Numerical simulation of generalized second-grade fluids using a 1D hierarchical model

FERNANDO CARAPAU
Universidade de Évora
Dept. Matemática and CIMA/UE
R. Romão Ramalho, 7000 Évora
PORTUGAL
flc@uevora.pt

ADÉLIA SEQUEIRA
Instituto Superior Técnico
Dept. Matemática and CEMAT/IST
Av. Rovisco Pais, 1049 Lisboa
PORTUGAL
adelia.sequeira@math.ist.utl.pt

JOÃO JANELA
Inst. Sup. Economia e Gestão
Dept. Matemática and CEMAT/IST
R. Quelhas, 1200 Lisboa
PORTUGAL
jjanela@iseg.utl.pt

Abstract: We consider the flow of a non-Newtonian incompressible second-grade fluid in an uniform rectilinear pipe and generalize it by introducing a shear-dependent viscosity function of power law type. The full 3D set of equations is reduced to a one-dimensional problem involving only time and one spatial variable. This is done using a director theory for fluid dynamics, also called Cosserat theory. An axisymmetric unsteady relationship between mean pressure gradient and volume flow rate over a finite section of the pipe and the corresponding equation to the wall shear stress are derived from this theory.

Key–Words: Cosserat theory, generalized second-grade fluid, axisymmetric motion, volume flow rate, pressure gradient, unsteady rectilinear flow, power law viscosity.

1 Introduction

The Cauchy stress tensor for a general incompressible and homogeneous Rivlin-Ericksen fluid of second-grade is given by (see e.g. Colemann and Noll [7])

$$T = -p^* I + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \qquad (1)$$

where p^* is the pressure, $-p^* I$ is the spherical part of the stress due to the constraint of incompressibility, μ is the constant viscosity, and α_1, α_2 are material constants usually called normal stress moduli. The kinematical first two Rivlin-Ericksen tensors A_1 and A_2 are defined through (see Rivlin and Ericksen [15])

$$A_1 = \nabla v^* + \big(\nabla v^*\big)^T \qquad (2)$$

and

$$A_2 = \frac{d}{dt} A_1 + A_1 \nabla v^* + \big(\nabla v^*\big)^T A_1 \qquad (3)$$

where v^* is the velocity field and $\frac{d}{dt}(\cdot)$ denotes the material time derivative. In equation (3) the material time derivative of the tensor A_1 is given by

$$\frac{d}{dt} A_1 = \frac{\partial A_1}{\partial t} + v^* \cdot \nabla A_1.$$

The model associated to the constitutive equation (1) has been studied by several authors (see e.g. [1], [8], [10]) under different perspectives. In this work, we consider an extension of the Rivlin-Ericksen fluid model of second-grade by introducing a shear-dependent viscosity (see e.g. [14], [17]). This means that the constitutive equation (1), becomes

$$T = -p^* I + \mu\big(|\dot\gamma|\big) A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \qquad (4)$$

where

$$\mu\big(|\dot\gamma|\big) : \mathbb{R}^+ \to \mathbb{R}^+$$

is the shear-dependent viscosity function and $\dot\gamma$ is a scalar measure of the rate of shear defined by $\dot\gamma = \sqrt{2D : D}$ with

$$D := \frac{1}{2}\big(\nabla v^* + \big(\nabla v^*\big)^T\big)$$

being the rate of deformation tensor. The particular functional dependence of the viscosity on shear-rate is generally chosen in order to fit experimental data and, in the case of a power law fluid model, is given by

$$\mu(|\dot\gamma|) = k |\dot\gamma|^{n-1} \qquad (5)$$

where the parameters k and n are called the consistency and the flow index (positive constants), respectively. If $n = 1$ in (5), the Cauchy stress tensor (4) corresponds to the second-grade constitutive equation (1) with constant viscosity $\mu = k$. If $n < 1$ in (5) then

$$\lim_{|\dot\gamma| \to +\infty} \mu(|\dot\gamma|) = 0, \quad \lim_{|\dot\gamma| \to 0} \mu(|\dot\gamma|) = +\infty,$$

and we have a shear-thinning fluid behaviour (viscosity decreases monotonically with shear rate). For $n > 1$ in (5), we get

