The Stability of the Single Population’s System Under Environmental Toxin Pulse Control

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Abstract: In this paper, we investigate a single population model in a polluted environment with impulsive toxicant input at fixed moment. We show that the population is extinct when the impulsive period is less than some critical value. Otherwise the population is permanent. We have drawn the universal law under this kind of situation. Finally we simulated the solution that we drawn in this paper, and witch verified exactness of the conclusion.

Key-Words: Impulsive, Permanent, Periodic solution, Stability, Control, System model.

1 Introduction

In recent years, a lot of scholars have adopted mathematics modeling approach [1,2,4-9] to study the influence of environmental pollution on the surviving of biological population; all the previous models established assumed that the exogenous input of toxicant was continuous. The toxicants, however, are often emitted to the environment with regular pulse. A lot of data have indicated that the use of agriculture chemicals may cause potential harm to the health of both human beings and living beings. If the spraying of agriculture chemicals can be regarded as timing-pulse discharge, the continuous input of toxin can be regarded as being discharged and replaced by an impulsive perturbation. In this case, though the discharge of toxin is transient, the influence of the toxin will last long. It is of great significance to control the toxin-discharging pulse cycle, so to make biological population survive continuously.

After having taken into account not only the environmental toxin pulse-inputs, but also the toxin that biological population has absorbed from food chain, this paper has set up a model for single population system. Systematic research on the model established enables the author to attain the key factor, i.e., threshold value, for controlling the continuous surviving of living beings, which has been verified through digital simulation. The finding in this research is of great referential significance not only to environmental pollution control, but to the sustainable development of ecological environment.

2 Model Formulation

Let \( x(t) \) is the density of the species at time \( t \)
\( c_o(t) \) is the concentration of toxicant in the organism at time \( t \)
\( c_e(t) \) is the concentration of toxicant in the environment at time \( t \)
\( r_i \) is the intrinsic growth rate of the population in the environment without toxicant \( r_i > 0 \) is the decreasing rate of the intrinsic growth rate associated with the uptake of the toxicant \( a_i c_e(t) \) represents the organism’s net uptake of toxicant from the environment \( d \beta k c_e(t) \) represents the quantity of toxicant which the population absorbs from the food chain \( -g c_o(t) \) and \( -mc_o(t) \) represent the elimination and depuration rates of toxicant in the organism; \( -h c_e(t) \) represents the toxicant loss from the environment itself by volatilization and so on. Here, we assume that the capacity of the environment is so large that the change of toxicant in environment that comes from uptake and elimination by the organisms can be neglected. Now we may set up the following simplified single species system model in a polluted environment with pulse toxicant input at fixed moment,
\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t)[r(c_o(t))B(C + r_o - a) - ax(t)] \\
\frac{dc_o(t)}{dt} &= \alpha c_o(t) + \frac{d_i}{d_i} - \beta c_o(t) - g_c(t) - m_c(t) \\
\frac{dc_i(t)}{dt} &= -h_c(t)
\end{align*}
\]
\[\Delta t = 0, \Delta c_o(t) = 0, \Delta c_i(t) = \beta, \quad t = n \tau, n \in \mathbb{Z}^+\]
\[x(0) > 0, 0 \leq c_o(t) \leq 1, 0 \leq c_i(t) \leq 1\]

where

\[\Delta t = x(t^+) - x(t^-), \Delta c_o(t) = c_o(t^+) - c_o(t^-), \Delta c_i(t) = c_i(t^+) - c_i(t^-)\]

\[r_o, r_i, B, C, a, a_i, d_i, k, g, m, h\] in model (1) are positive constants and \(Z^+ = \{1, 2, 3, \cdots\}; \tau\) is the period of the impulsive effect about the exogenous input of toxicant and \(b\) is the toxicant input amount at every time

**Remark:** Each of \(c_o(t)\) and \(c_i(t)\) is a concentration, and thus, these variables satisfy the inequalities \(0 \leq c_o(t) \leq 1\) and \(0 \leq c_i(t) \leq 1\). In order that \(c_o(t), c_i(t)\) remain less than one, it is necessary that

\[a_i + \frac{d_i}{d_i} \beta k \leq g + m, \quad b \leq 1 - e^{bc}\]  \hspace{1cm} (2)

In the following, we always assume that (2) holds true (see [8]).

First of all, let’s consider the following periodic logistic equation

\[
\frac{du(t)}{dt} = \alpha(t) - \beta(t)u(t))
\]

where \(\alpha(t), \beta(t)\) are \(\tau\)-periodic continuous functions on \(R\).

