Determinant of Mixed Matrices: The Analog Notion of Convex Geometry Notion of Volume of Mixed Bodies

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Abstract: We discuss the use of the Aleksandrov inequality for matrices. This inequality is a matrix version of the Aleksandrov-Fenchel inequality of convexity theory. The Aleksandrov-Fenchel inequality can be used to derive a number of other useful inequalities, among them the Minkowski inequality. By applying the Aleksandrov inequality to mixed determinants and determinants of mixed matrices, we obtain some important results analogous to those in convex geometry.


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1 Introduction

The notion of mixed determinants and mixed matrices for positive definite symmetric matrices give analog notions to the notions of mixed volumes for convex bodies and mixed bodies in convex geometry, respectively. Many convex geometry notions and inequalities have been developed for more than a century (for references see Lutwak [2] and Schneider [3]). Also many notions in convex geometry have analogs in linear algebra and matrix theory (see [4, 5, 6] for some of such examples). These linear algebra (or matrix) analog notions can be employed with other notions of the theory of matrix inequalities in order to obtain new inequalities for positive definite symmetric matrices.

In this paper we state a well-known theorem for inequalities of mixed determinants of positive definite symmetric matrices. This very useful theorem, which has many applications, called the Aleksandrov inequality is the matrix version of the Aleksandrov inequality of convex geometry. By applying the Aleksandrov inequality to mixed matrices, we obtain a number of other useful inequalities, among them the Minkowski inequality.

2 Materials and Methods

We begin by stating the definition of cofactor matrix, Blaschke summation, definition and axiomatic properties of mixed determinant, then we quote the very useful Aleksandrov inequality in subsection 2.1. In subsection 2.2 we introduce the concept of mixed matrix and state the properties of it. The final subsection of this section we present results. We use the materials of subsection 2.1 and 2.2 to obtain our results.

2.1 Blaschke summation, Mixed determinant and The Aleksandrov inequality

Definition 1 (Cofactor Matrix). The cofactor matrix of $A$, $C_A$, is the transpose of the classical adjoint of $A$, thus it is defined by

$$(C_A)_{ij} := (-1)^{i+j} DA(i|j)$$

where $A(i|j)$ denote the $(n-1) \times (n-1)$ matrix obtained by deleting row $i$ and column $j$ of matrix $A$.

Definition 2 (Blaschke Summation [2, 3]). The Blaschke summation of $n \times n$ matrices $A$ and $B$, denoted by $A \oplus B$, is defined as the matrix whose cofactor matrix is the sum of cofactor matrices of $A$ and $B$;
that is, it satisfies the following equality:

$$
C(A + B) = C(A) + C(B)
$$

**Definition 3** (Mixed Determinant$^1$[4, 5, 6]). Let $A_1, A_2, \ldots, A_r$ be $n \times n$ symmetric matrices, $\lambda_1, \lambda_2, \ldots, \lambda_r$ be nonnegative real numbers. Then the determinant of $\lambda_1A_1 + \cdots + \lambda_rA_r$ can be written as

$$
D(\lambda_1A_1 + \cdots + \lambda_rA_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} D(A_{i_1}, \ldots, A_{i_n}),
$$

where the sum is taken over all $n$-tuples of positive integers $(i_1, \ldots, i_n)$ whose entries do not exceed $r$. The coefficient $D(A_{i_1}, \ldots, A_{i_n})$, with $A_{i_k}, 1 \leq k \leq n$ from the set $\{A_1, \ldots, A_r\}$, is called the mixed determinant of the matrices $A_{i_1}, \ldots, A_{i_n}$.

**Properties of Mixed Determinants:** Let $A_1, A_2, \ldots, A_n, A, B$ and $B'$ be $n \times n$ symmetric matrices, $\lambda_1, \lambda_2, \ldots, \lambda_n$ be nonnegative real numbers.

1. 

$$
D(A, \ldots, A, B) = D(A, \ldots, A, B, A)
$$

$$
= \cdots
$$

$$
= D(A, B, A, \ldots, A)
$$

$$
= D(B, A, \ldots, A).
$$

In fact, the mixed determinant is symmetric in its arguments, so in a larger generality one has:

$$
D(A, \ldots, A, B, \ldots, B) = \cdots
$$

$$
= D(B, \ldots, B, A, \ldots, A).
$$

We use the notation $D(A, n-k; B, k)$ to represent any of $D(A, A, \ldots, A, B, \ldots, B), \ldots, D(B, \ldots, B, A, \ldots, A)$ in (1).

