \| v_m(s) \|^2) ds \leq C, \tag{18}
\]

where the constant \( C \) is independent of \( t \) and \( m \).

From (18) the functions \( u_m(t) \) and \( v_m(t) \) can be extended as a solution of the approximated problem, to the whole interval \([0,T]\).

From (18) we still can get that,

\[
\begin{align*}
(u_m), (v_m) \ &\text{ are bounded sequences in } L^\infty(0,T;W_0^p(\Omega)) \\
(u_m), (v_m) \ &\text{ are bounded sequences in } L^2(0,T;H_0^1(\Omega)) \\
(u_m), (v_m) \ &\text{ are bounded sequences in } L^\infty(0,T;L^2(\Omega))
\end{align*}
\tag{19}
\]

Furthermore, since the p-Laplacian \( A : W_0^{1,p}(\Omega) \rightarrow W^{-1,q}(\Omega) \) is a bounded operator ([6]), where \( \frac{1}{p} + \frac{1}{q} = 1 \) we obtain

\[
(Au_m), (Av_m) \ &\text{ are bounded sequences in } L^\infty(0,T;W^{-1,q}(\Omega)) \tag{20}
\]

Using the inequality ,

\[
\frac{1}{\rho + 2} \| u_m(t)v_m(t) \|_{L^{\rho + 2}(\Omega)} \leq \frac{1}{2p} \| u_m(t) \|_0^p + \frac{1}{2p} \| u_m(t) \|_0^p + C
\]

and the condition,

\[
p > 4(\rho + 2),
\]

we get,

\[
(u_m, v_m) \ &\text{ is a bounded sequence in } L^\infty(0,T;L^{\rho + 2}(\Omega)) \tag{21}
\]

**Estimate II.**

Using properly the projection operator, ([7]), the chain of Sobolev spaces

\[
H_0^0(\Omega) \hookrightarrow W_0^{1,p} \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow W^{-1,q} \hookrightarrow H^{-s}(\Omega)
\]

and both equations of the approximated problem, we get that,

\[
(u_m''), (v_m'') \ &\text{ are bounded sequences in } L^2(0,T;H^{-s}(\Omega)) \tag{22}
\]

**Passage to the Limit and Initial Conditions.**

The necessary results to carry out the passage to limit in the approximated
problem, and obtain a weak solution as in the Theorem, are, Aubin-Lions Compactness, and Lions’ Lemma ([6], ([7]). From these results and boundedness in (19) - (22) we obtain subsequences weak convergent and weak \( \star \) convergent that allow to pass to limit in the nonlinear term. Some properties of monotone operators,([6]), are necessary in order to complete the proof. The proof of initial conditions are standard and so we omit both them here.

References


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