Extensions of edge-coloured digraphs

HORTENSIA GALEANA-SÁNCHEZ
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Ciudad Universitaria
México, D.F. 04510

ROCÍO ROJAS-MONROY
Facultad de Ciencias
UAEMex
Instituto Literario No. 100, Centro
50000, Toluca, Edo. de México
MEXICO

Abstract: A digraph $D$ is said to be an $m$-coloured digraph, if its arcs are coloured with $m$ colours. A directed path (or a directed cycle) is called monochromatic if all of its arcs are coloured alike. A set $N \subseteq V(D)$ of vertices of $D$ is said to be a kernel by monochromatic paths of the $m$-coloured digraph $D$, if it satisfies the two following properties: (1) $N$ is independent by monochromatic paths; i.e. for any two different vertices $x, y \in N$, there is no monochromatic directed path between them, and (2) $N$ is absorbent by monochromatic paths; i.e. for each vertex $u \in V(D) - N$, there exists a $uv$-monochromatic directed path, for some $v \in N$. In this paper we present a method to construct a large variety of $m$-coloured digraphs with (resp. without a kernel) kernel by monochromatic paths; starting with a given $m$-coloured digraph $D_0$. A previous result is generalized.

Key–Words: kernel, kernel by monochromatic paths, $m$-coloured digraph

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1 Introduction

For general concepts we refer the reader to [1]. Let $D$ be a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of $D$ respectively. Let $S_1, S_2 \subseteq V(D)$, an arc $(u_1, u_2)$ of $D$ will be called an $S_1S_2$-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph of $D$ induced by $S_1$; $S_1 \subseteq V(D)$. A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel $N$ of $D$ is an independent set of vertices such that for each $z \in V(D) - N$ there exists a $zN$-arc in $D$.

A digraph $D$ is called a kernel-perfect digraph when every induced subdigraph of $D$ has a kernel. And $D$ is called a critical kernel imperfect digraph when $D$ has no kernel but every proper induced subdigraph of $D$ has a kernel.

A digraph $D$ is called an $m$-coloured digraph whenever its arcs are coloured with $m$ colours.

If $D$ is an $m$-coloured digraph; the closure of $D$, denoted $\mathcal{C}(D)$ is the $m$-coloured multidigraph defined as follows: $V(\mathcal{C}(D)) = V(D)$ and $(u, v) \in A(\mathcal{C}(D))$ with colour $i$ if and only if, there exists an $uv$-monochromatic directed path in $D$, coloured $i$.

Clearly $D$ has a kernel by monochromatic paths if and only if $\mathcal{C}(D)$ has a kernel. And $D$ has a kernel if and only if the $m$-coloured digraph $D$ (where every two different arcs have different colours) has a kernel by monochromatic paths.

In [4] was introduced the $s$-construction and was defined a digraph $s(S)$ associated to a given digraph $D_0$; in the same paper was proved that $s(S)$ has a kernel if and only if $D_0$ has a kernel. Also in [4] was proved that given a kernel-perfect digraph $D_0$, it is possible to construct a large variety of critical kernel imperfect digraphs containing $D_0$ as an induced subdigraph. More results concerning the existence of kernels in $s$-constructions and $s$-extensions of a given digraph $D_0$ can be found in [5] and [6].

Sufficient conditions for the existence of a kernel in a digraph have been investigated by several authors, namely Von Neumann and Morgenstern [13], Richardson [10], Duchet and Meyniel [2] and Galeana-Sánchez and Neumann-Lara [3]. The concept of a kernel is very useful in applications; and clearly the concept of a kernel by monochromatic paths generalizes those of a kernel. Sufficient conditions for the existence of a kernel by monochromatic paths in $m$-coloured digraphs have been investigated by several authors: for example in [12] Sands et al. proved that any 2-coloured digraph has a kernel by monochromatic paths; in [11] Shen Minggang proved that any $m$-coloured tournament in which each sub-tournament of order 3 is 2-coloured has a kernel by monochromatic paths. In [7] it was proved that if $T$ is an $m$-coloured tournament such that every directed cycle of length at most 4 is quasi-monochromatic then...
\( \mathcal{C}(T) \) is kernel perfect. A generalization of this result was obtained by Hahn, Illle and Woodrow in [9] they proved that if \( T \) is an \( m \)-coloured tournament such that every directed cycle of length \( k \) is quasi-monochromatic and \( T \) has no polychromatic directed cycles of length \( \ell, \ell < k \), for some \( k \geq 4 \); then \( T \) has a kernel by monochromatic paths. (A directed cycle is called quasi-monochromatic if with at most one exception all of its arcs are coloured alike, and a directed cycle is called polychromatic whenever its arcs allows at least three colours Kernels by monochromatic paths in bipartite tournaments were studied in [8], where it is proved that if \( T \) is a bipartite tournament such that every directed cycle of length \( 4 \) is monochromatic, then \( T \) has a kernel by monochromatic paths.