2. 

$$
D(\lambda_1A_1, \ldots, \lambda_nA_n) = \lambda_1 \cdots \lambda_n D(A_1, \ldots, A_n).
$$

$^1$The authors choose to quote this definition of mixed determinant in a way analogous to the definition of mixed volume in convex geometry [2, 3].

3. 

$$
D(A_1, \ldots, A_{n-1}, B + B') = D(A_1, \ldots, A_{n-1}, B) + D(A_1, \ldots, A_{n-1}, B').
$$

In particular,

$$
D(A_1, \ldots, A_n, B + B') = D(A_1, \ldots, A_n, B)
$$

$$
+ D(A_1, \ldots, A_n, B').
$$

The properties in (2) and (3) follow from the $n$-linearity of the mixed determinant.

One can show that for $n \times n$ matrices $A$ and $B$:

$$
D(A, \ldots, A, B) = \frac{1}{n} \left( \begin{array}{c|c|c|c}
\lambda_1 & \cdots & \lambda_n & b_1 \\
\vdots & & \vdots & \vdots \\
\lambda_{n-1} & \cdots & \lambda_n & a_2 \\
\lambda_n & \cdots & \lambda_n & a_n
\end{array} \right),
$$

of which the generalization gives an alternative definition of the mixed determinant as in the following remark:

**Remark 4** ([4, 5]). A mixed determinant $D(A_1, A_2, \ldots, A_n)$ of $n \times n$ matrices $A_1, A_2, \ldots, A_n$ can be regarded as the arithmetic mean of the determinants of all possible matrices which have exactly one row from the corresponding rows of $A_1, A_2, \ldots, A_n$.

If $A_1 = A_2 = \cdots = A_{n-1} = A$ and $A_n = B$, then $D(A_1, A_2, \ldots, A_n)$ is a mixed determinant of this special kind for which we assign the following notation.

**Definition 5** ([4, 6]). $D_1(A, B)$ is the mixed determinant of $n \times n$ matrices $A$ and $B$ defined as follows:

$$
D_1(A, B) := D(A, \ldots, A, B) = D(A, n-1; B, 1)
$$

**Definition 6** ([4, 6]). $W_i(A)$ is the mixed determinant of $n \times n$ matrices $A$ and the identity matrix $I$ defined as follows:

$$
W_i(A) := D(A, \ldots, A, I, \ldots, I) = D(A, n-i; I, i).
$$
We easily see that if \( \lambda_i, i = 1, \ldots, n \), are eigenvalues of \( A \), then

\[
W_0(A) = |A| = \lambda_1 \cdots \lambda_n
\]

\[
W_1(A) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i \cdots \lambda_{i-1} \lambda_{i+k} \cdot \lambda_{i+k+1} \cdots \lambda_n
\]

\[
\vdots
\]

\[
W_{n-1}(A) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i = \frac{1}{n} (\lambda_1 + \cdots + \lambda_n).
\]

**Proposition 7** ([4]). Let \( Q \) be an \( n \times n \) matrix, \( A_1, A_2 \) be positive definite symmetric matrices. Let \( \lambda_1, \lambda_2 \) be nonnegative real numbers. Then

\[
D_1(\lambda_1 \cdot A + \lambda_2 \cdot A_2, Q) = \lambda_1 D_1(A_1, Q) + \lambda_2 D_1(A_2, Q)
\]

where \( \cdot \) is called Blaschke scalar multiplication. The relation between Blaschke scalar multiplication and scalar multiplication is

\[
\lambda \cdot A = \lambda^{1/(n-1)} A.
\]

This is an alternative way of expressing the definition of Blaschke addition.