$$\lim_{|\dot{\gamma}| \to +\infty} \mu(|\dot{\gamma}|) = +\infty, \quad \lim_{|\dot{\gamma}| \to 0} \mu(|\dot{\gamma}|) = 0,$$

and the fluid shows a shear-thickening behaviour (viscosity increases with shear rate). This theoretical viscosity model has limited applications to real fluids due to the unboundedness of the viscosity function (see Figure 1), but is widely used and can be accurate for specific flow regimes. The theoretical study of the model associated to the constitutive equation (4), namely existence, uniqueness and regularity of classical and weak solutions with any $\alpha_1, \alpha_2 \in \mathbb{R}$ is an open problem. In this paper we are interested in the numerical study of the model associated to equation (4), using the director approach (also called Cosserat theory) developed by Caulk and Naghdi [6]. This theory includes an additional structure of directors (deformable vectors) assigned to each point on a space curve (Cosserat curve), where a 3D system of equations is replaced by a 1D system depending on time and on a single spatial variable. In the first half of the last century, this theory has been used in studies of rods, plates and shells, see e.g. Ericksen and Truesdell [9], Green et al. [11]. Later, the Cosserat theory has been developed by Caulk and Naghdi [6], Green and Naghdi [12], [13] in studies of unsteady and steady flows, related to fluid dynamics. Recently, this theory approach has been applied to blood flow in the arterial system by Robertson and Sequeira [16] and also by Carapau and Sequeira [2], [3], [4], [5] considering Newtonian and non-Newtonian flows.

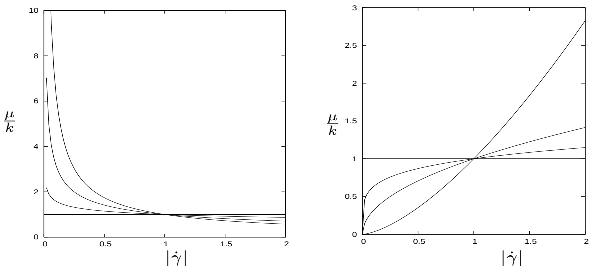

Figure 1: Power law model: (left) shear-thinning viscosity and (right) shear-thickening viscosity. In both cases the plots were obtained for different values of the flow index.

Using the director theory (see [6]) the velocity field, can be approximated by the following finite summation[1]:

$$\boldsymbol{v}^* = \boldsymbol{v} + \sum_{N=1}^{k} x_{\alpha_1} \dots x_{\alpha_N} \boldsymbol{W}_{\alpha_1 \dots \alpha_N}, \tag{6}$$

with

$$\boldsymbol{v} = v_i(z,t)\boldsymbol{e}_i, \quad \boldsymbol{W}_{\alpha_1 \dots \alpha_N} = W^i_{\alpha_1 \dots \alpha_N}(z,t)\boldsymbol{e}_i. \tag{7}$$

Here, $\boldsymbol{v}$ represents the velocity along the axis of symmetry z at time t, $x_{\alpha_1} \dots x_{\alpha_N}$ are polynomial weighting functions with order k (the number k identifies the order of the hierarchical theory and is related to the number of directors), the vectors $\boldsymbol{W}_{\alpha_1 \dots \alpha_N}$ are the director velocities which are completely symmetric with respect to their indices and $\boldsymbol{e}_i$ are the associated unit basis vectors. From this velocity field approach that we use to predict some of the main properties of the three-dimensional problem, we obtain the axisymmetric unsteady relationship between mean pressure gradient and volume flow rate over a finite section of a straight rigid and impermeable pipe with circular cross-section and constant radius. Also, we obtain the correspondent equation for the wall shear stress.

2 Flow modelling

We consider an homogeneous fluid moving within a straight, rigid and impermeable pipe with circular cross-section and constant radius ϕ, the domain Ω (see Figure 2) contained in $\mathbb{R}^3$, where the boundary $\partial\Omega$ is composed by Γ_1 (proximal cross-section), Γ_2 (distal cross-section) and by Γ_w the lateral wall of the pipe.