**Definition 1** the solution \(u(t)\) of equation (3) is said to be positive, if \(u(t) > 0\) for all \(t \geq 0\).

It’s easy to prove that the solution \(u(t)\) of equation (3) is positive if and only if the initial \(u(0) > 0\).

**Lemma 1** [3] If \(\beta(t) > 0\) and \(\int_0^t \alpha(t)dt > 0\), then equation (3) has a unique positive \(\tau\)-periodic solution \(\tilde{u}(t)\) which is globally asymptotically stable, that is to say \(u(t) \to \tilde{u}(t)\) as \(t \to \infty\) for any positive solution \(u(t)\) of equation (3)

**Definition 2** Let \(\alpha\) is a continuous function in \([0, +\infty) \to R\), we say that \(\alpha\) is asymptotic to \(\alpha\) (\(\alpha\) is \(\tau\)-periodic continuous function) if

\[\lim_{t \to \infty} \alpha(t) - \alpha(t) = 0\]

Let \(v(t)\) denote the unique solution of the Cauchy problem

\[
\begin{align*}
\frac{dv(t)}{dt} &= v(t)(\alpha(t) - \beta(t)v(t)) \\
v(0) &= v_0
\end{align*}
\]

where \(\alpha(t)\) is asymptotic to \(\alpha\), \(\beta(t)\) is a \(\tau\)-periodic continuous function

**Lemma 2** If \(\beta(t) > 0\) for all \(t\) and \(\int_0^t \alpha(t)dt > 0\), let \(\tilde{u}(t)\) denote the unique positive \(\tau\)-periodic solution of (3), \(v(t)\) denote the unique solution of (4) with initial \(v_0\). Then for any \(v_0 > 0\), we have

\[v(t) \to \tilde{u}(t) \text{ as } t \to \infty\]

The proof of this lemma is similar to Theorem 2.4 of [3].

Now we consider the following subsystem of model (1)

\[
\begin{align*}
\frac{dc_o(t)}{dt} &= \alpha c_o(t) + \frac{d_i}{d_i} - \beta c_o(t) - g_c(t) - m_c(t) \\
\frac{dc_i(t)}{dt} &= -h_c(t)
\end{align*}
\]

\[\Delta t = 0, \Delta c_o(t) = 0, \Delta c_i(t) = \beta, \quad t = n \tau, n \in \mathbb{Z}^+\]

\[x(0) > 0, 0 \leq c_o(t) \leq 1, 0 \leq c_i(t) \leq 1\]

where \(\Delta c_o(t) = c_o(t^+) - c_o(t^-), \Delta c_i(t) = c_i(t^+) - c_i(t^-)\).

**Lemma 3.** System (5) has a unique positive \(\tau\)-periodic solution \((\tilde{c}_o(t), \tilde{c}_i(t))\), and for every solution \((c_o(t), c_i(t))\) of (5), \(c_o(t) \to \tilde{c}_o(t)\) and \(c_i(t) \to \tilde{c}_i(t)\) as \(t \to \infty\). Moreover \(c_o(t) > \tilde{c}_o(t)\) and \(c_i(t) > \tilde{c}_i(t)\) for all \(t \geq 0\) if

\[c_o(0) > \tilde{c}_o(0) \text{ and } c_i(0) > \tilde{c}_i(0)\].
3 Main Result

From Lemma 3 we may find that \( c'_\tau(t) \) and \( c_\tau(t) \) can be solved for successively. Therefore, the model (1) is a one-dimensional system, that is

\[
\begin{aligned}
\frac{dx(t)}{dt} &= x(t)\left[r(c_\tau(t))B(C+r-a)-aCx(t)\right] \\
x(0) &= x_\tau > 0 
\end{aligned}
\]  

(7)

where \( c_\tau(t) \) satisfies the properties of system (5).

**Theorem 1** Let \( x(t) \) be any one solution of (7)

Then \( x(t) \to 0 \) as \( t \to \infty \) if one of the following conditions is satisfied:

(i) \( \tau < \frac{r_0 h(g + m)}{r_1 a_1 + \frac{d}{a_t} \beta k b} \)

(ii) \( \tau = \frac{r_1 a_1 + \frac{d}{a_t} \beta k b}{r_0 h(g + m)} \) and \( c_0(0) > \tilde{c}_0(0), c_\tau(0) > \tilde{c}_\tau(0) \).