In this paper we define the \( s \)-construction which generalizes de \( s \)-construction. Also we define an \( m \)-coloured digraph \( s(S) \) related to a given \( m \)-coloured digraph \( D_0 \); and we prove that \( s(S) \) has a kernel by monochromatic paths if and only if \( D_0 \) has a kernel by monochromatic paths. This results generalizes the main result of [4] (\( s(S) \) has a kernel if and only if \( D_0 \) has a kernel).

2 Systems and Extensions

In this section we introduce the \( \bar{s} \)-construction. The main result concerning the \( \bar{s} - \text{construction} \) is Theorem 2.2 which enables us to generate a large class of \( m \)-coloured digraphs with (resp. without) kernel by monochromatic paths.

**Definition 2.1** Let \( D_0 \) be an \( m \)-coloured digraph. A \( 4 \)-tuple \( S_0 = (\mathcal{C}(D_0), U, U_+, U_-) \) will be called an \( \tilde{s}_0 \)-system (over \( D_0 \)) if it satisfies: (i) \( U, U_+ \) and \( U_- \) are sets of vertices with the same cardinality and \( U \subseteq V(D_0) \). (ii) \( V(D_0), U_+ \) and \( U_- \) are mutually disjoint sets. (iii) \( \mathcal{C}(D_0) \) is the closure of \( D_0 \). (iv) If \( u \in U \), then there is no monochromatic directed cycle contained in \( D_0 \), passing by \( u \).

**Lemma 2.1** [1]. Let \( D_0 \) be a digraph and \( C \) a closed directed walk in \( D_0 \). Then for each \( u \in V(C) \), there exists a directed cycle contained in \( C \), passing by \( u \).

**Lemma 2.2** Let \( D \) be an \( m \)-coloured digraph, \( \mathcal{C}(D) \) its closure and \( u \in V(D) \). There exists a monochromatic directed cycle, coloured \( i \), contained in \( D \) and passing by \( u \) if and only if there is a monochromatic directed cycle coloured \( i \), contained in \( \mathcal{C}(D) \) and passing through \( u \).

**Proof**: If \( C \) is a monochromatic directed cycle coloured \( i \), contained in \( D \), and passing by \( u \). Then from the definition of \( \mathcal{C}(D) \), clearly we have \( C \subseteq \mathcal{C}(D) \).

Now suppose that there exists a monochromatic directed cycle, \( \gamma \), coloured \( i \), contained in \( \mathcal{C}(D) \) and passing by \( u \). Let \( \gamma = (u = z_0, z_1, \ldots, z_n = u) \); from the definition of \( \mathcal{C}(D) \), we have that for each \( j \in 1, 2, \ldots, n; \) there exists a \( z_{j-1}z_j \)-monochromatic directed path coloured \( i \), clearly the union of these paths is a closed monochromatic directed walk contained in \( D \) and passing through \( u \). Thus from Lemma 2.1 there exists a directed cycle coloured \( i \), contained in \( D \) and passing by \( u \).

In what follows, if \( \tilde{S}_0 = (\mathcal{C}(D), U, U_+, U_-) \) is an \( s_0 \)-system, we shall denote by \( u_+ \) (resp. \( u_- \)) the vertex in \( U_+ \) (resp. \( U_- \)) which corresponds to \( u \in U \) for any fixed bijection from \( U \) to \( U_+ \) (resp. from \( U \) to \( U_- \)).