It is trivial to verify that Blaschke addition is both commutative and associative and that

\[
1 \cdot A = A,
\]

\[
\lambda \cdot (A_1 + A_2) = \lambda \cdot A_1 + \lambda \cdot A_2,
\]

\[
(\lambda_1 \lambda_2) \cdot A = \lambda_1 \cdot (\lambda_2 \cdot A),
\]

\[
(\lambda_1 + \lambda_2) \cdot A = \lambda_1 \cdot A + \lambda_2 \cdot A.
\]

**Theorem 8** (The Aleksandrov inequality for matrices [1]). For any \( n \times n \) positive definite symmetric matrices \( A \) and \( B \). The Aleksandrov inequality states that

\[
D^s \left( \begin{array}{ccc} A, \ldots, A, \Xi \end{array} \right)_{s+t} D^t \left( \begin{array}{ccc} B, \ldots, B, \Xi \end{array} \right)_{s+t} \leq D^{s+t} \left( \begin{array}{ccc} A, \ldots, A, B, \ldots, B, \Xi \end{array} \right)_{s+t+n-s-t},
\]

or

\[
D^s(A, s + t; \Xi)D^t(B, s + t, \Xi) \leq D^{s+t}(A, s; B, t; \Xi)
\]

where \( \Xi \) is any \((n - s - t)\) tuples of matrices. 

**Remark 9.** The equality conditions in the Aleksandrov inequality are unknown unless very restrictive assumptions are made on the matrices in question. A special case of this general inequality, for which the equality conditions are known, is the Minkowski inequality which states as follows:

**Theorem 10** (Minkowski inequality [4, 5, 6]). If \( A \) and \( B \) are \( n \times n \) positive definite symmetric matrices, then

\[
D^{n-1}(A)D(B) \leq D^n(A, n - 1; B)
\]

with equality if and only if \( A \) and \( B \) are scalar multiples of each other.

In general the Minkowski inequality can be derived by applying the Aleksandrov inequality as follows.

Let \( A \) and \( B \) be \( n \times n \) positive definite symmetric matrices, \( s = n - 1, t = 1 \) in (9). Then

\[
D^{n-1}(A, n)D^1(B, n) \leq D^n(A, n - 1; B)
\]

\[
|A|^{(n-1)/n} |B|^{1/n} \leq D(A, n - 1; B)
\]

\[
\frac{1}{n} \Xi A \cdot B = D_1(A, B) = D(A, n - 1; B) \geq |A|^{(n-1)/n} |B|^{1/n}
\]

If \( A_1, \ldots, A_{n-1} \) are \( n \times n \) positive definite symmetric matrices, then by successively applying the Aleksandrov-Fehlce inequality, one obtains:

\[
D(A_1) \cdots D(A_n) \leq D^n(A_1, \ldots, A_n),
\]

with equality if and only if the matrices \( A_i \) are scalar multiples of each other.

**Theorem 11** (Uniqueness Theorem [4]). Let \( A, B \) be \( n \times n \) positive definite symmetric matrices. Then

\[
D_1(A, Q) = D_1(B, Q)
\]

for any \( n \times n \) matrix \( Q \) implies \( A = B \). 

**2.2 Mixed matrix**

**Definition 12** (Mixed Matrix [4]). Let \( A_1, \ldots, A_{n-1} \) be \( n \times n \) positive definite symmetric matrices. We call the mixed matrix of \( A_1, \ldots, A_{n-1} \), denoted
Let

\[ [A_1, \ldots, A_{n-1}], \text{ the matrix defined to satisfy} \]

(13)

\[ D_1([A_1, \ldots, A_{n-1}], Q) \]

\[ = D([A_1, \ldots, A_{n-1}], \ldots, [A_1, \ldots, A_{n-1}], Q) \]

\[ n-1 \text{ of same matrix} \]

\[ \text{def} \]

\[ D(A_1, \ldots, A_{n-1}, Q) \forall Q \text{ n x n matrix} \]

(14)

From Uniqueness Theorem it follows that \([A_1, \ldots, A_{n-1}] \) is symmetric in its arguments, and that if the \( A_i \) are replaced by scalar multiples, the resulting mixed matrix will be a scalar multiple of the original. It also follows that \([A, \ldots, A] = A \).

**Proposition 13.** Let \( A, B, A_2, \ldots, A_n \) be \( n \times n \) positive definite symmetric matrices. Then

(15)

\[ D_1([A + B, A_2, \ldots, A_{n-1}], A_n) \]

\[ = D_1([A, A_2, \ldots, A_{n-1}] + [B, A_2, \ldots, A_{n-1}], A_n), \]

and

(16)

\[ [A + B, A_2, \ldots, A_{n-1}] \]

\[ = [A, A_2, \ldots, A_{n-1}] + [B, A_2, \ldots, A_{n-1}]. \]

\( \square \)

