Constrained Inventory Allocation

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Abstract: This paper shows how to allocate a limited supply of inventory in a centralized distribution system with multiple retailers subject to minimal supply commitments and maximum delivery limits for each retailer. The objective is to maximize the number of units sold by all retailers in one time period. The analysis is done by building on a classical result of Derman. The demand at each retailer is described by a probability density function and unsold portions of the supply lose their value at the end of the period.

Key–Words: Inventory Control, Nonlinear Optimization

1 Introduction

We study the following allocation problem. Suppose at the beginning of a given time period there exists \( T \) units of some product (or service) which are to be allocated among \( k \) retailers. The demand at each retailer is a random variable with a distribution which depends on the retailer. At the end of the time period the unused product (or service) loses its value. The single supplier has contractual obligations to provide a minimum amount of \( l_i \geq 0 \) and a maximum amount of \( u_i \geq l_i \) to retailer \( i, i = 1, \ldots, k \). The problem of interest here is that of allocating the \( T \) units among the \( k \) retailers so that the amount \( a_i \) that retailer \( i \) receives satisfies the commitments, i.e., \( l_i \leq a_i \leq u_i, i = 1, \ldots, k \) and the expected number of units sold is maximized. This model is a generalization of [2] where the case \( l_i = 0, u_i = \infty \) was considered.

This paper’s central motivation follows [11], in that there is a clear desire to have simple and effective models to enhance management reasoning. How may we balance expected supply and expected demand of a retailer given contract constraints? What if the supplier has a different expected supply function than the retailer’s expected demand function?

The paper [10] gives a generalized framework for interpreting and solving nonlinear-additive functions constrained by the sum of their inputs. The focus is on the algorithmic solution to such problems. Variations on these problems are very close to the one considered by the current paper. Furthermore, [1] characterizes and extends probabilistically constrained convex programming on discrete distributions. Single source supplies to several retailers or locations are considered in [3] with finite time periods. The focus is on minimizing cost for the system. An infinite horizon case is considered in [4]. In [4] it is shown it may be expressed as single location problems. The paper [12] considers the balancing of inventory with a horizon constraint. Other related work includes [6], [5], [11], [8] and [7].

2 Main Theorem

Let the \( D_i \) be the random demand for retailer \( i \). We assume that it is a nonnegative continuous random variable with continuous density \( f_i(\cdot) \) and c.d.f. \( F_i(\cdot) \). We will assume that \( F_i(0) = 0 \). Let \( a_i \) denote the amount of product allocated to retailer \( i \). Under an allocation \( a = (a_1, \ldots, a_k) \), the total expected sales at retailer \( i \) are:

\[
S_i(a_i) = \int_0^{a_i} xf_i(x)dx + a_i \int_{a_i}^{\infty} f(x)dx. \tag{1}
\]

The problem then is to maximize total sales of the system:

\[
G(a_1, \ldots, a_k) = \sum_{i=1}^{k} S_i(a_i) \tag{2}
\]

\[
l_i \leq a_i \leq u_i, \quad i = 1, \ldots, k, \tag{3}
\]

\[
\sum_{i=1}^{k} a_i \leq T. \tag{4}
\]

Here we allow some \( l_i \) to be equal to 0 and some \( u_i \) to be equal to \( \infty \).
Note that
\[
\frac{\partial G(a)}{\partial a_i} = \frac{dS_i(a_i)}{da_i} = \int_{a_i}^{\infty} f(x)dx. \tag{5}
\]

Thus
\[
\frac{\partial^2 G(a)}{\partial a_i^2} = \frac{d^2 S_i(a_i)}{da_i^2} = -f(a_i) \leq 0. \tag{6}
\]

Thus \( G(a) \) is non-decreasing in each of the variables and (4) can be replaced by (7).

\[
\sum_{i=1}^{k} a_i = T. \tag{7}
\]

Let \( \rho > 0 \) and consider the modified problem \((P_{\rho}):\)

\[
G(a_1, \ldots, a_k) = \sum_{i=1}^{k} S_i(a_i) \tag{8}
\]

subject to the constraints:

\[
l_i \rho \leq a_i \leq u_i / \rho, \quad i = 1, \ldots, k, \tag{9}
\]

\[
\sum_{i=1}^{k} a_i = T. \tag{10}
\]

Where we define \( \infty /0 = \infty \). Let \( v(\rho) \) denote the maximum of \( G \) for the modified problem \((P_{\rho})\) and let \( a^*_i(\rho) \) be the maximizing values. Note that \( a^*_i = a^*_i(1) \) are the solution of the original problem \((P_1)\).

