The maximal function on spaces that lie between $L^\infty$ and BMO

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Abstract: There are subspaces of $\text{BMO}(\mathbb{R}^n)$, $\text{BMO}(r)$, $1 \leq r < \infty$, introduced in [S] and defined by the growth condition,

$$\sup_{1 \leq p < \infty} \left\{ \frac{1}{|\mathbb{R}^n|} \left( \sup_{x \in \mathbb{R}^n} \left( \frac{1}{|\mathbb{R}^n|} \int_{\mathbb{R}^n} |f(x) - f_S|^p \, dx \right)^{1/p} \right) \right\} \leq C_0 < \infty.$$

These spaces are shown to have rearrangement invariant hulls that are similar to the space weak $L^\infty$, which was defined by Bennett, DeVore and Sharpley. It is proved that the Hardy-Littlewood maximal operator, if it is finite a.e., takes $\text{BMO}(r)$ into itself, with norm boundedness.

Key words: Maximal functions, functions of bounded mean oscillation, weak-type estimates.

1 Introduction
The space of functions of bounded mean oscillation, BMO, was first identified by Fritz John and Louis Nirenberg in 1961 [JN]. They proved that BMO is identical with functions in a certain exponential space. BMO functions appear in the theory of solutions to second order partial differential equations, both as solutions and as functions which can be used to define the boundary of domains for solving the Dirichlet problem and as the function class for coefficients of a second order operator. Jurgen Moser [M] used John and Nirenberg’s result in order to prove a Harnack inequality for non-negative solutions to elliptic equations of the form $Lu = 0$, when the coefficients of $L$ are only assumed to be bounded and measurable, and to satisfy $1/|\lambda| \leq \xi \leq \Sigma \xi_0(a_{ij}(x)\xi_j) \leq |\xi|$, for some $\lambda > 0$. In other words the function space BMO has been very useful in obtaining information about solutions to second order partial differential equations. This paper presents new information about certain subspaces of $\text{BMO}(\mathbb{R}^n)$, which are denoted by $\text{BMO}(r, \mathbb{R}^n)$. The main fact is that the Hardy-Littlewood maximal function takes $\text{BMO}(r, \mathbb{R}^n)$ to itself, when the maximal function is not identically infinite. In [S] the spaces $\text{BMO}(r, \mathbb{R}^n)$ were defined as being functions that are locally integrable, for which

$$\sup_{1 \leq p < \infty} \left\{ \frac{1}{|\mathbb{R}^n|} \left( \sup_{x \in \mathbb{R}^n} \left( \frac{1}{|\mathbb{R}^n|} \int_{\mathbb{R}^n} |f(x) - f_S|^p \, dx \right)^{1/p} \right) \right\} \leq C_0 < \infty.$$

For $r=1$ this space can be identified with $\text{BMO}(\mathbb{R}^n)$ by the John-Nirenberg theorem [G]. However, for $r>1$, $\text{BMO}(r) \subsetneq \text{BMO}(1)$, and $\text{BMO}(r)$ can be identified with the exponential space defined as follows:
A central fact about the $\text{BMO}(r,\mathbb{R}^n)$ is that these are the dual spaces of certain atomic spaces that lie between the Hardy space $H^1(\mathbb{R}^n)$ and the space of integrable functions $L^1(\mathbb{R}^n)$ [S]. One reason for studying the structure of $\text{BMO}(r,\mathbb{R}^n)$ is to use this information as a tool for further investigation of the atomic predual spaces. The exact relation of the function spaces $L^\infty(\mathbb{R}^n)$ and $\text{BMO}(\mathbb{R}^n)$, and of $H^1(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ has long been of interest to harmonic analysts. The spaces $\text{BMO}(r,\mathbb{R}^n)$ lie between $L^\infty(\mathbb{R}^n)$ and $\text{BMO}(\mathbb{R}^n)$. They are implicit in the work of John Garnett ([G] Chapter 6) and in unpublished work of Svante Janson ([CWW]). See also the work of Abu-Shammala & Torchinsky [A-ST], Frazier & Jawreth [FJ] and that of Lelievre [L]. The present paper provides a new window on these spaces, identifying some of their properties as subspaces of weak $L^\infty(\mathbb{R}^n)$. Weak $L^\infty(\mathbb{R}^n)$ was introduced by Bennett, DeVore and Sharpley ([BDVS]). The methods of Bennett, DeVore and Sharpley ([BDVS]) are used in this paper to prove that the local Hardy-Littlewood maximal function maps $\text{BMO}(r,\mathbb{R}^n)$ into itself for $1 \leq r < \infty$. The next question is, of course, - How do singular integral operators act on the $\text{BMO}(r,\mathbb{R}^n)$? The maximal function result of Theorem 3 may be helpful in answering this question.