**Proposition 14.** Let \( A_1, \ldots, A_{n-1} \) be \( n \times n \) positive definite symmetric matrices, and \( \lambda_1, \ldots, \lambda_{n-1} \) be nonnegative real numbers. Then

\[ \lambda_1 A_1, \ldots, \lambda_{n-1} A_{n-1} \]

\[ = (\lambda_1 \ldots \lambda_{n-1}) \cdot [A_1, \ldots, A_{n-1}]. \]

\( \square \)

For the mixed matrix \([A_1, \ldots, AB, \ldots, B] \), with \( i \) copies of \( B \) and \( n - i - 1 \) copies of \( A \), we write \([A, B]_i \). For the case when \( B = I \), we write \([A]_i \) rather than \([A, I]_i \). We note that \([A]_0 = A \), while \([A]_{n-1} = I \).

Obviously \( A + B = [A + B, \ldots, A + B] \). If we use (8), (14) and the fact that a mixed matrix is symmetric in its arguments, we get

**Proposition 15.** If \( A \) and \( B \) are \( n \times n \) positive definite symmetric matrices, then

(17)

\[ A + B = \sum_{i=0}^{n-1} \binom{n-1}{i} [A, B]_i, \]

where the sum on the right denotes a Blaschke sum. \( \square \)

If \( A_1, \ldots, A_{n-1} \) are \( n \times n \) positive definite symmetric matrices, since

\[ D([A_1, \ldots, A_{n-1}]) \]

\[ = D_1([A_1, \ldots, A_{n-1}], [A_1, \ldots, A_{n-1}]), \]

we can use (13) to conclude that

\[ D([A_1, \ldots, A_{n-1}]) \]

\[ = D(A_1, \ldots, A_{n-1}, [A_1, \ldots, A_{n-1}]), \]

On the other hand, if we use inequality (11) with \( A_n = [A_1, \ldots, A_{n-1}] \), we get

**Theorem 16.** If \( A_1, \ldots, A_{n-1} \) are \( n \times n \) positive definite symmetric matrices, then

(18)

\[ D(A_1) \cdots D(A_{n-1}) \leq D^{n-1}([A_1, \ldots, A_{n-1}]), \]

with equality if and only if the matrices \( A_i \) are scalar multiples of each other. \( \square \)

As a by-product, we note that from the Aleksandrov inequality one easily obtains an upper bound on the determinant of \([A_1, \ldots, A_{n-1}] \):

(19)

\[ D^{n-1}([A_1, \ldots, A_{n-1}]) = D^{n-1}([A_1, \ldots, A_{n-1}])D(I) \]

\[ \leq D(A_1, \ldots, A_{n-1}, I). \]

### 2.3 Main results

Recall that the Brunn-Minkowski inequality state that if \( A \) and \( B \) are \( n \times n \) positive definite symmetric matrices, then

(20)

\[ D^{\frac{1}{n}}(A) + D^{\frac{1}{n}}(B) \leq D^{\frac{1}{n}}(A + B), \]

with equality if and only if \( A \) and \( B \) are scalar multiples of each other. Thus Theorem 16 implies that following result:

**Theorem 17.** If \( A \) and \( B \) are \( n \times n \) positive definite symmetric matrices, then

(21)

\[ D^{\frac{1}{n}}(A) + D^{\frac{1}{n}}(B) \]

\[ \leq \left[ \sum_{i=0}^{n-1} \binom{n-1}{i} D([A, B]_i)^{\frac{n-1}{i}} \right] \frac{1}{n-1} \]

\[ \leq D^{\frac{1}{n}}(A + B), \]

with equality, in either of these inequalities, if and only if \( A \) and \( B \) are scalar multiples of each other.
Proof. Suppose $Q$ is an arbitrary $n \times n$ positive definite symmetric matrix. From (17) we have

$$D_1(A + B, Q) = D_1 \left( \sum_{i=0}^{n-1} \binom{n-1}{i} [A, B]_i, Q \right),$$

and, hence, from (7) we get

$$D_1(A + B, Q) = \sum_{i=0}^{n-1} \binom{n-1}{i} D_1([A, B]_i, Q).$$

The Minkowski inequality (10) can now be used to conclude that

$$D_1(A + B, Q) \geq \sum_{i=0}^{n-1} \binom{n-1}{i} D_{\frac{n+1}{n}}([A, B]_i) D_{\frac{n+1}{n}}(Q),$$

with equality if and only if $[A, B]_i$ is a scalar multiple of $Q$ for $i = 0, 1, \ldots, n-1$. For equality to occur it is necessary that $[A, B]_0 = A$ and $[A, B]_{n-1} = B$ both be scalar multiples to $Q$. However, the requirement that $A$ and $B$ be scalar multiples of $Q$ is sufficient to conclude that $[A, B]_i$ is a scalar multiple of $Q$ for any $i = 0, 1, \ldots, n-1$.

