REDUCTION OF THE CODIMENSION FOR DEGENERATE SUBMANIFOLDS

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Abstract. We give in this paper sufficient conditions for \( r \)-lightlike submanifolds \( M \) of dimension \( m \), which is not totally geodesic in an \((m+n)\)-dimensional semi-Riemannian manifold of constant curvature \( c \) to admit a reduction of codimension. We consider proper \( r \)-lightlike, coisotropic and totally lightlike submanifolds, generalizing thus previous results on isotropic submanifolds [1] as well as in the Riemannian case developed in [2, 5, 10].

1. Introduction and basic facts

This paper deals with the reduction of the codimension of lightlike submanifolds in semi-Riemannian manifolds. Assume \((M, g)\) is an \( m \)-dimensional \( r \)-lightlike submanifold which is not totally geodesic in an \((m+n)\)-dimensional \((n \neq m)\) semi-Riemannian manifold of constant curvature \( c \). The reduction of the codimension consists of finding a sufficient condition for \( M \) to be immersed into an \((m+p)\)-dimensional totally geodesic submanifold of constant curvature, where \( p < n \). The substantial codimension is then the smallest codimension that an immersion can be reduced to. We generalize results obtained on the subject when the ambient space is Riemannian [5, 10] and the ones obtained in the semi-Riemannian case [1] where lightlike isotropic submanifolds have been considered. We also give a sufficient condition for a totally umbilical coisotropic submanifold [4] of pseudo-Euclidean space to admit a reduction of codimension.

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Reduction of codimension is often used in geometry. The following classical property of curves in Euclidean $n$-space $\mathbb{R}^n$ is a motivating example for our study. Consider a curve $c : (a, b) \to \mathbb{R}^n$. Suppose for $j < n$, its curvatures $k_1, \ldots, k_{j-1}$ do not vanish and $k_j$ is identically null. It is well known that $c$ is then contained in a $j$-dimensional affine subspace. From a physics point of view, the universe we live in is usually represented as a 4-dimensional subspace embedded into a $(4 + d)$-dimensional spacetime. This idea has attracted and still attracts the attention of many physicists and cosmologists. Also, the imbedding of the exact solutions of Einstein equations into higher dimensional semi-Euclidean space is expected to provide a better understanding of their intrinsic geometry. In both cases, the problem to be solved is to find out the lowest codimension of the imbedding under consideration in order to obtain a theoretical framework in which fundamental laws of physics might present some unification. The Kaluza-Klein scheme that takes into account the mutual interaction between matter and metric is a stimulating example. Remind that the smallest codimension which that of an immersion can be reduced to is referred to as the substantial codimension. The present paper aims to furnish a contribution to studies in those directions. It is organized as follows. We give in the preliminaries in section 2, basic formulas concerning geometric objects on lightlike submanifolds, which is now the classic reference in this subject. Proofs of the main results are given in section 3 and finally we construct examples to illustrate our motivations in section 4.

2. Preliminaries

2.1. Preliminary Recollections. For the convenience of the reader, we start with an overview of the geometry of lightlike submanifolds, using notations and results of [3]. The fundamental difference between the theory of lightlike (or degenerate) submanifolds $(M, g)$, and the classical theory of submanifolds of a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ comes from the fact that

$$\text{Rad} TM = TM \cap TM^\perp \neq \{0\}. \quad (1)$$
Given an integer \( r > 0 \), the submanifold \( M \) is said to be \( r \)-lightlike (or \( r \)-degenerate) if the rank of \( \text{Rad}(TM) \) is equal to \( r \) everywhere. We have four cases of lightlike submanifolds:

- The proper \( r \)-lightlike submanifolds, where \( 0 < r < \min(m, n) \). In this case, we have \( \text{Rad}(TM) \subseteq TM \) and \( \text{Rad}(TM) \subseteq TM^\perp \) then there exist non-degenerate screen distributions \( S(TM) \) and \( S(TM^\perp) \), complementary vector subbundle to \( \text{Rad}(TM) \) in \( TM \) and in \( TM^\perp \) respectively such that,

\[
TM = \text{Rad}(TM) \perp S(TM),
\]

\[
TM^\perp = \text{Rad}(TM) \perp S(TM^\perp).
\]

The subbundle \( S(TM^\perp) \) is called transversal screen distribution. Let \( tr(TM) \) and \( ltr(TM) \) be complementary vector bundles to \( TM \) in \( TM \) and to \( \text{Rad}(TM) \) in \( S(TM^\perp) \), respectively. Then we have

\[
\left. TM \right|_M = TM \oplus tr(TM)
\]

\[
= S(TM) \perp S(TM^\perp) \perp (\text{Rad}(TM) \oplus ltr(TM))
\]

where

\[
tr(TM) = ltr(TM) \perp S(TM^\perp)
\]

- The coisotropic submanifolds, with \( 1 \leq r = n < m \). In this case, we have The relation (2) becomes

\[
\left. TM \right|_M = TM \oplus ltr(TM) = S(TM) \perp (\text{Rad}(TM) \oplus ltr(TM))
\]

- The isotropic submanifold case, with \( 1 \leq r = m < n \) In this case, \( \text{Rad}(TM) = TM \subsetneq TM^\perp \) and \( S(TM) = \{0\} \). The relation (2) is expressed as

\[
\left. TM \right|_M = TM \oplus tr(TM) = (TM \oplus ltr(TM)) \perp S(TM^\perp).
\]

Null curves are examples of isotropic submanifolds.