**Definition 2.2** If \( \tilde{S}_0 = (\mathcal{C}(D_0), U, U_+, U_-) \) is an \( \tilde{s}_0 \)-system, we denote by \( \tilde{s}_0(S_0) \) the digraph defined as follows: \( V(\tilde{s}_0(S_0)) = (V(D_0) - U) \cup U_+ \cup U_-; \) \( (z,w) \) is an arc of \( \tilde{s}_0(S_0) \) coloured \( i \) if and only if one of the following conditions holds:

(i) \( \{z,w\} \subseteq V(D_0) - U \) and there exists an arc from \( z \) to \( w \) in \( \mathcal{C}(D_0) \), coloured \( i \).

(ii) \( z \in V(D_0) - U, w = u_- \) for some \( u \in U; \) and there exists an arc from \( z \) to \( u \) in \( \mathcal{C}(D) \), coloured \( i \).

(iii) \( z = u_+ \) for some \( u \in U \), \( w \in V(D_0) - U; \) and there exists an arc from \( u \) to \( w \), in \( \mathcal{C}(D) \), coloured \( i \).

(iv) \( z = u_+ \) and \( w = v_- \), for some \( \{u,v\} \subseteq U; \) and there exists an arc from \( u \) to \( v \), in \( \mathcal{C}(D) \), coloured \( i \).

**Definition 2.3** An \( \bar{s} \)-system is a 4-tuple \( \bar{S} \) = \( (\tilde{S}_0, \bar{\beta}, \bar{U}_+, \bar{U}_-) \) where:

(i) \( \tilde{S}_0 = (\mathcal{C}(D_0), U, U_+, U_-) \) is an \( \tilde{s}_0 \)-system and \( \bar{U}_+ \) and \( \bar{U}_- \) are \( m \)-coloured multigraphs such that \( V(\bar{U}_+ + V(\bar{U}_-) = U_- \).

(ii) \( \bar{\beta} = \{\beta_u \mid u \in U\} \) is a set of mutually disjoint directed paths where each \( \beta_u \) is a \( u_-u_+ \)-directed path of positive even length and \( V(\beta_u) \cap V(\tilde{s}_0(S_0)) = \{u, u_+\} \).

(iii) for each \( u \in U; \) any two consecutive arcs of \( \beta_u \) have different colours and the arc of \( \beta_u \) which incides from \( u_- \) (resp. incides to \( u_+ \)) has a colour which is different from the colour of each arc which incides to \( u \) (resp. from \( u \)) in \( D_0 \).

**Definition 2.4** Let \( D_0 \) be an \( m \)-coloured digraph. If \( \bar{S} = (\tilde{S}_0, \bar{\beta}, \bar{U}_+, \bar{U}_-) \) is an \( \bar{s} \)-system, we denote by \( \bar{s}(\bar{S}) \) the edge-coloured multidigraph defined as follows: \( \bar{s}(\bar{S}) = \tilde{s}_0(S_0) \cup U_i \beta_u \cup \bar{U}_+ \cup \bar{U}_- \). The
multidigraph $\tilde{s}(\hat{S})$ will be called an extension of $D_0$. Notice that if $U = \emptyset$ then $\tilde{s}(\hat{S}) \cong \tilde{s}_0(\hat{S}_0) \cong C(D_0)$.

**Definition 2.5** We will say that the multidigraph $\tilde{s}(\hat{S})$ satisfies property $(A)$ if the following condition holds: If there exists an arc from $u_+$ to $u_-$ (resp. from $v_-$ to $v_+$) in $\tilde{U}_+$ (resp. in $\tilde{U}_-$) coloured $i$, then there exists an arc from $u$ to $v$ in $C(D_0)$, coloured $i$.

**Definition 2.6** We will denote by $g$ the function defined as follows:

$$g : V(\tilde{s}_0(\hat{S}_0)) \to V(D_0);$$

$$g(z) = \begin{cases} 
    u & \text{if } z = u_- \text{ or } z = u_+ \\
    z & \text{if } z \notin U_+ \cup U_-
\end{cases}.$$ 

**Theorem 2.1** Let $\hat{S} = (\tilde{s}_0, \tilde{\beta}, \tilde{U}_+, \tilde{U}_-) \text{ be an } \tilde{s}\text{-system where } \tilde{s}_0 = (C(D_0), U, U_+, U_-)$. Suppose that $\tilde{s}(\hat{S})$ satisfies property $(A)$. Then the following conditions hold.