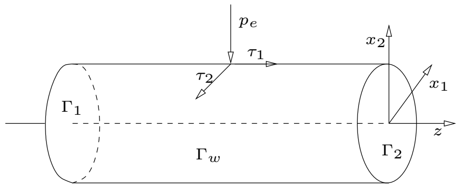

Figure 2: Fluid domain Ω with the components of the surface traction vector τ_1, τ_2 and p_e. The rectangular cartesian coordinates are denoted by x_i $(i = 1,2,3)$ and for convenience we set $x_3 = z$.

Using the notation adopted in Caulk and Naghdi [6], the components of the three-dimensional equations governing the axisymmetric motion of an incompressible generalized

[1] Latin indices subscript take the values $1,2,3$, Greek indices subscript $1,2$. Summation convention is employed over a repeated index.

second-grade viscous fluid, without body forces, in an uniform rectilinear pipe, are given by

$$\begin{cases} \rho\Big(\dfrac{\partial \boldsymbol{v}^*}{\partial t} + v^*_{,i}\boldsymbol{v}^*_i\Big) = t_{i,i}, & \\ & in \;\; \Omega\times(0,T), \\ v^*_{i,i} = 0, & \\ t_i = -p^*\boldsymbol{e}_i + \sigma_{ij}\boldsymbol{e}_j,\; \boldsymbol{t} = \vartheta^*_i t_i, & \end{cases} \tag{8}$$

with the initial condition

$$\boldsymbol{v}^*(x,0) = \boldsymbol{v}_0(x) \;\text{ in }\; \Omega, \tag{9}$$

and the homogeneous Dirichlet boundary condition

$$\boldsymbol{v}^*(x,t) = 0 \;\text{ on }\; \Gamma_w \times (0,T), \tag{10}$$

where $\boldsymbol{v}^* = v^*_i\boldsymbol{e}_i$ is the velocity field and ρ is the constant fluid density. Equation $(8)_1$ represents the balance of linear momentum and $(8)_2$ is the incompressibility condition. In equation $(8)_3$, $\boldsymbol{t}$ denotes the Cauchy stress tensor on the surface whose outward unit normal is $\vartheta^* = \vartheta^*_i\boldsymbol{e}_i$, and t_i are the components of $\boldsymbol{t}$, and σ_{ij} are the components of the extra stress tensor, given by

$$\sigma_{ij} = \mu\big(|\dot\gamma|\big)\big(A_1\big)_{ij} + \alpha_1\big(A_2\big)_{ij} + \alpha_2\big(A_1\big)_{ik}\big(A_1\big)_{kj} \tag{11}$$

where the viscosity function is given by (5). The components of the first two Rivlin-Ericksen tensors are given by

$$\big(A_1\big)_{ij} = \frac{\partial v^*_i}{\partial x_j} + \frac{\partial v^*_j}{\partial x_i}, \tag{12}$$

and

$$\begin{aligned} \big(A_2\big)_{ij} &= \frac{\partial \big(A_1\big)_{ij}}{\partial t} + v^*_k \frac{\partial \big(A_1\big)_{ij}}{\partial x_k} \\ &+ \big(A_1\big)_{ik}\frac{\partial v^*_k}{\partial x_j} + \frac{\partial v^*_k}{\partial x_i}\big(A_1\big)_{kj}. \end{aligned} \tag{13}$$

We assume that the lateral surface Γ_w of the axisymmetric pipe is defined by

$$\phi^2 = x_\alpha x_\alpha, \tag{14}$$

and the components of the outward unit normal to this surface are

$$\vartheta^*_\alpha = \frac{x_\alpha}{\phi}, \;\; \vartheta^*_3 = 0. \tag{15}$$

Since equation (14) defines a material surface, the velocity field must satisfy the kinematic condition

$$-x_\alpha v^*_\alpha = 0 \tag{16}$$

on the boundary (14). Averaged quantities such as flow rate and average pressure are needed to study 1D models. Consider $S(z,t)$ as a generic axial section of the pipe at time t defined by the spatial variable z and bounded by the circle defined in (14) and let $A(z,t)$ be the area of this section $S(z,t)$. Then, the volume flow rate Q is defined by

$$Q(z,t) = \int_{S(z,t)} v^*_3(x_1,x_2,z,t)da, \tag{17}$$

and the average pressure $\bar p$, by

$$\bar p(z,t) = \frac{1}{A(z,t)}\int_{S(z,t)} p^*(x_1,x_2,z,t)da. \tag{18}$$

3 Director theory approach

Using the director theory approach (6) it follows (see [6]) that the approximation of the velocity field $\boldsymbol{v}^* = v^*_i(x_1,x_2,z,t)\boldsymbol{e}_i$, with nine directors, is given by