- The totally lightlike submanifolds, where \( 1 < r = n = m \). We have in this case \( \text{Rad}(TM) = TM = TM^\perp \), \( S(TM) = S(TM^\perp) = \{0\} \) and

\[
\left. TM \right|_M = TM \oplus ltr(TM).
\]
Null curves of two dimensional manifolds are examples of totally lightlike submanifolds.

2.2. The Induced Connection. Let $\nabla$ be the Levi-Civita connection on $M$. Then we have

\[ \nabla_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM) \]  

and

\[ \nabla_X V = -A_V X + \nabla^t_X V \quad \forall X \in \Gamma(TM), \quad V \in \Gamma(tr(TM)) \]  

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ are in $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. We suppose $S(TM^\perp) \neq \{0\}$ and we denote by $L$ and $S$ the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ respectively. Using the relation (3), relations (7) and (8) become respectively

\[ \nabla_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall X, Y \in \Gamma(TM) \]  

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$ and

\[ \nabla_X V = -A_V X + D^l_X V + D^s_X V \]  

$\forall X \in \Gamma(TM), \quad \forall V \in \Gamma(tr(TM))$, where

\[ D^l_X V = L(\nabla^l_X V) \quad D^s_X V = S(\nabla^s_X V). \]

Then we have for all $X \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$

$\nabla^l_X (LV) = D^l_X (LV)$,  
$\nabla^s_X (SV) = D^s_X (SV)$,  
$D^l(X, SV) = D^l_X (SV)$  
and  
$D^s(X, LV) = D^s_X (LV)$

The applications $\nabla^l$ and $\nabla^s$ are linear connections on $ltr(TM)$ and $S(TM^\perp)$, respectively. We call them respectively lightlike connection and the screen transversal connection on $M$. Relation (10) can also be written as

\[ \nabla_X V = -A_V X + D^l(X, SV) + D^s(X, LV) + \nabla^l_X (LV) + \nabla^s_X (SV). \]
These geometric objects verify the following relations [3, p. 156]:

(11) \[ \mathfrak{g}(h^s(X, Y), W) + \mathfrak{g}(Y, D^l(X, W)) = \mathfrak{g}(A_W X, Y) \]

(12) \[ \mathfrak{g}(h^l(X, Y), \xi) + \mathfrak{g}(Y, h^l(X, \xi)) + \mathfrak{g}(Y, \nabla_X \xi) = 0 \]

(13) \[ \mathfrak{g}(W, D^s(X, N)) = \mathfrak{g}(A_W X, N) \]

(14) \[ \mathfrak{g}(A_N X, N') = \mathfrak{g}(A_{N'} X, Y) \]

(15) \[ \mathfrak{g}(A_N X, P Y) = \mathfrak{g}(N, \nabla_X P Y) \]

(16) \[ h^l_i(X, \xi_j) = h^l_j(X, \xi_i) \]

where

\[ X, Y \in \Gamma(TM), \quad N \in \Gamma(ltr(TM)), \quad \xi \in \Gamma(Rad(TM)), \quad W \in \Gamma(S(TM^\perp)) \]

and \( h^l_i \) are such that \( h^l_i(X, Y) = g(\nabla_X Y, \xi_i) \). Then \( h^l_i \) does not depend on the choice of \( S(TM) \), \( S(TM^\perp) \) and \( ltr(TM) \) and are zero on \( Rad(TM) \). Consequently the second fundamental form \( h^l \) is identically equal to zero on an isotropic and on a totally lightlike submanifolds.

So, \( D^l \) is a shape application form of \( r \)-lightlike and isotropic submanifolds.

We have

(17) \[ (\nabla_X g)(X, Y) = \mathfrak{g}(h^l(X, Y), Z) + \mathfrak{g}(h^l(X, Z), Y) \]

(18) \[ (\nabla_X^l \mathfrak{g})(V, V') = -(\mathfrak{g}(A_V X, V') + \mathfrak{g}(A_{V'} X, V)). \]

Hence, the induced connections \( \nabla \) and \( \nabla^l \) are not metric in general. As a consequence, we have

(1) The induced connection \( \nabla \) of the Levi-Civita connection \( \overline{\nabla} \) of \((M, \overline{g})\) is metric on isotropic and totally lightlike submanifolds \((M, g)\).