Proof. Supposed (i) is satisfied. Then we can choose a \( \varepsilon > 0 \), such that

\[
\delta = -\tau + \frac{r_1 a_1 + \frac{d}{a_t} \beta k b}{r_0 h(g + m)} > 0
\]

Note that \( c_\tau(t) \to \tilde{c}_\tau(t) \) as \( t \to \infty \) hence

\[
c_\tau(t) > \tilde{c}_\tau(t) - \varepsilon
\]

for all \( t \) large enough. For simplification, we assume (8) holds for all \( t > 0 \). From (7), we get

\[
\frac{dx(t)}{dt} = x(t)\left[r(c_\tau(t))B(C+r-a)-aCx(t)\right] \\
< x(t)\left[r_0 - r_1\tilde{c}_\tau(t)\right] \\
\leq x(t)\left[r_0 - r_1(\tilde{c}_\tau(t) - \varepsilon)\right]
\]

which leads to

\[
x((n+1)\tau) < x(n\tau) \exp\left[\int_{n\tau}^{(n+1)\tau} r_0 - r_1(\tilde{c}_\tau(t) - \varepsilon) dt\right]
\]

Therefore \( x(t) \to 0 \) as \( n \to \infty \). Therefore \( x(t) \to 0 \) as \( t \to \infty \) since \( 0 < x(t) < x(n\tau) \) for

\[
n\tau < t < (n+1)\tau
\]

Next suppose (ii) is satisfied. From Lemma (3), we have

\[
c_\tau(t) > \tilde{c}_\tau(t) \text{ for all } t > 0.
\]

Hence,

\[
\frac{dx(t)}{dt} = x(t)\left[r(c_\tau(t))B(C+r-a)-aCx(t)\right] \\
< x(t)\left[r_0 - r_1\tilde{c}_\tau(t)\right] - \frac{aC}{T_0}x(t)
\]

(9)

Integrating it from \( n\tau \) to \( (n+1)\tau \), we have

\[
x((n+1)\tau) < x(n\tau) \exp\left[\int_{n\tau}^{(n+1)\tau} r_0 - r_1\tilde{c}_\tau(t) - \frac{aC}{T_0}x(t) dt\right]
\]

where \( r_n = \frac{aC}{T_0} \int_{n\tau}^{(n+1)\tau} x(t) dt > 0 \).
We will prove \( r_n \to 0 \) as \( n \to \infty \). Otherwise, 
\[ \limsup_{n \to \infty} r_n = d > 0, \]
then there exists a subsequence \( r_{n_k} \) of \( r_n \) such that \( \lim r_{n_k} = d \). Thus there exists a \( k_0 \in \mathbb{Z}^+ \) we have \( r_{n_k} > d/2 \) for \( k > k_0 \) and \( \sum_{n=1}^{\infty} r_n > \sum_{n=k_0}^{\infty} r_{n_k} = \infty \). So
\[ x(n\tau) \leq x(0^+) \exp \left( - \sum_{i=1}^{\infty} r_i \right) \to 0 \text{ as } n \to \infty. \]
Which implies \( \lim r_n = 0 \), this is a contradiction. Hence, we have \( \lim r_n = 0 \) which implies \( x(t) \to 0 \) as \( t \to \infty \).

**Remark.** Let
\[ r_0 = 0.01, r_1 = 0.3, B = 0.4, C = 0.04, \]
\[ a = 0.06, a_i = 0.57, d_i = 0.6, \beta = 0.05, k = 0.05, \]
\[ g = 0.4, m = 0.2, b = 0.05, h = 0.071, \]
we simulated above conclusions (see figure 1, figure 2, and figure 3), and obtained the threshold \( \tau = 40.6568 \), then we choose the period of the impulsive input of toxicant, that is \( 10T `' \). From figure 1, figure 2, and figure 3, there exists periodic solution of \( c_0(t) \) and \( c_1(t) \), but the density of the biological population is reducing constantly, until trending towards 0 that is to say, the biological population will extinct.

Fig.1 the concentration of toxicant in the environment. Fig.2 the concentration of toxicant in the organism. Fig.3 the density of the species.

**Theorem 2.** If \( \tau > \frac{r_i \left( a_i + \frac{d_i}{a_i} \beta k \right) b}{r_i h (g + m)} \) then the species \( x(t) \) of system 7 is permanent. That is to say, there exist a positive number \( \delta \) and a finite time \( T \) such that \( x(t) \geq \delta \) for all \( t > T \).

**Proof.** Let \( \delta_1, \varepsilon > 0 \) be small enough such that
\[ \sigma = r_i \tau - \frac{r_i \left( a_i + \frac{d_i}{a_i} \beta k \right) b}{h (g + m)} - r_i \varepsilon \tau - f \delta_1 > 0 \]
Since \( c_0(t) \to \tilde{c}_0(t) \) as \( t \to \infty \), there exist a \( T_1 > 0 \) such that \( c_0(t) < \tilde{c}_0(t) + \varepsilon \) for \( t > T_1 \). We will prove \( x(t) \leq \delta_1 \) can not hold for all \( t > T_1 \).