$$\begin{aligned} \boldsymbol{v}^* &= \Big[\, x_1(\xi + \sigma(x_1^2+x_2^2)) - x_2(\omega + \eta(x_1^2+x_2^2)) \,\Big]\boldsymbol{e}_1 \\ &+ \Big[\, x_1(\omega + \eta(x_1^2+x_2^2)) + x_2(\xi + \sigma(x_1^2+x_2^2)) \,\Big]\boldsymbol{e}_2 \\ &+ \Big[\, v_3 + \gamma(x_1^2+x_2^2) \,\Big]\boldsymbol{e}_3 \end{aligned} \tag{19}$$

where $\xi,\omega,\gamma,\sigma,\eta$ are scalar functions of the spatial variable z and time t. The physical significance of these scalar functions in (19) is the following: γ is related to transverse shearing motion, ω and η are related to rotational motion (also called swirling motion) about $\boldsymbol{e}_3$, while ξ and σ are related to transverse elongation. Also, from Caulk and Naghdi [6], the expression of the stress vector on the lateral surface in terms of its normal and τ_1,τ_2,p_e is given by

$$\begin{aligned} \boldsymbol{t}_w &= \Big[\frac{1}{\phi}\Big(-p_e x_1 - \tau_2 x_2\Big)\Big]\boldsymbol{e}_1 \\ &+ \Big[\frac{1}{\phi}\Big(-p_e x_2 + \tau_2 x_1\Big)\Big]\boldsymbol{e}_2 \\ &+ \Big[\tau_1\Big]\boldsymbol{e}_3. \end{aligned} \tag{20}$$

Now, taking into account the boundary conditions (10) and incompressibility condition $(8)_2$, the velocity field (19), for a flow in a rigid pipe, without rotation, becomes

$$\boldsymbol{v}^* = \Big[\frac{2Q(t)}{\pi\phi^2}\big(1 - \frac{x_1^2+x_2^2}{\phi^2}\big)\Big]\boldsymbol{e}_3, \tag{21}$$

where the volume flow rate is a function of time t, given by

$$Q(t) = \frac{\pi}{2}\phi^2 v_3(z,t).$$

Instead of satisfying the momentum equation $(8)_1$ pointwisely in the fluid, we impose the following integral conditions

$$\int_{S(z,t)} \Big[t_{i,i} - \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v^*_i\Big)\Big] da = 0, \qquad (22)$$

$$\int_{S(z,t)} \Big[t_{i,i} - \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v^*_i\Big)\Big] x_{\alpha_1}\dots x_{\alpha_N} da = 0, \qquad (23)$$

where $N = 1, 2, 3$.

Using the divergence theorem and integration by parts, equations (22) – (23) for nine directors, can be reduced to the four vector equations:

$$\frac{\partial \boldsymbol{n}}{\partial z} + \boldsymbol{f} = \boldsymbol{a}, \qquad (24)$$

$$\frac{\partial \boldsymbol{m}^{\alpha_1\dots\alpha_N}}{\partial z} + \boldsymbol{l}^{\alpha_1\dots\alpha_N} = \boldsymbol{k}^{\alpha_1\dots\alpha_N} + \boldsymbol{b}^{\alpha_1\dots\alpha_N}, \qquad (25)$$

where $\boldsymbol{n}$, $\boldsymbol{k}^{\alpha_1\dots\alpha_N}$, $\boldsymbol{m}^{\alpha_1\dots\alpha_N}$ are resultant forces defined by

$$\boldsymbol{n} = \int_S \boldsymbol{t}_3 da, \quad \boldsymbol{k}^\alpha = \int_S \boldsymbol{t}_\alpha da, \qquad (26)$$

$$\boldsymbol{k}^{\alpha\beta} = \int_S \Big(\boldsymbol{t}_\alpha x_\beta + \boldsymbol{t}_\beta x_\alpha\Big) da, \qquad (27)$$

$$\boldsymbol{k}^{\alpha\beta\gamma} = \int_S \Big(\boldsymbol{t}_\alpha x_\beta x_\gamma + \boldsymbol{t}_\beta x_\alpha x_\gamma + \boldsymbol{t}_\gamma x_\alpha x_\beta\Big) da, \qquad (28)$$

$$\boldsymbol{m}^{\alpha_1\dots\alpha_N} = \int_S \boldsymbol{t}_3 x_{\alpha_1}\dots x_{\alpha_N} da. \qquad (29)$$