(2) A proper \( r \)-lightlike or a coisotropic submanifold \((M, g)\) admits a metric connection if and only if \( h^l \) vanishes identically on \( M \).

Let \( f : M^m \rightarrow M^{m+n} \) be an isometric immersion of an \( m \)-dimensional \( r \)-lightlike \((1 \leq r \leq \min(m, n))\) submanifold into an \((m + n)\)-dimensional semi-Riemannian manifold. The first transversal space of \( f \) at \( x \in M \) is the subspace

\[ T_1(x) = \text{span}\{h^l(X, Y) + h^s(X, Y), \quad X, Y \in T_x M\}, \quad x \in M \]
Proposition 2.1. Suppose that the induced connection $\nabla$ on $M$ is a metric connection, then the first transversal space of $f$, $T_1(x)$ is characterized by

$$T_1(x) = \{ W \in S(T_x M) \subset tr(T_x M), D^l(., W) = 0 \text{ and } A_W = 0 \}^\perp$$

for all $x \in M$

Proof
Recall that $\nabla$ is metric $\iff h^l = 0$ and we have

$$T_1(x) = \text{span}\{ h^s(X, Y) \mid X, Y \in T_x M \} \quad x \in M.$$ 

Suppose

$$N(x) = \{ W \in S(T_x M) \subset tr(T_x M), D^l(., W) = 0 \text{ and } A_W = 0 \}^\perp.$$ 

Let $V \in T_1(x)$ and $W \in N^\perp(x)$ such that $V = h^s(X, Y)$.

$$\overline{g}(V, W) = \overline{g}(h^s(X, Y), W) = \overline{g}(A_W X, Y) - \overline{g}(D^l(X, W), Y) = 0.$$ 

We have for every $V \in T_1(x)$, $\overline{g}(V, W) = 0$, $\forall W \in N^\perp(x)$ and $\forall x \in M$. Then $V$ lies in $(N^\perp(x))^\perp = N(x)$, $T_1(x) \subset N(x)$.

Conversely, taken $V \in T_1^\perp(x)$ such that $\forall X, Y \in T_x M$

$$\overline{g}(h^s(X, Y), V) = \overline{g}(A_V X, Y) - \overline{g}(D^l(X, V), Y) = 0.$$ 

If $Y \in \text{Rad}(T_x M)$, then $\overline{g}(D^l(X, V), Y) = 0$ and $D^l(X, .) = 0$ for all $X$.

If $Y \in S(T_x M)$, then $\overline{g}(A_V X, Y) = 0$ and $A_V = 0$.

Hence $V \in N^\perp(x)$ and $N(x) = (N^\perp)^\perp(x) \subset T_1(x) \square$

Let $f : M^m \to \mathbb{R}^{m+n}$ be an isometric immersion of an $m$-dimensional coisotropic submanifold into an $(m+n)$-dimensional semi-Riemannian manifold. Define the first radical space of $f$ at $x \in M$ to be the subspace

$$R_1(x) = \text{span}\{ \xi \in \text{Rad}(T_x M), \exists X \in T_x M, \dot{A}_\xi X \neq 0 \}, \quad x \in M.$$ 

The first transversal space then becomes

$$T'_1(x) = \{ h^l(X, Y) \mid X, Y \in T_x M \}, \quad x \in M.$$ 

Proposition 2.2. If $(M^m, g, S(TM))$ is a non totally geodesic coisotropic submanifold, then for $x \in M$, $T'_1(x)$ is characterized by $R_1(x)$. 


proof
Let \( x \in M \) and \( \pi_x \) a projection of \( T_x M \) on \( S(T_x M) \). If \( U \in T^1_x(x) \) and \( U \neq 0 \), then there exists \( X, Y \in T_x M \) such that \( U = h^l(\pi_x(Y), X) \). Moreover there exists \( \xi \in rad(T_x M) \) and \( (\xi \neq 0) \), such that \( g(h^l(\pi_x(Y), X), \xi) \neq 0 \).

Hence from relation (12), one has \( g(h^l(X, \pi_x(Y)), \xi) = g(\dot{A}_{\xi}X, \pi_x(Y)) \neq 0 \) and \( \xi \in R_1(x) \). Conversely, if \( \xi \in R_1(x) \) then there exists \( X \in T_x M \) such that \( \dot{A}_{\xi}X \neq 0 \). Hence \( g(\dot{A}_{\xi}X, \dot{A}_{\xi}X) = g(h(X, \dot{A}_{\xi}X), \xi) \neq 0 \) and \( U = h(X, \dot{A}_{\xi}X) \in R_1(x) \). \( \square \)

Let \( x \in M \) and \( P \) and \( \tilde{P} \) be subbundle in \( \text{Rad}(TM) \) and in \( \text{ltr}(TM) \) respectively. We say that \( P \) and \( \tilde{P} \) are corresponding subbundles, if for all \( \xi_x \in P(x) \) there exists \( N_x \in \tilde{P}(x) \) such that \( g(\xi_x, N_x) = 1 \) and \( g(\xi_x, N'_x) = 0 \) for all \( N'_x \in \text{ltr}(T_x M) \backslash \tilde{P}(x) \) and vice versa.

