Closed Form Solution for Dynamic Fund Protection under CEV

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Abstract: This paper studies the pricing of dynamic fund protection (DFP) in insurance contracts under the constant elasticity of variance (CEV) model. As empirical studies document that stock price volatility is not a constant but changes randomly, CEV model has been a popular alternative to model stock price movements in practice. We derive the first closed form solution to DFP under CEV in terms of Laplace transform.

Key–Words: Dynamics Fund Protection, Constant Elasticity of Variance, Option Pricing, Laplace Transform

1 Introduction

The concept of dynamic fund protection (DFP) in insurance was introduced by Gerber and Shiu (1999). DFP ensures that the fund value is upgraded if it goes below a pre-specified guarantee level at any time during the life of the contract. DFP can be extended to incorporate the performance of a financial index. Gerber and Shiu (2003) considered perpetual equity index annuity with dynamic protection, where the guarantee level is another stock index.

There is a close link between DFP and lookback options. Using lookback option pricing techniques, Imai and Boyle (2001) derived the mid-contract valuation under the Black-Scholes (1973) assumption. Under the Black-Scholes model, Chu and Kwok (2004) investigated the reset and withdrawal rights in DFP. They showed that the proposed scheme of Gerber and Shiu (2003) is related to a lookback payoff.

There is empirical evidence against the price dynamics assumed in the original Black Scholes (BS) model and in particular there is strong empirical evidence that the volatility is not constant but stochastic. As volatility “smile” is commonly observed in financial markets, various methods are proposed to capture it. One popular way considers that the asset price evolves as a CEV model proposed by Cox (1975). The implications of the CEV model are studied by Becker (1980). It has been shown that this model can fit the volatility skew of equity options. Emanuel and MacBeth (1982) derive a closed form solution for European option under a positive elasticity. Davydov and Linetsky (2001) study the pricing and hedging of path-dependent options under CEV.

In the valuation of DFP, Imai and Boyle (2001) provide a finite element method and a Monte Carlo simulation for the CEV model. However, no closed form solution is obtained so far. Wong and Chan (2006) point out that DFP can actually be viewed as a quanto lookback option. The notion of quanto lookback option can be found in Dai, Wong and Kwok (2004). We shall see shortly that this provides an important insight into pricing DFP in closed form.

In this paper, we demonstrate the use of replicating portfolio approach of Wong and Kwok (2003) in replicating DFP as a quanto lookback option under CEV. Then, we obtain the closed form solution in terms of Laplace transform. Valuing exotic options with Laplace transform is very common in the finance literature. For instance, Davydov and Linetsky (2001) used Laplace transform to value barrier and lookback options under CEV. The computation usually takes less than 1 second.

The remainder of this paper is organized as follows. In Section 2, we give a brief introduction to the option pricing under CEV model. In Section 3, we apply the model to value DFP and derive the closed form solution. Section 4 concludes.

1.1 Option Pricing under CEV

The CEV model assumes that, under the risk-neutral probability measure \( \mathbb{Q} \), the underlying asset price evolves according to the stochastic differential equation:

\[
\frac{dS}{S} = (r - q)dt + \delta S^{\alpha} \, dZ,
\]

where \( r \) is the risk-free rate, \( q \) is the dividend yield, \( Z \) is a Wiener process, \( \delta \) is known as the scale parameter and \( \alpha \) is constant elasticity parameter. The CEV model nests several asset price processes as special cases. For instance, it incorporates a local volatility given by \( \sigma(S) = \delta S^{\alpha - 1} \). The notion of
constant elasticity can then be interpreted through 
\( (S/\sigma)(\partial \sigma/\partial S) = \alpha - 1 \). When \( \alpha = 1 \), the process (1) becomes the geometric Brownian motion, which is the Black-Scholes model. We have the Ornstein-Uhlenbeck process by setting \( \alpha \) to zero.