The quantities $\boldsymbol{a}$ and $\boldsymbol{b}^{\alpha_1\dots\alpha_N}$ are inertia terms defined by

$$\boldsymbol{a} = \int_S \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v^*_i\Big) da, \qquad (30)$$

$$\boldsymbol{b}^{\alpha_1\dots\alpha_N} = \int_S \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v^*_i\Big) x_{\alpha_1}\dots x_{\alpha_N} da, \qquad (31)$$

and $\boldsymbol{f}$, $\boldsymbol{l}^{\alpha_1\dots\alpha_N}$, which arise due to surface traction on the lateral boundary, are defined by

$$\boldsymbol{f} = \int_{\partial S} \boldsymbol{t}_w ds, \qquad (32)$$

$$\boldsymbol{l}^{\alpha_1\dots\alpha_N} = \int_{\partial S} \boldsymbol{t}_w x_{\alpha_1}\dots x_{\alpha_N} ds. \qquad (33)$$

The equation relating the mean pressure gradient (wall shear stress, respectively) with the volume flow rate will be obtained using the above equations.

4 Results and discussion

Replacing the results (26) – (33) obtained by the Cosserat theory, with nine directors, into equations (24) – (25), we get the following unsteady relationship[2]

$$\begin{aligned}\bar{p}_z(z,t) &= -\frac{4k\big(2^{\frac{5n+1}{2}}\big)Q^n(t)}{(n+3)\pi^n\phi^{3n+1}} \\ &- \frac{4\rho}{3\pi\phi^2}\Big(1 + 6\frac{\alpha_1}{\rho\phi^2}\Big)\dot{Q}(t), \qquad (34)\end{aligned}$$

and the axial component τ_1 of the stress tensor on the lateral surface of the domain Ω is the wall shear stress, given by

$$\begin{aligned}\tau_1 &= \frac{k\big(2^{\frac{5n+1}{2}}\big)Q^n(t)}{(n+3)\pi^n\phi^{3n}} \\ &+ \frac{\rho}{6\pi\phi^2}\Big(1 + 24\frac{\alpha_1}{\rho\phi^2}\Big)\dot{Q}(t). \qquad (35)\end{aligned}$$

Now, integrating equation (34), over a finite section of the pipe, between z_1 and z_2 with $z_1 < z_2$, we obtain the following equation.

$$\begin{aligned}G(t) &= \frac{\bar{p}(z_1,t) - \bar{p}(z_2,t)}{z_2 - z_1} \qquad (36) \\ &= \frac{4k\big(2^{\frac{5n+1}{2}}\big)Q^n(t)}{(n+3)\pi^n\phi^{3n+1}} + \frac{4\rho}{3\pi\phi^2}\Big(1 + 6\frac{\alpha_1}{\rho\phi^2}\Big)\dot{Q}(t).\end{aligned}$$

Solving equation (36), we can compute the volume flow rate in terms of the mean pressure gradient, $G(t)$. Setting $n = 1$ in (36) we recover the solution for a Rivlin-Ericksen viscous fluid of second-grade (see [4]), while setting $n > 1$ or $n < 1$ we obtain the results in the shear-thickening or shear-thinning cases, respectively. Also, in (36) if $\alpha_1 = 0$, we recover the results obtained by Carapau and Sequeira [3]. In the next sections we have fixed the parameters $\rho = k = \phi = \alpha_1 = 1$ and $\alpha_2 \in \mathbb{R}$, i.e. the only free parameter is n. This is enough to derive the qualitative behaviour of the 1D reduced model. In the full 3D problem or in curved geometries the normal stress moduli α_1 and α_2 would also have a significative impact.