Proposition 2.3. Let \( P \) be vector subbundle of constant rank in \( \text{Rad}(TM) \) which contains \( R_1(x) \) for all \( x \in M \) and \( \overline{P}(x) \) the complementary of \( P(x) \) in \( \text{Rad}(T_x M) \). If \( P(x) \) and \( \overline{P}(x) \) are parallel w.r.t the \( \tilde{\nabla}^l \) then their corresponding subbundles \( \tilde{P}(x) \supset T_1(x) \) and \( \overline{P}(x) \) in \( \text{ltr}(TM) \) respectively are parallel w.r.t. \( \nabla^l \).

Proof
Let \( x \in M \) and \( \xi \in \overline{P}(x) \). Then \( \dot{A}_\xi = 0 \) and for all \( U \in \tilde{P}(x) \), \( \overline{g}(\xi, U) = 0 \).

Let \( X \in T_x M \), we have
\[
\overline{\nabla}_X \overline{g}(\xi, U) = 0 \iff \overline{g}(\tilde{\nabla}^l_X \xi, U) + \overline{g}(\xi, \nabla^l_X U) = 0
\]
\[
\iff \overline{g}(\tilde{\nabla}^l_X \xi, U) = -\overline{g}(\xi, \nabla^l_X U) = 0
\]
\( \overline{g}(\xi, \nabla^l_X U) = 0 \iff \nabla^l_X U \in \tilde{P}(x) \). Thus \( \tilde{P}(x) \) is parallel.

It’s the same with \( \overline{P}(x) \) and \( \overline{P}(x) \) \( \square \)

2.3. The main results. Suppose that \( \overline{M^{m+n}} \) is an \( (m+n) \)-dimensional complete and simply connected semi-Riemannian manifold with constant sectional curvature \( c \) and \( f : M^m \rightarrow \overline{M^{m+n}} \) an isometric immersion of the lightlike submanifold \( M^m \) in \( \overline{M^{m+n}} \).
Theorem 2.1. Let $f : M^m \rightarrow \overline{M}^{m+n}$ be an isometric immersion of a $r$-lightlike submanifold ($1 \leq r \leq m$, $r \neq n$) $(M, g, S(TM), S(TM^\perp))$ into $(\overline{M}^{m+n}, \overline{g})$. Suppose that

1. the induced linear connection $\nabla$ on $M$ and the transversal linear connection $\nabla^t$ on the transversal subbundle $\text{tr}(TM)$ are metric ones.
2. there exists a screen transversal subbundle $P$ of $S(TM^\perp)$ of constant rank $p$ ($p < n$), parallel w.r.t the connection $\nabla^s$ on $S(TM^\perp)$, such that

$$T_1(x) \subset P(x), \quad \forall x \in M$$

where $T_1(x)$ is the first transversal space of $f$ at $x \in M$

Then the codimension of $f$ can be reduced to $p$.

The difference which exists between the theorem 1 of [1] and this theorem 2.1, is that the last one is more general. Because in this case the subbundle $S(TM^\perp) \neq \{0\}$ contrary to the isotropic case where $S(TM^\perp) = \{0\}$ (and $A_W$ is not defined).

Instead of a screen transversal subbundle as in theorem 2.1, in the coisotropic submanifold we use a radical subbundle. We have

Theorem 2.2. Let $f : M^m \rightarrow \mathbb{R}^{m+n}$ be an isometric immersion of a light-like coisotropic submanifold $(M, g, S(TM))$ into a pseudo-Euclidean space $(\mathbb{R}^{m+n}, \overline{g})$. Suppose there exists a radical subbundle $P$ of $\text{Rad}(TM)$ of constant rank $p$ ($p < n$), parallel w.r.t the connection $\hat{\nabla}^t$ on $\text{Rad}(TM)$, such that its complementary in $\text{Rad}(T_xM)$ is also parallel and

$$R_1(x) \subset P(x), \quad \forall x \in M$$

where $R_1(x)$ is the first radical space of $f$ at $x \in M$. Then the codimension of $f$ can be reduced to $p$.

Now suppose $(M, S(TM), g)$ be a coisotropic submanifold of semi-Riemannian $(\mathbb{R}^{m+n}, \overline{g})$. The submanifold $M$ is said to be totally umbilical on $\overline{M}$, if and only if there exists a smooth vector field, $N \in \text{tr}(TM)$ such that

$$h^I(X, Y) = \overline{g}(X, Y)N, \quad \forall X, Y \in \Gamma(TM).$$
and \( h \) the second fundamental form \([7]\). Then \( N \) is called an umbilical vector field.