1. Each $U_+ U_+$-directed path contained in $\tilde{s}(\hat{S})$ contains a directed path $\tilde{\beta}_u$, for some $u \in U$.
2. If $T$ is a monochromatic directed path contained in $\tilde{s}(\hat{S})$ of length at least two, then $V(T) \cap (\bigcup_{u \in U} V(\tilde{\beta}_u) - (U_+ \cup U_-)) = \emptyset$.
3. For any $u \in U$; there is no monochromatic directed path between $u_+$ and $u_-$ contained in $\tilde{s}(\hat{S})$.
4. If there exists a $zw$-monochromatic directed path coloured $i$, contained in $\tilde{s}(\hat{S})$ and with length at least two then $\{z, w\} \subset V(\tilde{s}_0(\hat{S}_0))$, and there exists a $g(z)g(w)$-directed path coloured $i$ (monochromatic) contained in $D_0$.
5. If $T$ is a $u_+u_-$-monochromatic directed path contained in $\tilde{s}(\hat{S})$ and $\beta_u = (u_+ = z_0, z_1, \ldots, z_n = u_-)$, then $z = z_1$ or $V(T) \subseteq U_-$.

**Proof:** (1) Let $T = (v = z_0, z_1, \ldots, z_n = v')$ a $vv'$-directed path with $v \in U_-$ and $v' \in U_+$. Denote by $j_0 = \min \{j \in \{0, 1, \ldots, n\} \mid z_{j+1} \notin U_-\}$ and by $j_1 = \min \{j \in \{0, 1, \ldots, n\} \mid j > j_0 \text{ and } z_j \in U_+\}$; thus $z_{j_0} \in U_-, z_{j_0+1} \notin U_-, z_{j_1} \in U_+$. Now we consider $T_0 = (z_{j_0}, T, z_{j_1})$ (the $z_{j_0}z_{j_1}$-directed path contained in $T$). Let $u \in U$ be such that $z_{j_0} = u_-$, from the definition of $\tilde{s}(\hat{S})$ we have that every arc of $\tilde{s}(\hat{S})$ starting in $u_-$ ends in $U_-$ or in the second vertex of $\tilde{\beta}_u$; since $z_{j_0+1} \notin U_-$ we have $z_{j_0+1} \in \tilde{\beta}_u$. We conclude that $\tilde{\beta}_u = T_0$. (as $z_{j_i} \in U_+$ we have $z_{j_i} = u_+$).

(2) Suppose that $T$ is a $zw$-directed path coloured $i$, contained in $\tilde{s}(\hat{S})$ with length at least two. Let $u \in U$ be and $x \in V(\tilde{\beta}_u) - \{u_+, u_-\}$. Since the arcs incident with $x$ belong to $\tilde{\beta}_u$; and thus they have different colours; we obtain $x \notin V(T) - \{z, w\}$. Now: if $x = z$, then the next vertex of $T$ must be $u_+$; and from definition of $\tilde{s}$-system, the colour of $(x, u_+)$ is different from the colour of the next arc of $T$; a contradiction (as $T$ is monochromatic). If $x = w$, then $(u_-, x) \in A(T)$. From the definition of $\tilde{s}$-system we have that $(u_-, x)$ has a colour which is different of the colour of any arc incident toward $u_-$; so $T$ is not monochromatic; a contradiction.