2 Problem Formulation

Specifically one can define a functional (see Def.1 below) that exactly specifies the rearrangement hull of

$$\text{BMO}(Q, r) = \left\{ f \in L^1_{loc}(Q) : \left( \frac{1}{|Q|} \int_Q \exp(C|f(x) - f_Q|) \, dx \right)^{1/p} \leq C < \infty, \text{ where } Q \subseteq \mathbb{R}^n \right\}.$$ 

One can then prove that the Hardy-Littlewood maximal operator on $Q$ takes this rearrangement hull to itself, and maps $\text{BMO}(Q, r)$ into itself with norm boundedness. As in [BDVS] if the maximal function is not identically infinite for a function $f$ in $\text{BMO}(R^n, r)$, then $Mf$ will also lie in $\text{BMO}(R^n, r)$.

**Definition:** The collection of functions $W(Q, r)$ is the collection of all functions defined and measurable on $Q$, with $f^*(t)$ finite for all $t > 0$, such that the functional $F(r) =$

$$\sup_{1 \leq p < \infty} \left( \frac{1}{p} \sup_{t > 0} \left( \frac{1}{t} \int_0^t [f^*(s) - f^*(t)]^p \, ds \right)^{1/p} \right)$$

$$\leq C_0 < \infty.$$

$W(r)$ has the same definition with $Q$ replaced by $R^n$. $f^*(t)$ is the non-increasing rearrangement of $f(x)$. Notice that for $p=1$ the integral in the functional $F(r)$ becomes $\sup_{t > 0} (f^{**}(t) - f^*(t))$ with $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds$. being the defining functional for the "space" $W(Q) = \text{weak } L^\infty$ used by Bennett, DeVore and Sharpley. This fact corresponds with the $\text{BMO}(Q, r)$ being subspaces of $\text{BMO}(Q)$, so that $W(Q, r) \subseteq W(Q)$. The norm for $\text{BMO}(1)$ is usually bigger than the norm for $\text{BMO}$, but these norms are equivalent ([G], p.233).

The defining functional $F(r)$ has the following relation to the Peetre $K$-functional for $(L^\infty, L^p)$:

$$\left( \frac{1}{t} \int_0^t [f^*(s) - f^*(\tau)]^p \, ds \right)^{1/p} \lesssim K(f^*, t; L^\infty, L^p)$$

$$\lesssim \left( \frac{1}{t} \int_0^t [f^*(s) - f^*(\tau)]^p \, ds \right)^{1/p} + f^*(\tau),$$

where $K(f^*, t; L^\infty, L^p)$ is the Peetre $K$-functional.
where $\tau = \frac{1}{t^p}$ and we have used
\[K(f,t;L^p,L^q) = tK(f,1t;L^p,L^q),\text{ and}\]
\[K(f,t;L^p,L^q) = \left(\int_0^1 (f^*(s))^p ds\right)^{1/p},\]
with constants independent of $t$, as shown by Bergh & Löfström in [BL].

### 3 Problem Solution

The following three theorems are generalizations of results of [BDVS] and they can be proved by similar methods. Theorems 1 and 2 include the Bennett, DeVore, Sharpley results as special cases using the identification of $\text{BMO}$ with $\text{BMO}(1)$. However, Theorem 3 assumes their Theorem 4.2 is valid to obtain finiteness of certain $L^p$ integrals involved in the proof of Theorem 3.

The $p$-sharp function can be defined by
\[(f_{\#}^p(x) = \sup_{u \in \mathcal{G}} \left(\frac{1}{|Q|} \int_Q f(y) dy - f_{\#}^p dy \right)^{1/p}.\]

When the function is confined to a fixed cube $Q$, then
\[(f_{\#}^p(x) = \sup_{u \in \mathcal{G}} \left(\frac{1}{|R|} \int_R f(y) dy - f_{\#}^p dy \right)^{1/p}.\]

**Theorem 1:** i) $f \in L^1(Q)$, then for $1 \leq p < \infty$,
\[\left(\frac{1}{t} \int_0^t (f^*(s) - f^*(t))^p ds\right)^{1/p} \leq c(n)^{1/p} f_{\#}^p(t) \text{ if } t \leq \frac{1}{6} |Q|.\]

ii) $f \in L^1(Q)$, then $f \in \mathcal{W}(Q,r)$ if and only if $f$ is the rearrangement of a function from $\text{BMO}(Q,r)$.