If \( \xi \) be nonzero vector fields in \( \Gamma(Rad(TM)) \) such that \( \overline{\mathfrak{g}}(\xi, N) = 1 \) then

\[
\overline{\mathfrak{g}}(h^l(X, Y), \xi) = \overline{\mathfrak{g}}(X, Y), \quad \text{and} \quad \overline{\mathfrak{g}}(\xi, N) = 1.
\]

This definition does not depend on the choice of screen distribution \([3, \text{Theo } 2.1, \text{pg } 157]\).

Then we have the following.

**Theorem 2.3.** Let \( f : M^m \rightarrow \mathbb{R}^{m+n} \) be a totally umbilical isometric immersion of a lightlike coisotropic submanifold \((M, g, S(TM))\) into a pseudo-Euclidean space \((\mathbb{R}^{m+n}_q, \overline{g})\). Suppose that the umbilical vector field is parallel w.r.t. the connection \( \nabla^t \) on \( ltr(TM) \). Then the codimension of \( f \) can be reduced to 1.

### 3. Proof of Theorems

#### 3.1. Proof of Theorem 2.1

Recall that \( P \) is parallel w.r.t. \( \nabla^s \) if for all \( X \in \Gamma(TM) \) and \( W \in \Gamma(P) \), \( \nabla^s X W \in \Gamma(P) \).

As \( c \) is constant, then we have three possible cases.

**Case \( c = 0 \)**

Let \( x_0 \in M \), we have to prove that \( f(M) \subset T_{x_0}M \oplus P(x_0) \).

Let \( \eta \) be a vector of the complementary orthogonal bundle, \( P^\perp(x_0) \) to \( P(x_0) \) in \( S(TM^\perp) \) and \( \eta_t \) the parallel transport of vector \( \eta \) in \( \Gamma(P^\perp) \) along the regular curve \( \gamma : I \rightarrow M \) (\( I \subset \mathbb{R} \)) through \( x_0 \).

As \( \overline{\mathfrak{g}} \) is non degenerate on \( S(TM^\perp) \). If \( P^\perp \) is parallel then \( P \) is parallel.

Hence \( \eta_t = \nabla^s_\gamma \eta \in \Gamma(P^\perp(\gamma(t))), \forall t \in I \)

\[
\nabla^s_\gamma \eta_t = -A_{\eta_t} \gamma + D^l(\gamma, \eta_t) + \nabla^s_\gamma \eta_t
\]

\( \eta_t = \nabla^s_\gamma \eta \in \Gamma(P^\perp(\gamma(t))) \implies A_{\eta_t} \gamma = 0 \) and \( D^l(\gamma, \eta_t) = 0 \).

As \( \eta_t \) is parallel transport of \( \eta \) along \( \gamma \) in \( \Gamma(P^\perp) \), \( \nabla^s_\gamma \eta_t = 0, \forall t \in I \).

Thus, we have \( \nabla^s_\gamma \eta_t = 0 \implies \eta_t = \eta = \text{cste} \) in \( \mathbb{R}^{m+n}_q \).

\[
\frac{d}{dt}(\overline{\mathfrak{g}}(f(\gamma(t)) - f(x_0), \eta_t)) = \overline{\mathfrak{g}}(f_* \dot{\gamma}, \eta) = 0
\]

\[
\implies f(\gamma(t)) - f(x_0) \in (P^\perp(\gamma(t)))^\perp = P(\gamma(t))
\]
As $\gamma$ and $\eta$ are arbitrary then
\[ f(M) \subset T_{x_0}(M) \oplus P(x_0) \cong \mathbb{R}^{n+p} \]
and $\mathbb{R}^{n+p}$ is totally geodesic in $\mathbb{R}^{q+m}$.

**Case $c > 0$**

Then $M^m$ is isometrically immersed in the pseudosphere $\overline{M}^m = S_q^n$ by an immersion $f : M^m \rightarrow S_q^n$. Put $i : S_q^n \rightarrow \mathbb{R}^q$ the canonical injection of $S_q^n$ in $\mathbb{R}^{q}$. Then we consider the isometric immersion $\hat{f} = i \circ f : M^m \rightarrow \mathbb{R}^q$. 

We have corresponding vector spaces.
\[ tr(\hat{T}_xM) = tr(T_xM) \oplus < f(x) > \]
where
\[ < f(x) > := \text{span}\{f(x)\} \subset S(\hat{T}_xM) \]

We deduce
\[ \hat{T}_1(x) = T_1(x) \oplus < f(x) > \subset P(x) \oplus < f(x) > = \hat{P}(x). \]

The complementary to $\hat{P}(x)$ in $S(\hat{T}_xM)$ and to $P(x)$ in $S(T_xM)$ are coincide $\hat{P}^\perp(x) = P^\perp(x)$ and parallel w.r.t. the connection $\hat{\nabla} = \hat{\nabla}|_{S(T_xM)}$. 