(3) Assume by contradiction that there exists a monochromatic directed path, between $u_+$ and $u_-$, for some $u \in U$, contained in $\tilde{s}(\hat{S})$; and let $T_0$ be such a directed path of minimum length. Let $\tilde{u} \in U$ be such that $\tilde{u}_+$ and $\tilde{u}_-$ are the terminals of $T_0$. Since $T_0$ is monochromatic, coloured, say $i$; it follows from the definition of $\tilde{s}(\hat{S})$ that $\ell(T_0) \geq 2$. Therefore from (2) we have: $V(T_0) \cap (\bigcup_{u \in U} V(\tilde{\beta}_u) - (U_+ \cup U_-)) = \emptyset$; the function $g$ is defined on $V(T_0)$; and $T_0$ contains no $\tilde{\beta}_u$ for every $u \in U$. Thus from (1) we have that $T_0$ is a $\tilde{u}_+ \tilde{u}_-$-directed path coloured $i$. From the choice of $T_0$, we have that for any $u \in U - \{\tilde{u}\}$, $\{u_+, u_-\} \not\subseteq V(T_0)$. Thus, function $g$ restricted to $V(T_0) - \{\tilde{u}_-, \tilde{u}_+\}$ is an injective function. If $T_0 = (\tilde{u}_+ = z_0, z_1, \ldots, z_n = \tilde{u}_-)$ then it follows from the definition of $\tilde{s}_0(\hat{S}_0)$ that for each $j \in \{1, \ldots, n\}$ there exists an arc from $g(z_{j-1})$ to $g(z_j)$ coloured $i$, contained in $C(D_0)$. Therefore $C = (u = g(z_0), g(z_1), \ldots, g(z_n) = u)$ is a monochromatic directed cycle, contained in $C(D_0)$, an passing through $u$; contradicting that $\tilde{S}_0$ is an $\tilde{s}$-system.

(4) Let $T$ be a $zw$-directed path, coloured $i$, contained in $\tilde{s}(\hat{S})$, and with $\ell(T) \geq 2$. From (2) we have $V(T) \cap (\bigcup_{u \in U} V(\tilde{\beta}_u) - (U_+ \cup U_-)) = \emptyset$; thus $V(T) \subseteq V(\tilde{s}_0(\hat{S}_0))$ and $g$ is defined for every vertex of $T$. Now: if $T = (z = z_0, z_1, \ldots, z_n = w)$ then from the definition of $\tilde{s}_0(\hat{S}_0)$ we have that for each $j \in \{1, \ldots, n\}$ there exists in $C(D_0)$, an arc coloured $i$, from $g(z_{j-1})$ to $g(z_j)$; therefore there exists a $g(z_{j-1})g(z_j)$-directed path coloured $i$ contained in $D_0$. We conclude that there exists a $g(z)g(w)$-directed path coloured $i$, contained in $D_0$.

(5) Let $T$ be a $u_+u_-$-monochromatic directed path, contained in $\tilde{s}(\hat{S})$ and assume that $\tilde{\beta}_u = (u_+ = z_0, z_1, \ldots, z_n = u_-)$. Assume by contradiction that $T \neq (u_-, z_1)$ and $V(T) \not\subseteq U_-$. Let $T = (u_+ = w_0, u_1, \ldots, w_m = z)$. First observe that $w_1 \in U_-$ (otherwise, from the definition of $\tilde{s}(\hat{S})$ we have $w_1 = z_1$; and the coloring of the arcs of $\tilde{\beta}_u$ implies $T = (u_-, z_1)$, contradicting our assumption). Denote by
there exists a N defined as follows: if

\( \ell(T) \geq 2 \), then the assertion follows from (2) and (4). If \( T = (z, w) \), then \( w \notin \bigcup_{u \in U} V(\tilde{\beta}_u - \{u_+\}) \) (as \( z \in (V(\tilde{s}_0(\tilde{S}_0)) - U_-) \) and \( w \notin (\bigcup_{u \in U} V(\tilde{\beta}_u - (U_+ \cup U_-)) \). So \( V(T) \cap (\bigcup_{u \in U} V(\tilde{\beta}_u - (U_+ \cup U_-)) = \emptyset \). It follows from that function \( g \) is defined for \( z \) and \( w \); and from the definition of \( \tilde{s}(\tilde{S}) \), there exists an arc coloured \( i \) from \( g(z) \) to \( g(w) \) in \( C(D_0) \) i.e. there exists a \( g(z)g(w) \)-directed path coloured \( i \) in \( D_0 \).