**Lemma 1** \( v(\rho) \) is concave in \( \rho \).

**Proof:** This follows using standard methods of non-linear programming.

**Lemma 2** For problem \((P_{\rho})\), there exists a partition of the retailer set \( \{1, \ldots, k\} \) into three disjoint subsets \( L_{\rho}, U_{\rho} \) and \( R_{\rho} \) where an optimal allocation \( a^*_i(\rho) \) is such that \( a^*_i(\rho) = l_i \rho \) for \( i \in L_{\rho}, \ a^*_i(\rho) = u_i / \rho \) for \( i \in U_{\rho}, \) and for which \( a^*_i(\rho) \) is such that \( 1 - F(a^*_i(\rho)) = P(D_i \geq a^*_i(\rho)) = \text{constant} \) for all \( i \in R_{\rho} \).

**Proof:** For \( \rho = 0 \), the modified problem \((P_0)\) does not have the constraints (9). For this problem the result of [2] holds i.e., there exit maximizing \( a^*_i(0) > 0 \) such that \( \sum_{i=1}^{k} a^*_i(0) = T, \) where \( P(D_i \geq a^*_i(0)) = c_0, \) for all \( i \).

As \( \rho \) increases \( l_i \rho \) increases and \( u_i / \rho \) decreases. Thus, from continuity and (5), (6), the same solution remains optimal for an interval \((0, p_1)\), where \( p_1 \) is the smallest value of \( \rho \) for which some of the constraints (9) become binding. Let \( L_{p_1} = \{i : l_i p_1 = a^*_i(p_1) = a^*_i(0)\} \) be the set of the lower limit binding constraints at \( p_1 \) and let \( U_{p_1} = \{i : u_i / p_1 = a^*_i(p_1) = a^*_i(0)\} \) be the set of the upper limit binding constraints at \( p_1 \).

Again, following [2], from continuity and (5), (6) means \( G(a) \) is concave. This means \( G(a) \)’s maximum is either at a critical point of \( G(a) \) where an \( a^*_i(\rho) \) is such that

\[
l_i \rho < a^*_i(\rho) < u_i / \rho \]

or at an endpoint: \( a^*_i(\rho) = l_i \rho \) or \( a^*_i(\rho) = u_i / \rho \).

Since, \( G(a) \) is non-decreasing on each of its inputs, then it follows that for \( \rho = p_1 + \epsilon \) (and \( \epsilon \) sufficiently small) the optimal solution of (8)-(10) has \( a^*_i(\rho) = l_i \rho, \) for \( i \in L_{p_1} \) and \( a^*_i(\rho) = u_i / \rho \) for \( i \in U_{p_1} \). That is, let \( a^*_i(p_1 + \epsilon) = l_i(p_1 + \epsilon) \) and \( G(a) \) is non-decreasing in any variable thus, for sufficiently small \( \epsilon > 0 \), the value \( a^*_i(p_1 + \epsilon) \) still maximizes \( G \). Moreover, if \( a^*_i(p_1) = u_i / p_1 \) maximizes \( G \), then \( G \) is non-increasing on decreasing inputs, thus for sufficiently small \( \epsilon > 0 \), then \( a^*_i(p_1 + \epsilon) = u_i / (p_1 + \epsilon) \) is still a binding solution.

As for the remaining \( i \notin L_{p_1} \cup U_{p_1}, \) the maximizing \( a^*_i(\rho) \) computed by solving the restricted problem: maximize:

\[
\sum_{i \notin L_{p_1} \cup U_{p_1}} S_i(a_i) \tag{11}
\]

subject to the constraints:

\[
\sum_{i \notin L_{p_1} \cup U_{p_1}} a_i = T - \sum_{i \in L_{p_1}} l_i \rho - \sum_{i \in U_{p_1}} u_i / \rho. \tag{12}
\]

Note that the restricted problem of (11) - (12), satisfies the conditions of [2], i.e., its solution \( \{a^*_i(\rho)\}_{i \notin L_{p_1} \cup U_{p_1}} \) satisfies \( P(D_i \geq a^*_i(\rho)) = \text{const} \) for all \( i \notin L_{p_1} \cup U_{p_1}. \)

The same argument can be repeated, i.e., the “same” solution remains optimal for an interval \((0, p_2)\), where \( p_2 \) is the smallest value of \( \rho > p_1 \) for which some additional constraints (9) become binding. Let \( L_{p_2} = \{i : l_i p_2 = a^*_i(p_2)\} \) be the set of the lower limit binding constraints at \( p_2 \) and let \( U_{p_2} = \{i : u_i / p_2 = a^*_i(p_2)\} \) be the set of the upper limit binding constraints at \( p_2 \). The proof can be completed since there are a finite number of constraints as in (9).