There exist interesting interpretations for the CEV model. When \( \alpha < 1 \), the local volatility increases as the stock price decreases. This produces a probability distribution similar to that observed for equities with a heavy left tail and less heavy right tail. When \( \alpha > 1 \), the local volatility increases as the stock price increases. This creates a probability distribution with a heavy right tail and a less heavy left tail. This corresponds to a volatility smile where the implied volatility is an increasing function of the strike price. This type of volatility smile often appears for options on futures.

Under the CEV model, the closed-form pricing formulas for European call and put options are available. Specifically, Cox (1975) shows that

\[
c = S_0 e^{-rT} [1 - \chi^2(a, b + 2, c)] - Ke^{-rT} \chi^2(c, b, a),
\]

\[
p = Ke^{-rT} [1 - \chi^2(c, b, a)] - S_0 e^{-rT} \chi^2(a, b + 2, c),
\]

when \( \alpha < 1 \), and Emanuel and MacBeth (1982) derive that

\[
c = S_0 e^{-qT} [1 - \chi^2(c, -b, a)] - Ke^{-rT} \chi^2(a, 2 - b, c),
\]

\[
p = Ke^{-rT} [1 - \chi^2(a, 2 - b, c)] - S_0 e^{-qT} \chi^2(c, -b, a),
\]

when \( \alpha > 1 \), with

\[
a = \frac{[Ke^{-(r-q)T}]^{2(1-\alpha)}}{(1-\alpha)^2 w},
\]

\[
b = \frac{1}{1-\alpha},
\]

\[
c = \frac{S^{2(1-\alpha)}}{(1-\alpha)^2 w},
\]

where

\[
w = \frac{\delta^2}{2(r-q)(\alpha-1)} \left[e^{2(r-q)(\alpha-1)T} - 1\right].
\]

and \( \chi^2(z, k, w) \) is the cumulative distribution function (cdf) of a noncentral chi-square random variable with noncentrality parameter \( w \) and \( k \) degrees of freedom.

The CEV model can generate volatility smile consistent with the market. The volatility smile is a curve on the implied volatility-moneyness plan, where the moneyness stands for \( K/S \) in the standard call or put option. The curve shows a downward sloping shape with skewing up in the right tail. Figure 1 shows several typical volatility smiles from S&P 500 index option data, where the \( x \)-axis represents the log-moneyness-to-maturity ratio and the \( y \)-axis is the implied volatility. Implied volatility is obtained by solving the volatility from the equation that sets a market observed option price equal to the Black-Scholes formula. It is the market practice to quote implied volatility instead of the option price itself.

Figure 2 shows the implied volatility skew generated by the CEV model. The graph is produced in the following way. We first regard CEV as the correct model and produce option prices. Then, we solve the implied volatility by matching the CEV-price with the Black-Scholes ones. To generate different slopes, the local volatility is fixed to be 20%, i.e. \( 20\% = \delta S(0)^{\alpha-1} \), and then vary the \( \alpha \). It can be seen that the CEV model can capture most parts of the volatility smile except the right tail.

2 Dynamic Fund Protection

Dynamic fund protection (DFP) guarantees a predetermined protection level to an investor who owns the underlying fund. Let \( K \) denote the constant protec-
The payoff of a fund holder is then given by, [see Imai and Boyle (2001)]

\[ S(T) \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{K}{S(\tau)} \right\}, \]

where \( S(t) \) is the value of the fund without the protection. Hence, the terminal payoff for DFP should be

\[ \text{DFP}(T) = S(T) \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{K}{S(\tau)} \right\} - S(T). \]  

Our aim is to determine the fair present value of DFP.