**Case 4(a).** \( \{z, w\} \subset V(\tilde{s}_0(\tilde{S}_0)) \).

From (4) in Theorem 2.1 and the definitions of \( \tilde{s}(\tilde{S}) \) and \( g \); we have that there exists in \( D_0 \) a \( g(z)g(w) \)-monochromatic directed path when \( \ell(T) \geq 2 \). When \( \ell(T) = 1 \) it follows from condition (A) that there exists a \( g(z)g(w) \)-monochromatic directed path in \( D_0 \). Since \( \{z, w\} \subset V(\tilde{s}_0(\tilde{S}_0))\), then \( \{z, w\} \subset (N_0 - U) \cup ((\bigcup_{u \in U} N_u) \cap (U_+ \cup U_-)) \). And we have three subcases:

Case 4(a.1). \( \{z, w\} \subseteq N_0 - U \).

In this case \( g(z) = z, g(w) = w \). And there exists a \( zw \)-monochromatic directed path in \( D_0 \); a contradiction (as \( N_0 \) is a kernel by monochromatic paths of \( D_0 \)).

Case 4(a.2). \( z \in (N_0 - U) \) and \( w \in ((\bigcup_{u \in U} N_u) \cap (U_+ \cup U_-)) \) (analogously \( z \in ((\bigcup_{u \in U} N_u) \cap (U_+ \cup U_-)) \) and \( w \in (N_0 - U) \).

Now \( w \in \{u_+, u_-\} \) for some \( u \in U \) and \( w \in N_0 \). From 1 and 2, \( \{u_-, u_+\} \subset N_u \setminus \{u_+, u_-\} \subset N \) and from (3) \( u \in N_0 \). Therefore there exists a \( zw \)-monochromatic directed path \( g(z) = z, g(w) = u \) in \( D_0 \), a contradiction.

Case 4(a.3). \( \{z, w\} \subset (\bigcup_{u \in U} N_u) \cap (U_+ \cup U_-)). \)

\( g(z) = u \) and \( g(w) = v \) for some \( u, v \subset N_0 \). If \( u = v \), then \( \{z, w\} = \{u_+, u_-\} \subset N_0 \cap U \). And there exists a monochromatic directed path in \( \tilde{s}(\tilde{S}) \) between \( u_- \) and \( u_+ \), contradicting (3) in Theorem 2.1. If \( u \neq v \), then there exists a uv-monochromatic directed path contained in \( D_0 \) and with \( \{u, v\} \subset N_0 \); a contradiction.

Case 4(b). \( \{z, w\} \cap V(\tilde{s}_0(\tilde{S}_0)) = \emptyset \).

In this case \( \{z, w\} \subseteq (\bigcup_{u \in U} N_u) - (U_+ \cup U_-) \).

Since each \( N_u \) is a kernel by monochromatic paths; if follows that \( z \in N_u \) and \( w \in N_v \), for some \( u, v \in U \), \( u \neq v \). Thus \( z \in (\tilde{\beta}_u) \) and \( w \in V(\tilde{\beta}_v) \). Clearly, \( z \notin \{u_+, u_-\} \) from the definition of \( \tilde{s}(\tilde{S}) \) we have \( \ell(T) = 1 \), moreover since the only arc which incides from \( z \) is in \( \beta_0 \), we have \( w \in V(\tilde{\beta}_v) \); contradicting that \( V(\tilde{\beta}_u) \cap V(\tilde{\beta}_v) = \emptyset \).

Case 4(c). \( z \notin V(\tilde{s}_0(\tilde{S}_0)) \) and \( w \in V(\tilde{s}_0(\tilde{S}_0)) \) (analogously the case \( z \in V(\tilde{s}_0(\tilde{S}_0)) \) and \( w \notin V(\tilde{s}_0(\tilde{S}_0)) \)).

In this case \( z \in N_u \setminus \{u_+, u_-\} \) for some \( u \in U \) and \( w \notin N_0 \cap U \).

From the definition of \( \tilde{s}(\tilde{S}) \) there is no monochromatic directed path between \( z \) and \( w \) in \( \tilde{s}(\tilde{S}) \); a contradiction.

5. N is absorbent by monochromatic paths.

Let \( x \in (V(\tilde{s}(\tilde{S})) - N) \); we will prove that there exists a \( x \)x-monochromatic directed path, for some \( x \in N \).