Flow under constant pressure gradient

In the particular case of a constant mean pressure gradient $G(t) = G_0$ the system converges toward a steady state solution. In Figure 3 this steady state volume flow rate is obtained solving the time dependent problem but, if we are not interested in

[2] Were the notation $\dot{Q}$ is used for time differentiation.

the behaviour during the initial transient phase, the steady (asymptotic) value of the volume flow rate can be obtained directly from (36) setting $\dot{Q}(t) = 0$, since at constant pressure gradient $\dot{Q}(t)$ converges to zero as t goes to infinity. Therefore the steady solution is characterized by

$$Q(t) = \frac{\phi^3}{4\sqrt{2}\sqrt[n]{4}} \sqrt[n]{\frac{G_0(n+3)\phi}{4k}}, \quad (37)$$

$$\tau_1 = \frac{G_0\phi}{4}, \quad (38)$$

which is in excellent agreement with the numerical results shown in Figure 3.

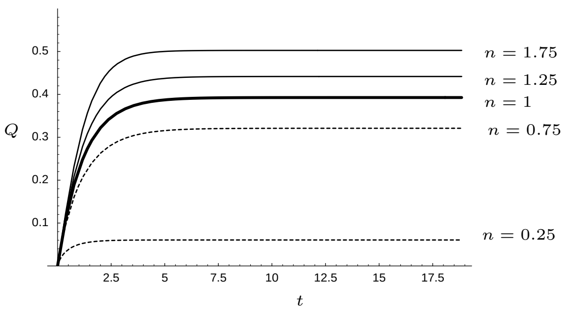

Figure 3: Time evolution of the volume flow rate for different values of n.

We see from these results that there is no qualitative difference between the shear-thinning and the shear-thickening cases, except from the fact that the corresponding curves become more dense as n increases.

Figure 4 shows how the volume flow rate varies with the mean pressure gradient. For $n = 1$ there is a linear relation between these quantities, but when a variable viscosity is introduced the relation becomes non-linear. The growth of volume flow rate is faster for the shear-thinning viscosity (doted lines).

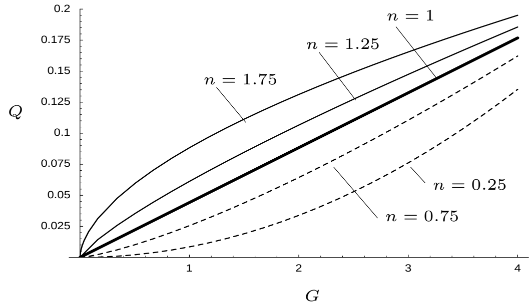

Figure 4: Mean pressure gradient vs asymptotic volume flow rate.

Time dependent mean pressure gradient

In the general situation of imposing a time dependent mean pressure gradient, the theory still holds, but additional conditions must be imposed in order to get convenient solutions. We will only briefly show results indicating some difficulties that can occur in this case, leaving detailed analysis for a future publication. In Figure 5 we can observe from the doted lines that if the power index n is too low, one can obtain unphysical solutions. The theory only holds for values of n above a certain threshold that depends on the lower bound of $G(t)$.

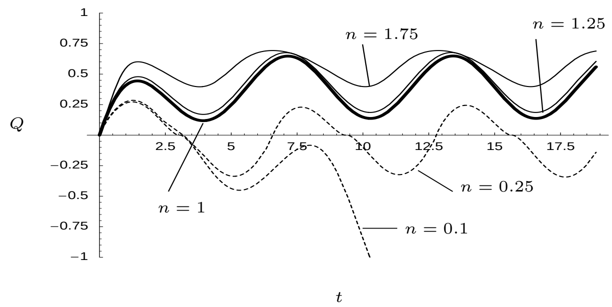

Figure 5: Time evolution of volume flow rate for several values of n with $G(t) = 1 + \cos(t)$.

5 Conclusions

A generalization of the constitutive equation for second-grade fluids has been obtained by considering a shear-dependent viscosity function. Although there is some controversly related to the existence of real fluids of second-grade type, the model has been successfully applied to particular flow situations of polymeric liquids and, therefore, the use of a variable viscosity is an important contribution to the applicability of the second-grade model to real flows. A nine-director theory has been applied to the generalized second-grade model in a straight, rigid and impermeable pipe and some steady and unsteady relationships between mean pressure gradient (wall shear stress, respectively) and volume flow rate were obtained. Lacking reference solutions for this model, the quality of the obtained relations is not easy to assess. Nonetheless, analogy with similar application of Cosserat theory (see [3]) seems to indicate an error of about 1%. In the case of time dependent mean pressure gradient, the theory may fail for low values of the power index n if the mean pressure gradient is too small.