**Lemma 3** The following are true.

1. If \( \rho \leq \rho' \) then \( L_{\rho} \subset L_{\rho'} \) and \( U_{\rho} \subset U_{\rho'} \)
2. \( P(D_i \geq a^*_i(\rho)) \) is increasing in \( \rho \) for all \( i \in U_{\rho} \)
Proof:

(1) Note in the constructive proof of Lemma 2 that \( \emptyset \subset \{1, \ldots, k\} \) into three disjoint subsets \( L, U \) and \( R \) such that an optimal allocation \( a^* \) is such that \( a^*_{ij} = l_{ij} \) for \( i \in L \), \( a^*_{ij} = u_{ij} \) for \( i \in U \) and for which \( a^*_{ij} \) is such that \( 1 - F(a^*_{ij}) = P(D_{ij} \geq a^*_{ij}) = \) constant for all \( i, j \in R \).

3. \( P(D_1 \geq a^*_1(\rho)) \) is decreasing in \( \rho \) for all \( i \notin U_\rho \)

4. \( P(D_i \geq a^*_i(\rho)) \leq P(D_j \geq a^*_j(\rho)) \leq P(D_k \geq a^*_k(\rho)) \), for all \( i \in L_\rho, j \in R_\rho \) and \( k \in U_\rho \)

**Proof:**

(1) Note in the constructive proof of Lemma 2 that \( \emptyset = L_0 \subset L_{\rho_1} \subset L_{\rho_2} \) and \( \emptyset = U_0 \subset U_{\rho_3} \subset U_{\rho_4} \) etc.

(2): Note that for \( i \in U_\rho, a^*_i(\rho) = u_i/\rho \) is decreasing in \( \rho \).

(3): Note that for \( i \notin U_\rho, a^*_i(\rho) \) is increasing in \( \rho \).

(4): The result follows again from the constructive proof of Lemma 2.

The main result in the paper is the following:

**Theorem 4** For problem \( (P_1) \), there exists a partition of the retailer set \( \{1, \ldots, k\} \) into three disjoint subsets \( L, U \) and \( R \) such that an optimal allocation \( a^* \) is such that \( a^*_{ij} = l_{ij} \) for \( i \in L \), \( a^*_{ij} = u_{ij} \) for \( i \in U \) and for which \( a^*_{ij} \) is such that \( 1 - F(a^*_{ij}) = P(D_{ij} \geq a^*_{ij}) = \) constant for all \( i, j \in R \).

**Proof:**

The result follows as we let \( \rho \to 1 \), using Lemmas 2 and 3.

3 A Computational Example

Note,

\[ S_i(a_i) = a_i \mathbb{P}[D_i > a_i] + \mathbb{E}[D_i \leq a_i] \]

for each \( i : 5 \geq i \geq 1 \).

Figure 3 has five instances \( (D_1, \ldots, D_5) \) of the exponential distribution where the instances have the following parameters:

\[ \lambda_1 = 1/4, \lambda_2 = 1/12, \lambda_3 = 1/24, \lambda_4 = 1/36, \lambda_5 = 1/48. \]

Figure 3 maximizes the total expected supply. Moreover, the total available supply is limited to at most \( T = 10 \) units.

Generalizing this situation, consider the seven cases of different upper bounds on the total units available. Figure 4 shows the affects of increasing the total inventory \( T \).

The upper bounds for each \( a^*_i \) are in the last column of the table in Figure 3. For all values of \( T \geq 90 \), the upper bounds in Figure 4 are all satisfied, thus maximizing the objective functions.

4 Conclusions

Extending Derman’s work [2] seems like a fruitful area of research. This is particularly interesting for work on supply chain challenges as started in the current paper. Extending this work over multiple time periods seems interesting.

5 Acknowledgements

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**References:**


Maximizing Value Upper Bound

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<td>$a_3 \mathbb{P}[D_3 &gt; a_3^<em>] + \mathbb{E}[D_3 \leq a_3^</em>]$</td>
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<td>$a_4 \mathbb{P}[D_4 &gt; a_4^<em>] + \mathbb{E}[D_4 \leq a_4^</em>]$</td>
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<td>$a_5 \mathbb{P}[D_5 &gt; a_5^<em>] + \mathbb{E}[D_5 \leq a_5^</em>]$</td>
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Figure 3: Maximizing $G$ where $a_1^* + \cdots + a_5^* \leq 10$.

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Figure 4: Maximizing $G$ where $a_1^* + \cdots + a_5^* \leq T$.