Case 5(a). \( x \in V(\tilde{s}_0(\tilde{S}_0)) \).

In this case \( g(x) = x \) and \( x \in (V(D_0) - N) \);
then there exists an \( xy \)-monochromatic directed path, for some \( y \in N_0 \). Thus there exists an arc from \( x \) to \( y \) in \( C(D_0) \). When \( y \notin U \), clearly we have in \( s(\tilde{S}) \) an arc from \( x \) to \( y \) with \( y \in N \). When \( y \in U \), then there exists an arc from \( x \) to \( y \_ \) in \( s(\tilde{S}) \), and since \( y \in N_0 \), we have \( y \_ \in N \).

Case 5(b). \( x \in V(\tilde{\beta}_u) - \{u_+, u_-\} \) for some \( u \in U \).

Since \( x \notin N \), then \( x \notin N_u \); and there exists an \( xz \)-monochromatic directed path for some \( z \in N_u \) (as \( N_u \) is a kernel by monochromatic paths of \( \beta_u \) or of \( (\tilde{\beta}_u) - \{u_+\} \)).

Case 5(c). \( x \in U_+ \) i.e. \( x = u_+ \) for some \( u \in U \).

Since \( x \notin N \); then we have \( u \notin N_0 \). From the definition of \( N_0 \); there exists an \( uy \)-monochromatic directed path in \( D_0 \), for some \( y \in N_0 \). Thus, there exists an arc from \( u \) to \( y \) in \( C(D_0) \). When \( y \notin U \); we have that there exists an \( s(\tilde{S}) \) an arc from \( u_+ \) to \( y \), and from the definition of \( N \); \( y \in N \). When \( y \in U \) we obtain that there exists an arc from \( u_+ \) to \( y \_ \) in \( s(\tilde{S}) \), and \( y \_ \in N \) (as \( y \in N_0 \)).

Case 5(d). \( x \in U_- \) i.e. \( x = u_- \) for some \( u \in U \).

Since \( x \notin N \), then from the definition of \( N \); we have \( x \notin N_u \) i.e. \( u \notin N_u \). Therefore \( N_u \) is not a kernel by monochromatic paths of \( \tilde{\beta}_u \) and then \( N_u \) is a kernel by monochromatic paths of \( \beta_u - \{u_+\} \).

Thus, there exists a \( u_-z \)-monochromatic directed path in \( \tilde{\beta}_u - \{u_+\} \), with \( z \in N_u \). N \).

We conclude from 4 and 5 that \( N \) is a kernel by monochromatic paths of \( s(\tilde{S}) \).

Now suppose that \( N \) is a kernel by monochromatic paths of \( s(\tilde{S}) \). First we prove the following assertion:

6. \( u_+ \in N \) if and only if \( u_- \in N \).

First suppose \( u_+ \in N \) and let \( \tilde{\beta}_u = (u_- = z_0, z_1, \ldots, z_n = u_+) \), it follows from the definition of \( \tilde{s} \)-system that \( z_1 \notin N \). Now assume by contradiction \( u_- \notin N \). Since \( N \) is a kernel by monochromatic paths of \( s(\tilde{S}) \), then there exists a \( u_-z \)-monochromatic directed path, say, \( T \), in \( s(\tilde{S}) \) for some \( z \in N \); since \( z_1 \notin N \) it follows from (5) Theorem 2.1 that \( V(T) \subseteq U_- \). Therefore the function \( g \) restricted to \( V(T) \) is injective and from the condition (A) we have that there exists a \( g(u_-)g(z) \)-monochromatic directed path contained in \( C(D_0) \). It follows that there exists an arc from \( g(u_-) \) to \( g(z) \) in \( C(D_0) \), and from the definition of \( s(\tilde{S}) \), there exists an arc from \( u_+ \) to \( \tilde{u} \_ \) in \( s(\tilde{S}) \) with \( \tilde{u} \in U \); \( z = \tilde{u} \_ \); i.e. there exists an arc from \( u_+ \) to \( z \) in \( s(\tilde{S}) \) with \( \{u_+, z\} \subseteq N \); a contradiction.