Acknowledgements: This work has been partially supported by the research centers CEMAT/IST and CIMA/UE, through FCT´s funding program and by project FCT/POCTI/MAT/61792/2004.

References:

[1] P.H. Boulanger, H. Hayes, and K.R. Rajagopal, Some unsteady exact solutions in the Navier-Stokes and the second-grade fluid theories, *SAACM*, Vol.1, No.2, 1991, pp.185-203.

[2] F. Carapau, *Development of 1D Fluid Models Using the Cosserat Theory. Numerical Simulations and Applications to Haemodynamics*, PhD Thesis, IST, 2005.

[3] F. Carapau, and A. Sequeira, 1D Models for blood flow in small vessels using the Cosserat theory, *WSEAS Transactions on Mathematics*, Issue 1, Vol.5, 2006, pp.54-62.

[4] F. Carapau, and A. Sequeira, Axisymmetric motion of a second order viscous fluid in a circular straight tube under pressure gradients varying exponentially with time, *Advances in Fluid Mechanics VI*, WIT Trans. Eng. Sci., Vol.52, WIT Press, Southampton, 2006, pp.409-419.

[5] F. Carapau, and A. Sequeira, Unsteady flow of a generalized Oldroyd-B fluid using a director theory approach, *WSEAS Transactions on Fluid Mechanics*, Issue 2, Vol.1, 2006, pp.167-174.

[6] D.A. Caulk, and P.M. Naghdi, Axisymmetric motion of a viscous fluid inside a slender surface of revolution, *Journal of Applied Mechanics*, Vol.54, 1987, pp.190-196.

[7] B.D. Coleman, and W. Noll, An approximation theorem for functionals with applications in continuum mechanics, *Arch. Rational Mech. Anal.*, Vol.6, 1960, pp.355-370.

[8] V. Coscia, and G.P. Galdi, Existence, uniqueness and stability of regular steady motions of a second-grade fluid, *Int. J. Non-Linear Mechanics*, Vol.29, No.4, 1994, pp.493-506.

[9] J.L. Ericksen, and C. Truesdell, Exact theory of stress and strain in rods and shells, *Arch. Rational Mech. Anal.*, Vol.1, 1958, pp.295-323.

[10] G.P. Galdi, and A. Sequeira, Further existence results for classical solutions of the equations of a second-grade fluid, *Arch. Rational Mech. Anal.*, Vol.128, 1994, pp.297-312.

[11] A.E. Green, N. Laws, and P.M. Naghdi, Rods, plates and shells, *Proc. Camb. Phil. Soc.*, Vol.64, 1968, pp.895-913.

[12] A.E. Green, and P.M. Naghdi, A direct theory of viscous fluid flow in pipes: I Basic general developments, *Phil. Trans. R. Soc. Lond. A*, Vol.342, 1993, pp. 525-542.

[13] A.E. Green, and P.M. Naghdi, A direct theory of viscous fluid flow in channels, *Arch. Ration. Mech. Analysis*, Vol.86, 1984, pp.39-63.

[14] M. Massoudi, and T.X. Phuoc, Flow of a generalized second grade non-Newtonian fluid with variable viscosity,*J. Continuum Mechanics and Thermodynamics*, Issue 6, Vol.16, 2004, pp.529-538.

[15] R.S. Rivlin, and J.L. Ericksen, Stress-deformation relations for isotropic materials, *J. Rational Mech. Anal.*, Vol.4, 1955, pp.323-425.

[16] A.M. Robertson, and A. Sequeira, A Director Theory Approach for Modeling Blood Flow in the Arterial System: An Alternative to Classical 1D Models, *Mathematical Models & Methods in Applied Sciences*, 15, nr.6, 2005, pp.871-906.

[17] E. Walicki, and A. Walicka, Convergent Flows of Molten Polymers Modeled by generalized Second-Grade Fluids of Power-Law type, *J. Mechanics of Composite Materials*, Vol.38, nr1, 2002, pp.89-94.