Otherwise, we have
\[ \frac{dx}{dt} \geq x(t) \left( r_0 - r_i \left( \tilde{c}_0(t) + \varepsilon \right) - f \delta_1 \right) \]
for \( t > T_1 \). Let \( N_1 \in \mathbb{Z}^+ \) and \( N_i \tau \geq T_1 \). Integrating 12 on \( (n \tau, (n + 1) \tau) \), \( n > N_1 \), we have
\[ x(n \tau + 1) \tau \geq x(n \tau) \exp \left\{ \int_{n \tau}^{(n + 1) \tau} \left[ r_0 - r_i \left( \tilde{c}_0(t) + \varepsilon \right) - f \delta_1 \right] dt \right\} \\
= x(n \tau) \exp \left\{ r_i \tau - r_i \frac{kb}{h (g + m)} - r_i \varepsilon \tau - f \delta_1 \tau \right\} \\
= x(n \tau) \exp(\sigma). \]
Then \( x((N_i + k) \tau) \geq x(N_i \tau) \exp(\sigma) \to \infty \) as \( k \to \infty \), which is a contradiction. Hence there exist a \( T > T_1 \) such that \( x(T) > \delta \).

Now we claim \( x(t) \geq \delta \exp(-\beta \tau) \) for all \( t > T \) where
\[ \beta = \sup \left\{ |r_0 - r_i (\tilde{c}_0(t) + \varepsilon) - f \delta_1| : t \in [0, +\infty) \right\} \]
\[ 0 < \beta < \infty. \]

Otherwise there exist a \( t_0 > T \) such that \( x(t_0) < \delta_1 \exp(-\beta \tau) \) so there exist a \( t_i \in (T, t_0) \) such that
\[ x(t_i) = \delta \text{ and } x(t_i) < \delta \text{ for } t \in (t_i, t_{i+1}] \text{ let } \]
\[ N_2 \in \mathbb{Z}^+ \text{ and } t_0 \in (t_i + N_2 \tau, t_i + (N_2 + 1) \tau] , \text{ we have } \]
\[ \delta \exp(-\beta \tau) > x(t_0) = x(t_i) \int_{t_i}^{t_0} (r_0 - r_1 c_{\delta} - f) dt \]
\[ \geq \delta \exp \left( \int_{t_i}^{t_0} \left( r_0 - r_1 (\tilde{c}_0 + \varepsilon) - f \delta \right) dt \right) \]
\[ \geq \delta \exp \left( -\beta \tau \right) \]
which is a contradiction. Let \( \delta = \delta \exp(-\beta \tau) \), we obtain that \( x(t) \geq \delta \) for all \( t > T \). \( \square \)

**Remark.** Let \( \gamma = 0.05, r_i = 0.1, B = 0.4, C = 0.04, a = 0.06, a_i = 0.57, d_i = 0.6, \beta = 0.05, k = 0.05, g = 0.4, \]
\( m = 0.2, h = 0.05, b = 0.071 \), we simulated above conclusions (see figure 4, figure 5 and figure 6), and obtained the threshold \( \tau = 2.7105 \), then we choose the period of the impulsive input of toxicant \( T = 6 \), that is \( T > \tau \) from Figure 4, Figure 5 and Figure 6, there exists periodic solution of \( c_{\delta}(t) \) and \( c_{\varepsilon}(t) \), but the density of the biological population is rising constantly, after reaching a certain degree, \( x(t) \) tends towards stability, that is to say, the biological population will permanent.

Fig. 4 the concentration of toxicant in the environment

4 Conclusion

In the paper, we discussed improved single specie Smith model in polluted environment with impulsive toxicant input, and obtained the key factor, i.e., threshold value, for controlling the continue surviving of living beings, that is, when the period of the impulsive input of toxicant more than \( \tau = \left( \frac{a_i + d_i}{a_i} \beta k \right) b \)
\( \frac{r_i h (g + m)}{r_i h (g + m)} \) the biological population will permanent, or the biological population will extinct, when \( \beta = 0 \), that is not considering the factor of the toxin from the food chain, then the model (1) may be simplified. In this case, the key factor for controlling the continue surviving of living beings: threshold value \( \tau = \frac{r_i a_i b}{r_i h (g + m)} \), this is as same as the result of the Logistic model under the effects of toxin impulsive input in the environment. So the finding in this research is of great referential significance not only to environmental pollution control, but to the sustainable development of ecological environment.

Acknowledgment

This work is supported by the National sciences Foundation of China (10471040) and Youth Science Foundations of Shan xi Province (20041004).

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