As $< f(x) > \subset \hat{P}(x)$ and $P(x)$ is parallel w.r.t. the connection $\hat{\nabla}|_{S(T_xM)}$ in $S(\hat{T}_xM)$ then $\forall X \in \Gamma(TM)$ and $W \in \hat{P}(x)^\perp$,
\[ \bar{g}(\hat{\nabla}^s_Xf(x), W) = \nabla_X\bar{g}(f(x), W) + \bar{g}(f(x), \hat{\nabla}^s_XW) \]
\[ = 0 \ (\text{car} \hat{\nabla}^s_XW \in \hat{P}^\perp(x)) \]
thus $\hat{\nabla}^s_Xf(x) \in \hat{P}(x)$. Hence $\hat{P}(x)$ is parallel w.r.t. the connection $\hat{\nabla}^s$.

As $ltr(T_xM) = ltr(\hat{T}_xM)$ is parallel, then $\forall N \in ltr(T_xM)$
\[ \nabla_X < f(x), N > = 0 \]
\[ = < \nabla_Xf(x), N > + < f(x), \nabla_XN > \]
\[ = - < \hat{A}_{f(x)}X, N > \]

Hence $\hat{\nabla}^t$ is metric connection. As in the case $c = 0$ we obtain
\[ \hat{f}(M) \subset \hat{T}_xM \oplus \hat{P}(x) = T_xM \oplus P(x) \oplus f(x) \cong \mathbb{R}^{m+p+1} \]
\[ f(M) \subset S_q^{m+n} \cap \mathbb{R}^{m+p+1}. \]
This ends the proof of the case $c > 0$

**Case** $c < 0$

The proof of this case is similar to the second case $c > 0$. We consider an immersion $\hat{f} = M^m \rightarrow \mathbb{R}^{m+n+1}$ such that $\hat{f} = i \circ f$

where $i : \mathbb{H}^{m+p} \rightarrow \mathbb{R}^{m+n+1}$ is the canonical injection of pseudo-hyperbolic $\mathbb{H}^{m+n}$ into $\mathbb{R}^{m+n+1}$, we have

$$\hat{f}(M) \subset \hat{T}_x M \oplus \hat{P}(x) = T_x M \oplus P(x) \oplus f(x) \cong \mathbb{R}^{m+p+1}$$

$$f(M) \subset \mathbb{H}^{m+n} \cap \mathbb{R}^{m+p+1}.$$
Proof
Since $h^l = 0$ and $T_1(x) \subset P(x) \subset S(TM^\perp)$, then $\forall x \in M$, $T_xM \oplus P(x)$ has
$r$ lightlike vectors fields. Hence $Q$ is $r$-degenerate.
Since
$$T_xQ = T_xM \oplus P(x) = S(T_xM) \oplus \text{Rad}(T_xM) \oplus P(x), \forall x \in M.$$ Moreover
$$T_xM = T_xM \oplus P(x) \oplus P^\perp(x) \oplus \text{ltr}(T_xM) = T_xQ + \overline{\text{tr}(T_xM)}, \forall x \in M.$$ where $\text{tr}(T_xM) = P^\perp(x) \oplus \text{ltr}(T_xM)$.
If $p < n - r$, then the rank of $P^\perp(x)$ is zero. Hence $Q$ is an $r$ degenerate manifold.
If $p = n - r$, then the rank of $P^\perp(x)$ is zero and $\overline{\text{tr}(T_xM)} = \text{ltr}(T_xM)$. Hence $Q$ is lightlike coisotrope submanifold. □

Proof of theorem 2.2
The idea of proof is identical to that of theorem 2.1 apart from some technical
use for radical subbundle. Let $x \in M$, we will prove that $f(M) \subset T_xM \oplus \tilde{P}(x)$ and that $T_xM \oplus \tilde{P}(x)$ is totally geodesic in $\mathbb{R}^{m+n}$.
Let $\eta$ be a vector of $\tilde{P}(x)$ and $\eta_t$ parallel transport of $\eta$ in $\tilde{P}$ along an
arbitrary smooth curve $\gamma : I \rightarrow M (I \subset \mathbb{R})$ through $x$. The relation (7)
gives
$$\nabla_\gamma \eta_t = \nabla_\gamma \eta_t + h^l(\dot{\gamma}, \eta_t) \forall I \in \mathbb{R}.$$ With the Weingarten relation we have
$$\nabla_\gamma \eta_t = -\dot{A}_n \dot{\gamma} + \ddot{\gamma} \eta_t$$
$\eta_t \in \Gamma(\tilde{P}) \implies \dot{A}_n \dot{\gamma} = 0$. As $\eta_t$ is obtained by parallel transport of $\eta$ along
$\gamma$ in $\Gamma(\tilde{P})$, then $\nabla_\gamma \eta_t = 0, \forall t \in I$. Hence we have $\nabla_\gamma \eta_t = 0$.
With relation (12), $h(\eta_t, \dot{\gamma}) = 0$, and $\nabla_\gamma \eta_t = 0$ yields $\eta_t = \eta = \text{cste}$
$$\frac{d}{dt}(\overline{f(\gamma(t)) - f(x), \eta_t}) = \overline{f(\gamma(t)) - f(x), \eta}) = 0$$ As $\gamma$ and $\eta$ are arbitrary and $f(\gamma(t)) - f(x) \in \text{ltr}(M)$,
then $f(\gamma(t)) - f(x) \in \tilde{P}(\gamma(t))$. Hence
$$f(M) \subset T_x(M) \oplus \tilde{P}(x) \equiv \mathbb{R}^{m+p}$$
$\mathbb{R}^{m+p}$ is totally geodesic in $\mathbb{R}^{m+n}$ \Box