The Hardy-Littlewood maximal function on $\mathbb{R}^n$ and on a cube $Q$, is defined respectively by
\[Mf(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q f(x) dx \right)\]
\[M_{\#}f(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q f(x) dx \right)\]

and by

**Theorem 2:** i) $M_{\#}$ maps $\mathcal{W}(Q,r)$ to $\mathcal{W}(Q,r)$

ii) $M : \mathcal{W}(r) \to \mathcal{W}(r)$

**Theorem 3:** i) $M_{\#}$ maps $\text{BMO}(Q,r)$ into itself, with norm boundedness.

ii) $M$ maps $\text{BMO}(r)$ into itself if $Mf \neq \infty$

Remark: Since $\text{BMO}(Q,r)$ and $\text{BMO}(r)$ are subspaces of $\text{BMO}(Q)$ and $\text{BMO}$, the Hardy-Littlewood maximal function actually takes the spaces $\text{BMO}(Q,r)$ and $\text{BMO}(r)$ into their intersection with the space of functions of bounded lower oscillation, $\text{BLO}(Q)$ and $\text{BLO}$, as defined in ([BS2], 5.7 & 5.8).

The proofs of the first two theorems will be discussed only briefly, since they follow the line of argument in [BDVS] for Theorems 3.1 and 4.1. Theorem 3 will be proved in detail to make clear exactly what is needed for the case of the $\text{BMO}(r,\mathbb{R}^n)$. To prove Theorem 1i) one can assume that $f(x) \geq 0$ and use the covering lemma, Lemma 3.2 of [BDVS], for the set $\{x \in Q : f(x) > f^*(t)\}$.

For $f \in L^1(Q)$, then $f \in \mathcal{W}(Q,r)$ if and only if $f$ is the rearrangement of a function from $\text{BMO}(Q,r)$.

The Hardy-Littlewood maximal function on $\mathbb{R}^n$ and on a cube $Q$, is defined respectively by
\[O \ni \{x \in Q : f(x) > f^*(t)\} \cup \{x \in Q : (f_{\#}^p(x) > (f_{\#}^p)^*(t)\},\]
where $t$ is fixed, $0 < t < |Q|$. We can prove the estimate
\[\left(\frac{1}{t} \int_0^t (f^*(s) - f^*(t))^p ds\right)^{1/p} \leq C(t)^{1/p} f_{\#}^p(t),\]
for $t < |Q|$ by essentially the same methods used in [BDVS] to prove Theorem 3.1a). There
are some changes needed to deal with the $L^p$ norms.

This implies that

$$\sup_{1 < t < \infty} \left( \frac{1}{t} \right) \sup_{0 < t \leq \frac{1}{6}|Q|} \left( \frac{1}{t} \int_0^t \left( f^*(s) - f^*(t) \right)^p ds \right)^{1/p} \right)$$

$$\leq C \|f\|_{\text{BMO}(Q)}, \text{if } 0 < t \leq \frac{1}{6}|Q|$$

If $t > (1/6)|Q|$, then

$$\left( \frac{1}{t} \int_0^t \left[ f^*(s) - f^*(t) \right]^p ds \right)^{1/p} \leq$$

$$C \left\{ \left( \frac{|Q|}{6} \right)^{1/p} + f_{e} \right\}.$$  

When $f \in \text{BMO}(Q,r)$ these estimates imply that

$$\sup_{1 < t < \infty} \left( \frac{1}{t} \right) \sup_{0 < t \leq \frac{1}{6}|Q|} \left( \int_0^t \left( f^*(s) - f^*(t) \right)^p ds \right)^{1/p} \right)$$

$$\leq \sup_{1 < t < \infty} \frac{C}{t^{1/p}} \left( \left\| f^\# \right\|_{\infty} + f_{e} \right)$$

and this gives that

$$\|f\|_{W(Q,r)} \leq \|f\|_{\text{BMO}(Q,r)} + f_{e}.$$  

So $\text{BMO}(Q,r) \subseteq W(Q,r)$.