We recognize that the payoff resembles that of a quanto lookback option. To see this, we introduce the following variables

\[ S_F(t) = K/S(t) \quad \text{and} \quad M_F^\tau = \max_{0 \leq \tau \leq t} S_F(\tau). \]  

The payoff then becomes

\[ \text{DFP}(T) = S(T) \max(1, M_F^T) - S(T) \]

\[ = S(T) \max(M_F^T - 1, 0). \]  

If we view \( S \) as an exchange rate and hence \( S_F \) as an asset trading in the foreign currency world, then the payoff (5) represents a fixed strike lookback call on \( S_F \) with a unity strike price trading in the foreign currency world and is then translated back to the domestic currency by the exchange rate \( S(T) \). Hence, this option can simply be valued as the fixed strike lookback call in the foreign currency world followed by multiplying the exchange rate \( S(t) \). A fixed strike lookback call option on an asset \( S \) has the payoff function:

\[ c_{fix}(T, S_T, M_t^T; K) = \max(M_t^T - K, 0), \]

where \( M_t^T = \max_{0 \leq \tau \leq T} S_t \) and \( K \) is a constant strike price. Thus, we established a model-independent result as follows.

**Proposition 1.** The present value of DFP relates to the fixed strike lookback call option by

\[ \text{DFP}(t) = S(t) \times c_{fix}^\text{Q}_F(t, S_F, M_F^t), \]  

where \( c_{fix}^\text{Q}_F(t, S_F, M_F^t) \) is the fixed strike lookback call in the foreign currency world (or under the \( \text{Q}_F \)-measure), and \( S_F \) and \( M_F^t \) are defined in (7).

To value DFP, we concentrate on the process of \( S_F \) in the foreign currency world. As the process of the naked fund \( S(t) \), which is viewed as an exchange rate, follows the CEV model of (1), the interest rate \( r \) in (1) is the domestic interest rate, while the dividend yield \( q \) corresponds to the foreign interest rate. In the foreign currency world, \( S_F \) represents the exchange rate being translated from the domestic currency to the foreign currency because it is proportional to \( 1/S(t) \). Define \( \text{Q}_F \) to be the equivalent martingale (or risk-neutral) measure with respect to the numeraire \( S(T)e^{-qt} \).

**Proposition 2.** Under the measure \( \text{Q}_F \), the process of \( S_F \) is given by

\[ dS_F = (q - r)S_F dt + \delta_F S_F^\beta dW^F_t, \]  

where

\[ W^F_t = \int_0^t \delta S^{\alpha - 1}(\tau) d\tau - W_t, \]

\[ \delta_F = \delta K^{\alpha - 1}, \beta = 2 - \alpha. \]  

**Proof.** We restrict ourselves to the well-defined CEV diffusion processes in which the local volatility satisfies the Novikov condition. In other words, we should use an appropriate value of \( \alpha \) such that the local volatility is squared-integrable. Davydov and Linetsky (2001) gives possible choice of \( \alpha = 1 \).

Applying Itô’s lemma to \( S_F = K/S \), we obtain the stochastic differential equation (SDE) for \( S_F \):

\[ \frac{dS_F}{S_F} = - \left[(r - q)dt + \delta S^{\alpha - 1} dW_t - \delta^2 S^{2(\alpha - 1)} dt \right] = (q - r)dt + \delta S^{\alpha - 1} \left[ \delta S^{(\alpha - 1)} dt - dW_t \right]. \]

After recognizing \( S_F = K/S \) and defining \( \delta_F = \delta K^{\alpha - 1} \), the SDE becomes

\[ \frac{dS_F}{S_F} = (q - r)dt + \delta_F S_F^{1 - \alpha} \left[ \delta S^{(\alpha - 1)} dt - dW_t \right]. \]

As the original process of \( S(t) \) satisfies the Novikov conditions, an application of the Girsanov theorem ensures that

\[ W^F_t = \int_0^t \delta S^{\alpha - 1}(\tau) d\tau - W_t \]

is a Weiner process under the equivalent martingale measure \( \text{Q}_F \) of which \( S(T)e^{-qt} \) is the numeraire.