**Corollary 3.4.** Let $f : M^m \longrightarrow \mathbb{R}^{m+n}_q$ be a 1-regular immersion. If $R_1$ is a parallel subbundle of rank $p < n$, then $f$ has a substantial codimension $p$.

3.2. **Proof of theorem 2.3.** For totally umbilical coisotropic submanifold $M$, $T_1(x) = \text{span}\{N_x\}$, for each $x \in M$ and as $N$ is a parallel vector field, $T_1$ is then a distribution of constant rank 1. Then the first radical space $R_1$ is also parallel and of constant rank 1. Use theorem 2.2 to complete the proof \Box

**Remark 3.1.** In the theorem 2.3, one can replace the condition on the vector field $N$ by $\nabla^\xi_x N = \alpha(\xi) N$ (parallel along the Rad(TM) subbundle), where $\alpha(\xi)$ is a smooth function of $M$, because we have

\[
\nabla^\xi_x N = 0, \forall X \in \Gamma(S(TM)).
\]

4. **Examples**

4.1. **r-lightlike submanifold.** We consider the surface $M$ of Euclidean space $\mathbb{R}^4_2$ with semi-Riemannian metric of signature $\text{sig}(g) = (-,-,+,+)$ by equations:

\[
M \longrightarrow \mathbb{R}^4_2
\]

\[
(v^1, v^2) \longmapsto (x^1, x^2, x^3, x^4)
\]

where

\[
\begin{aligned}
  x^1 &= v^1 \\
  x^2 &= v^2 \\
  x^3 &= \frac{1}{\sqrt{2}}(v^1 + v^2) \\
  x^4 &= \frac{1}{2} \log(1 + (v^1 - v^2)^2)
\end{aligned}
\]

$TM = \text{span}\{V_1, V_2\}$

with

\[
V_1 = \frac{\partial}{\partial v^1} = \frac{\partial}{\partial x^1} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^3} + \frac{(x^1 - x^2)}{(1 + (x^1 - x^2)^2)} \frac{\partial}{\partial x^4}
\]

\[
V_2 = \frac{\partial}{\partial v^2} = \frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^3} - \frac{(x^1 - x^2)}{(1 + (x^1 - x^2)^2)} \frac{\partial}{\partial x^4}
\]
and $TM^\perp = \text{span}\{H_1, H_2\}$,

where

\[
H_1 = \frac{\partial}{\partial x^1} + \sqrt{2} \frac{\partial}{\partial x^3}
\]

\[
H_2 = 2(x^2 - x^1) \frac{\partial}{\partial x^2} + \sqrt{2}(x^2 - x^1) \frac{\partial}{\partial x^3} + (1 + (x^2 - x^1)^2) \frac{\partial}{\partial x^4},
\]

moreover $H_1 = V_1 + V_2$, and

\[
\text{Rad}(TM) = TM \cap TM^\perp = \text{span}\{\xi = H_1\}
\]
is a distribution of constant rank $1$.

Hence the surface $M$ is a $1$-lightlike surface of $\mathbb{R}^4$. The vector subbundle $S(TM^\perp)$, complementary to $\text{Rad}(TM)$ in $TM^\perp$ is spanned by $H_2$.

\[
S(TM^\perp) = \text{span}\{H_2\}.
\]

The construction of lightlike transversal vector bundle $ltr(TM)$ gives:

\[
ltr TM = \text{span}\{N = -\frac{1}{2} \frac{\partial}{\partial x^1} + \frac{1}{2} \frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^3}\}
\]

and $\bar{g}(N, N) = 0$, $\bar{g}(N, \xi) = 1$

Put $H_1 = \xi$, $H_2 = W_2$ and $U = \sqrt{2}(1 + (x^1 - x^2))V_2$.