The rearrangement invariance of $W(Q,r)$ means that the rearrangement hull of $\text{BMO}(Q,r)$ lies inside $W(Q,r)$. To prove that $W(Q,r)$ is contained in the rearrangement hull of $\text{BMO}(Q,r)$, one can proceed almost exactly as in [BDVS], replacing $L^1$ integrals by $L^p$ integrals. As in the case of BMO and W, one cannot use the above method to relate $W(R^n,r)$ to $\text{BMO}(R^n,r)$.

To prove Theorem 2i) one can start with the estimate of Bennett, DeVore and Sharpley

$$(M_{\mathcal{L}} f)^*(t) - f^*(t) \leq C \left( \frac{1}{t} \int_0^t \left[ f^*(s) - f^*(t) \right] ds \right)$$

for $f \in W(Q,r) \subseteq W(Q)$, and obtain the estimate

$$\frac{1}{t} \int_0^t \left[ M_{\mathcal{L}} f(s) - M_{\mathcal{L}} f(t) \right]^p ds \leq$$

$$C \sup_{s \geq 0} \left( \frac{1}{t} \int_0^t \left[ f^*(s) - f^*(t) \right]^p ds \right)$$

for all $t > 0$, by an elementary argument. Multiplying by $\frac{1}{(p^r)}$, taking $p$th roots and the sup over $0 < t < \infty$ on the left and then the sup over $p$, $1 \leq p < \infty$, gives that

$$\|M_{\mathcal{L}} f\|_{W(Q,r)} \leq C \|f\|_{W(Q,r)}.$$  

The same proof shows that, if $Mf \neq \infty$,

$$\|Mf\|_{W(Q)} \leq C \|f\|_{W(Q)}$$

with $Q$ replaced by $R^n$.

To prove Theorem 3, one can assume that $f(x) \geq 0$ on $Q$. If $R$ is any sub-cube of $Q$, we want to find a constant $\alpha_x$ so that (B)

$$\left( \frac{1}{|R|} \int_R |M_{\mathcal{L}} f(x) - \alpha_x|^p dx \right)^{1/p} \leq C \|f\|_{\infty}.$$  

Then

$$\left( \frac{1}{|R|} \int_R |M_{\mathcal{L}} f(x) - (M_{\mathcal{L}} f)^\#|^p dx \right)^{1/p} \leq$$

$$2 \left( \frac{1}{|R|} \int_R |M_{\mathcal{L}} f(x) - \alpha_x|^p dx \right)^{1/p},$$

so dividing by $p^{1/p}$ and taking sup over $R \cap Q$ and sup over $p$, $1 \leq p < \infty$, implies that when $f \in \text{BMO}(Q,r)$, we have $M_{\mathcal{L}} f \in \text{BMO}(Q,r)$, if $C_0$ in (B) is independent of $p$. The constant $C_0$ we will obtain depends on the constant for the maximal inequality,

$$2 \left( \frac{5^n}{p - 1} \right)^{1/p}$$

this means we must use the estimate
This causes no difficulty in the BMO(Q,r) norm estimate since
\[ \frac{1}{2} \leq \frac{1}{p^{1/r'}} \leq 1 \]
for all indices \( 1 \leq p \leq 2 \) and \( 1 \leq r < \infty \).

Consequently one can think of \( p \) as being \( \geq 2 \) in the following argument.

To begin with we can choose \( \alpha \) so that (A)
\[
\int_{\{ x \in Q : M_{Q}f(x) > \alpha \}} |M_{Q}f(x) - \alpha|^{p} \, dx \\
= \int_{\{ x \in Q : M_{Q}f(x) \leq \alpha \}} |M_{Q}f(x) - \alpha|^{p} \, dx.
\]

It is possible to do this because both integrals are continuous in \( \alpha \), at least when they are finite.

The integral on the right is finite for all \( \alpha < \infty \).

Using the result of Theorem 4.2 of [BDVS] and the identification of BMO(Q,1) with BMO(Q), there is at least one value for \( \alpha \), namely \( (M_{Q}f)_{Q} \), for which the left hand side is finite.