When \( \alpha = 1 \), the asset price dynamics is reduced to the standard BS dynamics. In such a situation, DFP can be immediately valued as

\[ \text{DFP}(t) = F(t) \times c_{fix}^{BS}(t, S_F, M_F^t; r \leftrightarrow q), \]
where $c_{tr}^{BS}$ is the Black-Scholes formula for the fixed strike lookback call and the notation $r \leftrightarrow q$ means that the roles of $r$ and $q$ are interchanged with each other. It is easy to verify that the above formula exactly equals the mid-contract valuation formula for DFP obtained by Imai and Boyle (2001).

It is now clear that the key of pricing DFP is to calculate the fixed strike lookback call under the $Q_F$ measure. We consider the following calculation:

$$e^{q(T-t)}c_{tr}^{BS}(t,S_F,M_T^F) = e^{-qT}E^{Q_F}[\max(M_T^F-1,0)] = \begin{cases} E^{Q_F}\left[\int_0^{\infty} 1_{\{M_T^F > 0\}} d\xi\right], & M_T^F \leq 1 \\ E^{Q_F}\left[M_T^F - 1 + \int_0^{\infty} 1_{\{M_T^F > 0\}} d\xi\right], & M_T^F > 1 \end{cases}$$

(9)

Thus, the key of determining the fixed strike lookback call price is to find an explicit expression for the probability $Q_F(M_T^F > \xi)$.

Given the process of $S_F$ in (7), we can use a result of Davydov and Linetsky (2001) to write down the probability in Laplace transform. Let $\tau = T - t$ be the remaining life of the DFP and $L_{\tau,\lambda}$ be the Laplace transform operator with respect to $\tau$. Then, we have

$$L_{\tau,\lambda}\left\{Q(M_T^F > \xi)\right\} = \frac{1}{\lambda} \psi_\lambda(S_F),$$

(10)

where $\lambda > 0$ is the independent variable of the Laplace transform, and

$$\psi_\lambda(S_F) = \begin{cases} S_F^{\frac{1}{2}} e^{\frac{1}{2}\lambda} M_{k,\tilde{m}}(\tilde{x}), & \beta < 1, \ r - q \neq 0, \\ S_F^{\frac{1}{2}} e^{\frac{1}{2}\lambda} W_{k,\tilde{m}}(\tilde{x}), & \beta > 1, \ r - q \neq 0, \\ S_F^{\frac{1}{2}} I_{V} \left(\sqrt{2\lambda \tilde{z}}\right), & \beta < 1, \ r - q = 0, \\ S_F^{\frac{1}{2}} K_{\nu} \left(\sqrt{2\lambda \tilde{z}}\right), & \beta > 1, \ r - q = 0, \end{cases}$$

(11)

$$\tilde{x} = \frac{|r - q|}{\delta_F^2 |\beta - 1|} S_F^{2(\beta-1)}, \quad \tilde{z} = \frac{1}{\delta_F |\beta - 1|} S_F^{1-\beta},$$

$$\tilde{\theta} = \text{sign}((\beta - 1)(q-r)), \quad \tilde{\nu} = \frac{1}{2|\beta - 1|},$$

$$\tilde{m} = \frac{1}{4|\beta - 1|},$$

(12)

The functions $M_{k,\tilde{m}}(x)$ and $W_{k,\tilde{m}}(x)$ are the Whittaker functions, and $I_{V}(x)$ and $K_{\nu}(x)$ are the modified Bessel functions.

We summarize the procedure of obtaining the CEV-DFP price. Firstly, the probability is obtained by inverting the Laplace transform in (10). Secondly, the probability is substituted into (9). A numerical integration should be carried out in this step. Finally, the DFP is priced with Proposition 1.

### 3 Conclusion

Dynamic fund protection is a possible protection scheme in insurance contract. As empirical studies reveal that the Black-Scholes dynamics is inadequate to describe the evolution of asset prices, CEV has become a popular alternative process in finance and insurance. We investigated the pricing of DFP using CEV and derived the first closed form solution in terms of Laplace transform.

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**References:**