Then

\[
tr(TM) = ltr(TM) \perp S(TM^\perp) = \text{span}\{N, W\}.
\]

An easy computation gives

\[
\nabla U = 2(1 + (x^2 - x^1)^2)\{2(x^2 - x^1) \frac{\partial}{\partial x^2} + \sqrt{2}(x^2 - x^1) \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4}\}
\]

\[
\nabla U = \nabla_X \xi = \nabla_X N = 0 \quad \forall X \in \Gamma(TM)
\]

Using the Gauss and Weigentern relations to obtain

\[
h^l = 0, \ h^s(X, \xi) = 0, \ h^s(U, U) = W
\]

\[
\nabla_X U = \frac{2\sqrt{2}(x^1 - x^2)^3}{(1 + (x^1 - x^2)^2)}X^2 U \text{ with } X = X^1 \xi + X^2 U \in \Gamma(TM)
\]

$A_\xi = 0$, $D^l(X, W) = 0$, $A_W \xi = 0$ and $A_W U = -2U$

So, the surface $M$ is non totally geodesic and the induced and transversal connections $\nabla$ and $\nabla^l$ respectively are metric ones. The first transversal space is given by

\[
T_1(x) = \{h^s(X, Y), X, Y \in \Gamma(TM)\} = S(T_x M^\perp).
\]
The distribution $T_1$ is of constant rank 1. Hence $M$ admits a reduction of its codimension to 1.

**4.2. Coisotropic submanifold.** Let $M$ be a submanifold of $\mathbb{R}^5$, Euclidean space of $\mathbb{R}^5$ with semi-Riemannian metric of signature $\text{sig}(g) = (-, -, +, +, +)$.

Suppose $M$ is defined by equations:

$$
\begin{aligned}
x^1 &= u \\
x^2 &= ((v^1)^2 + (v^2)^2)^{1/2} \\
x^3 &= v^1 \\
x^4 &= u \\
x^5 &= v^2
\end{aligned}
$$

(21)

The tangent bundle is given by $T M = \text{span}\{U_1, U_2, U_3\}$ where

$$
\begin{aligned}
U_1 &= \frac{\partial}{\partial u} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} \\
U_2 &= \frac{\partial}{\partial v^1} = \frac{x^3}{x^5} \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} \\
U_3 &= \frac{\partial}{\partial v^2} = \frac{x^2}{x^5} \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^5}
\end{aligned}
$$

and the cotangent bundle is $T M^\perp = \text{span}\{\xi_1 = U_1, \xi_2 = x^3 U_2 + x^5 U_3\}$.

Then the radical subbundle is given by $\text{Rad}(TM) = TM \cap TM^\perp = TM^\perp$.

Thus $M$ is coisotropic.

The construction of lightlike transversal subbundle, $ltr(TM)$ gives:

$$ltrTM = \text{span}\{N_1, N_2\}
$$

where

$$
N_1 = \frac{1}{2} \left( -\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} \right) \quad \text{and} \quad N_2 = \frac{1}{2(x^3)^2} \left( -x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} - x^5 \frac{\partial}{\partial x^5} \right)
$$

with $\bar{g}(N_i, N_j) = 0$, $\bar{g}(N_i, \xi_j) = \delta_{ij}$

Put $TM = \text{span}\{\xi_1, \xi_2, V\}$ where $V = x^2 U_3$. With a direct computation

$$\bar{\nabla}_V \xi_1 = \bar{\nabla}_{\xi_2} \xi_1 = \bar{\nabla}_{\xi_1} \xi_2 = \bar{\nabla}_{\xi_1} V = 0, \bar{\nabla}_V \xi_2 = V.
$$

Thus the Gauss and the Weigentern formulas give

$$\bar{\nabla}_V \xi_2 = V, \bar{\nabla}_{\xi_1} \xi_2 = \bar{\nabla}_{\xi_1} V = 0, \bar{\nabla}_{\xi_2} \xi_2 = \xi_2, \bar{\nabla}_{\xi_2} V = V,$$
\[ \nabla_V V = \frac{1}{2} \xi_2 \nabla_X \xi_1 = 0 \quad \forall X \in \Gamma(TM) \]

and

\[ \begin{align*}
h_1^l(X, Y) &= 0, \quad h_2^l(X, \xi) = 0, \\
h_2^l(V, V) &= -(x^3)^2 \neq 0 \quad \forall x \in M.
\end{align*} \]

Then the induced connection \( \nabla \) on \( M \), is not metric, thus \( M \) is no totally geodesic. Moreover we have \( \dot{A}_{\xi_1} X = \dot{A}_{\xi_2} \xi = 0 \), \( \dot{A}_{\xi_2} V = -V \), \( \forall X \in \Gamma(TM) \) and \( \xi \in \Gamma(TM^\perp) \). Then the first transversal space and the first radical space are

\[ T_1(x) = \text{span}\{h^l(V, V)\} = \text{span}\{N_2\} \quad \text{and} \quad R_1(x) = \text{span}\{\xi_2\}. \]

We have \( \overline{\nabla_V \xi_2, N_1} = 0 \implies R_1(x) \) is parallel \( \forall x \in M \) and the rank of \( R_1 \) is constant equal to 1. So \( f \) is 1-regular. Hence \( f \) admit a substantial codimension 1. Moreover, we have \( h^l(X, Y) = \overline{\nabla_X Y, N_2} \).

References

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