In fact an elementary calculation then shows the integral on the left is finite for all \( \alpha \). For \( \alpha \) such that (A) holds it is only necessary to estimate
\[
\int_{\{ x \in Q : M_{Q}f(x) < \alpha \}} |M_{Q}f(x) - \alpha|^{p} \, dx.
\]

One can proceed to use much the same argument as in [BDVS]: obtaining a Calderon-Zygmund decomposition of \( \overline{Q} \) which is the smallest subcube of \( Q \) that contains \( Q \setminus 3R \). 3R= the cube concentric with \( R \), of side length 3 times the side length of \( R \), where \( R \) is a fixed subcube of \( Q \).

So one has \( \{Q_{j}\} \) a disjoint collection of cubes such that

\[
\cup_{Q_{j}} \subset \overline{Q}, f_{Q_{j}} \leq \alpha \leq f_{Q_{j}}, |Q_{j}| = 2^{n}|Q_{j}|
\]

and \( f(x) \leq \alpha \) a.e. \( x \in \overline{Q} \cup Q_{j} \) (i.e. replace \( F_{R} \) by \( \alpha \) in the Bennett, DeVore, Sharpley proof). Then
\[
\mathcal{F}(x) = g(x) = \sum_{j} (f(x) - f_{Q_{j}}) \cdot \chi_{Q_{j}}(x) + \sum_{j} f_{Q_{j}} \cdot \chi_{Q_{j}}(x) + f(x) \cdot \chi_{\overline{Q} \cup Q_{j}}(x).
\]

Now \( g(x) \leq \alpha \) for a.e. \( x \in \overline{Q} \), and this means that \( M_{Q}g(x) \leq \alpha \).

Also \( M_{Q}g(x) \leq \alpha \) for a.e. \( x \in Q \), since \( g(x) = 0 \) if \( x \in \overline{Q} \).

Taking \( F_{1}(x) = M_{Q}f(x) \) and
\[
F_{2}(x) = \sup_{g \equiv \alpha, \mathcal{F} \in \mathcal{F}} \left( \frac{1}{|Q|} \int_{Q} |f(y)| \, dy \right)
\]
we can see that
\[
F_{1}(x) \leq M_{Q}(f \cdot \chi_{Q})(x) \leq M_{Q}b(x) + M_{Q}g(x) \leq M_{Q}b(x) + \alpha, \text{ a.e. } x \in \overline{Q}.
\]

Using the definitions
\[
\Omega = \{ x \in \overline{Q} : M_{Q}f(x) > \alpha \}, \quad \Omega_{1} = \{ x \in \Omega : M_{Q}f(x) = F_{1}(x) \},
\]
we have
\[
\int_{\Omega_{1}} |M_{Q}f(x) - \alpha|^{p} \, dx \leq \int_{\Omega_{1}} |M_{Q}b(x)|^{p} \, dx \\
\leq C(p,n) \int_{\Omega_{1}} |b(x)|^{p} \, dx =
\]
 Altogether

\[ C(p,n) \sum_j \int_{\Omega_j \cap \Omega_1} |f(y) - f_{2\Omega_j}|^p \, dy \leq \]

\[ C(p,n) \sum_j |\Omega_j| \cdot \frac{2^n}{|\Omega_j|} \int_{2\Omega_j} |f(y) - f_{2\Omega_j}|^p \, dy \]

\[ \leq C(p,n) \left\| f^\# \right\|_\infty^{p} |\Omega_1|. \]

Again this result can be extended to \( \mathbb{R}^n \) when \( Mf(x) < \infty \) a.e.

### 4 Conclusion

As mentioned in the introduction, BMO functions appear frequently in mathematical results about solutions to certain second order partial differential equations. Maximal function estimates are routinely employed in harmonic analysis in proving estimates on singular integrals of the kind that appear when a solution to the Dirichlet problem can be represented as the integral of the boundary function against a kernel function. The theorems presented above give a step towards obtaining estimates of this kind.

### References:

[A-S T] W.Abu-Shammala, A.Torchinsky

“Spaces between \( H^1(\mathbb{R}^n) \) and \( L^1(\mathbb{R}^n) \)”, preprint, arXiv:math.FA/0511207 v1 08/11/05

[BDVS] C.Bennett, R.DeVore, R.Sharpley


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[BS2] C.Bennett, R.Sharpley


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"Subspaces of \( L^1(\mathbb{R}^n) \)"