If we now take $Q$ to be $A + B$, our last inequality becomes

$$D_{\frac{n-1}{n}}(A + B) \geq \sum_{i=0}^{n-1} \binom{n-1}{i} D_{\frac{n-1}{n}}([A, B]_i),$$

with equality if and only if $A$ and $B$ are scalar multiples of each other. From Theorem 16 it follows that

$$\sum_{i=0}^{n-1} \binom{n-1}{i} D_{\frac{n-1}{n}}([A, B]_i) \geq \sum_{i=0}^{n-1} \binom{n-1}{i} D_{\frac{n-1}{n}}(A) D_{\frac{n-1}{n}}(B),$$

with equality if and only if $A$ and $B$ are scalar multiples of each other. Since the right-hand side of this equality is just

$$(D_{\frac{n-1}{n}}(A) + D_{\frac{n-1}{n}}(B))^{n-1},$$

the proof is complete.

Recall that for $n \times n$ positive definite matrices $A$, $B$, the Blaschke inequality states that

$$D \left( \frac{n-1}{n} \right)(A) + D \left( \frac{n-1}{n} \right)(B) \leq D \left( \frac{n-1}{n} \right)(A + B),$$

with equality if and only if $A$ and $B$ are scalar multiples of each other.

We would like to point out that by applying the Blaschke inequality to (17) followed by Theorem 16, one would get the Brunn-Minkowski inequality, along with the conditions for equality.

A better result than Theorem 16 is possible for the case where $A_1 = \cdots = A_{n-i-1} = A$ and $A_{n-i} = \cdots = A_{n-1} = I$; namely:

Theorem 18. If $A$ is an $n \times n$ positive definite symmetric matrix and $0 < i < n - 1$, then

$$W_i \left( \frac{n-i-1}{n} \right)(A) \leq D([A]_i) \left( \frac{n-1}{n} \right) \leq W_{i+1}(A),$$

with equality, in either inequality, if and only if $A$ is a scalar matrix.

Proof. Since $D([A]_i) = D_1([A]_i, [A]_i)$, from (13) it follows that

$$D([A]_i) = D(A, n - i - 1; I, i; [A]_i).$$

The Aleksandrov inequality (9), with $B = [A]_i$, $s = n - i - 1$ and $t = 1$, can be used to conclude that

$$D([A]_i, n - i; I, i) D^{n-i-1}(A, n - i; I, i) \leq D^{n-1}(A|_i).$$

However, from (11) we have

$$D^{(n-1)/n}(A|_i) \leq D([A]_i, n - i; I, i),$$

with equality if and only if $[A]_i$ is a scalar matrix. If we combine the two previous inequalities, we obtain the left inequality of our theorem. To obtain the right inequality we combine the observation that

$$W_{i+1}(A) = D(A, n - i - 1; I, i; I)$$

with (13) to conclude that

$$W_{i+1}(A) = D_1([A]_i, I).$$

The Minkowski inequality (10) will then yield

$$D^{n-1}(A|_i) \leq W_{i+1}(A),$$

with equality if and only if $[A]_i$ is a scalar matrix. This is the right inequality of our theorem. In both of the inequalities of our theorem equalities occur if and only if $[A]_i$ is a scalar matrix. □
Proposition 19. If \( A \) is an \( n \times n \) positive definite symmetric matrix and \( 0 < i < n - 1 \), then
\[
D^{n-1}([A]_i) \leq W^n_i(A)/D(A),
\]
with equality if and only if \( A \) and \([A]_i\) are scalar multiple of each other.

Proof. Since \( W^i(A) = D(A, n - i - 1; I, i; A) \), it follows from (13) that
\[
W^i(A) = D_1([A]_i, A).
\]
From the Minkowski inequality we get
\[
D^{n-1}([A]_i)D(A) \leq W^n_i(A),
\]
with equality if and only if \( A \) and \([A]_i\) are scalar multiple of each other. \( \square \)

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