Now suppose \( u_- \in N \); and let \( \tilde{\beta}_u = (u_- = z_0, z_1, \ldots, z_n = u_+) \). Since \( N \) is independent by monochromatic paths, we have \( z_1 \notin N \). Now \( z_2 \notin N \) (from the definition of \( \tilde{s} \)-system), and \( z_3 \notin N \); continuing this way, we get \( u_+ \in N \).

Now we will prove that \( N_0 = \{g(z) \mid z \in N - (\bigcup_{u \in U} V(\tilde{\beta}_u) - (U_+ \cup U_-))\} \) is a kernel by monochromatic paths of \( D_0 \).

7. \( N_0 \) is independent by monochromatic paths in \( D_0 \).

Let \( z, w \in (N - (\bigcup_{u \in U} \tilde{\beta}_u) - (U_+ \cup U_-)) \) be such that \( g(z) = x \) and \( g(w) = y \). Assume by contradiction that there exists an \( xy \)-monochromatic directed path in \( D_0 \); this implies that there exists an arc from \( x \) to \( y \) in \( C(D_0) \).

We will analyze the following four possible cases:

7(a). \( x \notin U \) and \( y \notin U \).

In this case \( z = x \), \( w = y \). From definition of \( s(\tilde{S}) \); there exists an arc from \( z \) to \( w \) in \( s(\tilde{S}) \) with \( \{z, w\} \subseteq N \); a contradiction.

7(b). \( x \in U \) and \( y \notin U \).

Now, \( w = y \) and we may assume (from 6) that \( z = x_+ \). From definition of \( s(\tilde{S}) \); there exists an arc from \( x_+ \) to \( y \) in \( s(\tilde{S}) \) i.e. \( z \) to \( w \) with \( \{z, w\} \subseteq N \); a contradiction.

7(c). \( x \notin U \), \( y \in U \).

In this case \( z = x \) and from 6 we may assume \( w = y \). From definition of \( s(\tilde{S}) \); there exists an arc from \( z \) to \( w \) with \( \{z, w\} \subseteq N \); a contradiction.

7(d). \( x \in U \), \( y \in U \).

From 6 we may assume \( z = x_+ \) and \( w = y \). From definition of \( s(\tilde{S}) \); there exists an arc from \( z \) to \( w \) with \( \{z, w\} \subseteq N \); a contradiction.

8. \( N_0 \) is absorbent by monochromatic paths in \( D_0 \).

Let \( x \in (V(D_0) - N_0) \) be; we will prove that there exists an \( xy \)-monochromatic directed path in \( D_0 \), for some \( y \in N_0 \).

8(a). \( x \notin U \).

In this case \( x \in (V(s_0(\tilde{S}_0)) - (U_+ \cup U_-)) \) and then \( g(x) = x \). Since \( x \notin N_0 \), then \( x \notin N \); and there exists an \( xw \)-monochromatic directed path, say, \( T \), in \( s(\tilde{S}) \), for some \( w \in N \). Now from (6) in Theorem 2.1 we have \( V(T) \cap (\bigcup_{u \in U} V(\tilde{\beta}_u) - (U_+ \cup U_-)) = \emptyset \) and there exists a \( g(x)g(w) \)-monochromatic directed path in \( D_0 \). Thus \( g(w) \in N_0 \) (recall \( w \in N \)), and there exists an \( xy \)-monochromatic directed path in \( D_0 \), with \( y \in N_0 \); \( y = g(w) \).

8(b). \( x \in U \).

Since \( x \notin N_0 \), then \( x_+ \notin N \) and \( x_- \notin N \). Therefore there exists an \( x_+w \)-monochromatic directed path in \( s(\tilde{S}) \), say \( T \), for some \( w \in N \). From (6) in Theorem 2.1, we have \( V(T) \cap (\bigcup_{u \in U} V(\tilde{\beta}_u) - (U_+ \cup U_-)) = \emptyset \) and there exists a \( g(x_+)g(w) \)-monochromatic directed path in \( D_0 \), (as \( x_+ \notin U_- \)); moreover \( g(w) \in N_0 \) (because \( w \in N \)). We conclude
that there exists an $xy$-monochromatic directed path in $D_0$ with $y = g(w) \in N_0$.

References:


