A Proof of $\sum a_{j_1} \cdots a_{j_{n-i}} \geq \sum \lambda_{j_1} \cdots \lambda_{j_{n-i}}$, $0 \leq i \leq n - 1$ where $a_{jk} \in \{a_{11}, a_{22}, \ldots, a_{nn}\}$, $\lambda_{jk} \in \sigma(A)$ for $A = [a_{ij}] > 0 \in \mathbb{R}^{n \times n}$.

Using Mixed Determinants

**PORAMATE (TOM) PRANAYANUNTANA**
King Mongkut’s Institute of Technology Ladkrabang (KMITL)
Department of Control Engineering
Faculty of Engineering
3 Moo 2, Chalongkrung Rd., Ladkrabang, Bangkok
THAILAND 10520

**PATTRAWUT CHANSANGIAM**
King Mongkut’s Institute of Technology Ladkrabang (KMITL)
Department of Mathematics and Computer Science
Faculty of Science
3 Moo 2, Chalongkrung Rd., Ladkrabang, Bangkok
THAILAND 10520

**Abstract:** We discuss the use of mixed determinant of special form $D(A, n - i; I, i)$ for positive definite symmetric matrix $A$, in particular the operator concavity property of the map $f : A \mapsto D^{1/(n-i)}(A, n - i; I, i)I$ together with unital positive linear map $\Phi : A \mapsto A \circ f$ the Hadamard product of $A$ with the identity matrix $I$, in deriving an inequality of the form $\sum a_{j_1} \cdots a_{j_{n-i}} \geq \sum \lambda_{j_1} \cdots \lambda_{j_{n-i}}$, $0 \leq i \leq n - 1$, where the sums are taken over all $(n - i)$-tuples of positive integers $(j_1, \ldots, j_{n-i})$ whose entries do not exceed $n$, with $\lambda_{jk}$ and $a_{jk}$, $1 \leq k \leq n$ from the set of all $n$ positive eigenvalues of $A$ and the set of main diagonal entries of $A$, respectively.

**Key Words:** Mixed determinant, Aleksandrov inequality, Operator monotone function, Operator convex function, Operator concave function, Unital positive linear map, Matrix Hadamard product

October 9, 2006

1 Introduction

The notion of mixed determinants for positive definite symmetric matrices gives an analog notion to the notion of mixed volumes for convex bodies in convex geometry. Many convex geometry notions and inequalities have been developed for more than a century (for references see Lutwak [4] and Schneider [5]). Also many notions in convex geometry have analogs in linear algebra and matrix theory (see [6, 7, 8] for some of such examples). These linear algebra (or matrix) analog notions can be employed with other notions of the theory of matrix inequalities in order to obtain new inequalities for positive definite symmetric matrices.

In this paper we employ the notion of mixed determinant for positive definite symmetric matrices. The particular form of mixed determinant used is $D(A, n - i; I, i)$ in the scalar matrix operator $f : A \mapsto D^{1/(n-i)}(A, n - i; I, i)I$ for $A = [a_{ij}] > 0 \in \mathbb{R}^{n \times n}$. Then the operator concavity property of the operator $f$ was proved then used in conjunction with a unital positive linear map $\Phi : A \mapsto A \circ I$, where $\circ$ stands for the matrix Hadamard product, to obtain $D(A \circ I, n - i; I, i) \geq D(A, n - i; I, i)$ which in turn yields our main result: $\sum a_{j_1} \cdots a_{j_{n-i}} \geq \sum \lambda_{j_1} \cdots \lambda_{j_{n-i}}$, $0 \leq i \leq n - 1$, where the sums are taken over all $(n - i)$-tuples of positive integers $(j_1, \ldots, j_{n-i})$ whose entries do not exceed $n$, with $\lambda_{jk}$ and $a_{jk}$, $1 \leq k \leq n$ from the set of all $n$ positive eigenvalues of $A$ and the set of main diagonal entries of $A$, respectively. The main tools used here are the Aleksandrov inequality [1, 4, 5, 6], and operator monotonicity, operator convexity and operator concavity as found in the matrix inequalities book by Zhan [10].

2 Materials and Methods

We begin by stating the definition and axiomatic properties of mixed determinant and then we quote the very useful Aleksandrov inequality in Section 2.1. In Section 2.2 we introduce the concepts of operator monotone, operator convex and operator concave functions. Also we introduce the concepts of unital, positive and linear maps. Then we state the applications and relation between them. The final subsection
of this section we present results. We use the materials of section 2.1 and 2.2 to obtain our main theorem, Theorem 16.

2.1 Mixed Determinants and Aleksandrov Inequality

Definition 1 (Positive Semidefinite Symmetric Matrix). An $n \times n$ matrix $A$ is said to be positive semidefinite symmetric if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$. An equivalent condition is that $A$ is symmetric and have all eigenvalues nonnegative. For symmetric matrices $A$, $B$ we write $B \leq A$ or $A \geq B$ to mean that $A - B$ is positive semidefinite symmetric.

In particular, $A \geq 0 \in \mathbb{R}^{n \times n}$ indicates that $A$ is positive semidefinite symmetric. If $A$ is positive definite symmetric, that is positive semidefinite symmetric and invertible, we write $A > 0 \in \mathbb{R}^{n \times n}$.

Definition 2 (Mixed Determinant). Let $A_1, \ldots, A_r$ be $n \times n$ symmetric matrices, $\lambda_1, \ldots, \lambda_r$ be nonnegative real numbers. Then the determinant of $\lambda_1 A_1 + \cdots + \lambda_r A_r$ can be written as

$$D(\lambda_1 A_1 + \cdots + \lambda_r A_r) = \sum_{(i_1, \ldots, i_r)} \lambda_{i_1} \cdots \lambda_{i_r} D(A_{i_1}, \ldots, A_{i_r}),$$

where the sum is taken over all $n$-tuples of positive integers $(i_1, \ldots, i_n)$ whose entries do not exceed $r$. The coefficient $D(A_{i_1}, \ldots, A_{i_n})$, with $A_{i_k} = 1 \leq k \leq m$ from the set $\{A_1, \ldots, A_r\}$, is called the mixed determinant of the matrices $A_{i_1}, \ldots, A_{i_n}$.

Properties of Mixed Determinants [6, 7]: Let $A_1, \ldots, A_n$, $A$, $B$ and $B'$ be $n \times n$ symmetric matrices, $\lambda_1, \ldots, \lambda_n$ be nonnegative real numbers.

1. $D(A, \ldots, A, B) = D(A, \ldots, A, B, A)$

2. $D(\lambda_1 A_1, \ldots, \lambda_n A_n) = \lambda_1 \cdots \lambda_n D(A_1, \ldots, A_n)$.

3. $D(A_1, \ldots, A_{n-1}, B + B') = D(A_1, \ldots, A_{n-1}, B) + D(A_1, \ldots, A_{n-1}, B')$.


The properties in (2) and (3) follow from the nonlinearity of the mixed determinant.

Using the three axiomatic properties of mixed determinant above, one can show that for $n \times n$ symmetric matrices $A$ and $B$:

$$D(A, \ldots, A, B) = \frac{1}{n} \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_{n-1} & b_{n-1} \\ a_n & b_n \end{pmatrix},$$

of which the generalization gives an alternative definition of the mixed determinant as in the following remark:

Remark 3 ([6, 7]). A mixed determinant $D(A_1, A_2, \ldots, A_n)$ of $n \times n$ symmetric matrices $A_1, A_2, \ldots, A_n$ can be regarded as the arithmetic mean of the determinants of all possible matrices which have exactly one row from the corresponding rows of $A_1, A_2, \ldots, A_n$.

Now we state a well-known theorem for inequalities of mixed determinants of positive definite symmetric matrices. This very useful theorem, which has many applications, is called the Aleksandrov inequality is the matrix version of the Aleksandrov-Fenchel inequality of convex geometry.

Theorem 4 (Aleksandrov Inequality [1, 4, 5, 6]). For any $n \times n$ positive definite symmetric matrices $A$ and $B$, the Aleksandrov inequality states that

$$D^n(A, s + t; \Xi) D^k(B, s + t; \Xi) \leq D^{n+k}(A, s; B, t; \Xi),$$

where $D^n(A, s + t; \Xi)$ represents any of $D(A_1, A_2, \ldots, A_n)$ in (1).

The authors choose to quote this definition of mixed determinant in a way analogous to the definition of mixed volume in convex geometry [4, 5].
where $\Xi$ is any $(n - s - t)$ tuples of matrices.

2.2 Maps on Matrix Spaces-Operator Monotonicity, Operator Convexity, Operator Concavity and Unital Positive Linear Map.

**Definition 5** (Operator Monotonicity, Operator Convexity, Operator Concavity [2, 10]). A real-valued continuous function $f(t)$ defined on a real interval $\Omega$ is said to be operator monotone if

$$A < B \implies f(A) \leq f(B)$$

for all such Hermitian matrices $A, B$ of all orders whose eigenvalues are contained in $\Omega$. $f$ is called operator convex if for any $0 < \varepsilon < 1,$

$$f(\varepsilon A + (1 - \varepsilon) B) \leq \varepsilon f(A) + (1 - \varepsilon) f(B)$$

holds for all Hermitian matrices $A, B$ of all orders with eigenvalues in $\Omega$. $f$ is called operator concave if $-f$ is operator convex.

In general we have the following useful integral representations for operator monotone and operator convex functions.

**Theorem 6** (Integral Representations of Operator Monotone Function and Operator Convex Function [2, 3, 10]). If $f$ is an operator monotone function on $[0, \infty)$, then there exists a positive measure $\mu$ on $[0, \infty)$ such that

$$f(t) = \alpha + \beta t + \int_0^\infty \frac{st}{s + t} d\mu(s)$$

where $\alpha$ is a real number and $\beta \geq 0$. If $g$ is an operator convex function on $[0, \infty)$ then there exists a positive measure $\mu$ on $[0, \infty)$ such that

$$g(t) = \alpha + \beta t + \gamma t^2 + \int_0^\infty \frac{st^2}{s + t} d\mu(s)$$

where $\alpha, \beta$ are real numbers and $\gamma \geq 0$.

The three concepts of operator monotone, operator convex and operator concave functions are intimately related. One of such examples is as follows, which we will state without proof:

**Theorem 7** (Relation between Operator Monotone and Operator Concave Functions [10]). A nonnegative continuous function on $[0, \infty)$ is operator monotone if and only if it is operator concave.

**Definition 8** (Positive Map [2, 10]). A map $\Phi : M_n \rightarrow M_m$ is called positive if it maps positive semidefinite matrices to positive semidefinite matrices: $A \geq 0 \implies \Phi(A) \geq 0$.

**Definition 9** (Unital Map [2, 10]). Denote by $I_n$ the identity matrix in $M_n$. A map $\Phi : M_n \rightarrow M_m$ is called unital if $\Phi(I_n) = I_m$.

**Definition 10** (Linear Map). A map $L : M_n \rightarrow M_m$ is called linear if for all $A$ and $B$ in $M_n$ and every scalar $\alpha$

$$T(\alpha A + B) = \alpha T(A) + T(B).$$

For the following material in this subsection we will follow the treatment as in [10]. We will first derive some inequalities involving unital positive linear maps with operator monotone functions and with operator convex functions, then use these results to obtain our main inequalities of Theorem 16 in the next Section.

**Lemma 11** ([10]). Let $A > 0$. Then

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \succeq 0$$

if and only if the Schur complement

$$C - B^* A^{-1} B \succeq 0.$$

**Lemma 12** ([9, 10]). Let $\Phi$ be a unital positive linear map from $M_m$ to $M_n$. Then

$$\Phi(A^2) \succeq \Phi(A)^2 \quad (A \succeq 0),$$

$$\Phi(A^{-1}) \succeq \Phi(A)^{-1} \quad (A > 0).$$

**Theorem 13** ([9, 10]). Let $\Phi$ be a unital positive linear map from $M_m$ to $M_n$ and $f$ an operator monotone function on $[0, \infty)$. Then for every $A \geq 0$,

$$f(\Phi(A)) \succeq \Phi(f(A)).$$

**Theorem 14** ([9, 10]). Let $\Phi$ be a unital positive linear map from $M_n$ to $M_m$ and $g$ an operator convex function on $[0, \infty)$. Then for every $A \geq 0$,

$$g(\Phi(A)) \succeq \Phi(g(A)).$$
2.3 Main Results

By symmetry in arguments and n-linearity properties of mixed determinant together with applications of Aleksandrov inequality one can show that for any \( n \times n \) positive definite symmetric matrices \( A, B \), and for \( 0 \leq i \leq n - 1 \),

\[
D^{1/(n-i)}(\varepsilon A + (1-\varepsilon)B, n-i; I, i) \\
\geq \varepsilon D^{1/(n-i)}(A, n-i; I, i) \\
+ (1-\varepsilon)D^{1/(n-i)}(B, n-i; I, i).
\]

This tells us that \( A \mapsto D^{1/(n-i)}(A, n-i; I, i) \) is operator concave. So is the scalar matrix operator \( f: A \mapsto D^{1/(n-i)}(A, n-i; I, i) \). \( f \) is also operator monotone by Theorem 7.

**Definition 15** (Matrix Hadamard Product \([2, 3, 10])\). Given \( A = [a_{ij}], B = [b_{ij}] \in M_n \), the **Hadamard product** of \( A \) and \( B \) is defined as the entry-wise product:

\[
A \circ B := [a_{ij}b_{ij}] \in M_n.
\]

It is easy to see that \( \Phi: A \mapsto A \circ I \) is a unital positive linear map. Therefore by Theorem 13 we have

\[
D^{1/(n-i)}(A \circ I, n-i; I, i) \\
\geq D^{1/(n-i)}(A, n-i; I, i)I \circ I \\
= D^{1/(n-i)}(A, n-i; I, i)I.
\]

This implies

\[
D^{1/(n-i)}(A \circ I, n-i; I, i) \\
\geq D^{1/(n-i)}(A, n-i; I, i).
\]

Since \( 0 \leq i \leq n - 1 \) or \( 1 \leq n - i \leq n \), then (10) is equivalent to

\[
D(A \circ I, n-i; I, i) \\
\geq D(A, n-i; I, i).
\]

One can easily show that \( D(A + B) \) can be expanded as (see [6])

\[
D(A + B) = \sum_{i=0}^{n} \binom{n}{i} D(A, n-i; B, i).
\]

So \( D(A + \varepsilon I) \) can be written as

\[
D(A + \varepsilon I) = \sum_{i=0}^{n} \binom{n}{i} \varepsilon^i D(A, n-i; I, i).
\]

Differentiating (13) \( i \) times with respect to \( \varepsilon \) and setting \( \varepsilon = 0 \), we obtain

\[
d_i^i \frac{d^n}{d\varepsilon^n} D(A + \varepsilon I)_{|\varepsilon=0} = \frac{n!}{(n-i)!} D(A, n-i; I, i)
\]

or

\[
D(A, n-i; I, i) = \frac{(n-i)!}{n!} \frac{d^n}{d\varepsilon^n} D(A + \varepsilon I)_{|\varepsilon=0}.
\]

For positive definite symmetric matrix \( A \), there always exists an orthogonal matrix \( P \) such that \( A = P\Lambda P^{-1} = P\Lambda^T \), where \( P \) is the matrix whose columns form an orthonormal eigenbasis of \( A \) and \( \Lambda \) is the diagonal matrix whose diagonal entries are the corresponding eigenvalues of \( A \). In which case (15) yields

\[
D(A, n-i; I, i) = \frac{(n-i)!}{n!} \frac{d^n}{d\varepsilon^n} D(P\Lambda P^{-1} + P\varepsilon IP^{-1})_{|\varepsilon=0}
\]

\[
= \frac{(n-i)!}{n!} \frac{d^n}{d\varepsilon^n} D(\Lambda + \varepsilon I)_{|\varepsilon=0}
\]

\[
= D(\Lambda, n-i; I, i)
\]

\[
= \frac{1}{(n-i)!} \sum_{j_1, \ldots, j_{n-i}} \lambda_{j_1} \cdots \lambda_{j_{n-i}}
\]

where the sums are taken over all \( (n-i) \)-tuples of positive integers \( (j_1, \ldots, j_{n-i}) \) whose entries do not exceed \( n \), with \( \lambda_{j_k}, 1 \leq k \leq n \) from the set of all \( n \) positive eigenvalues of \( A \). The last equality of (16) was by Remark 3.

Using (16) in (11) we obtain the result of our main theorem as follows:

**Theorem 16.** Let \( A \) be an \( n \times n \) positive definite symmetric matrix, \( I \) be the identity matrix of order \( n \). Then

\[
\sum a_{j_1} \cdots a_{j_{n-i}} \geq \sum \lambda_{j_1} \cdots \lambda_{j_{n-i}}, \quad 0 \leq i \leq n-1,
\]

where the sums are taken over all \( (n-i) \)-tuples of positive integers \( (j_1, \ldots, j_{n-i}) \) whose entries do not exceed \( n \), with \( \lambda_{j_k}, 1 \leq k \leq n \) from the set of all \( n \) positive eigenvalues of \( A \) and the set of main diagonal entries of \( A \), respectively.

**Corollary 17** (Hadamard Inequality for a Positive Definite Symmetric Matrix \([3, 10])\). Let \( A = [a_{ij}] \) be an \( n \times n \) positive definite symmetric matrix. Then (see also \([3, 10])\)

\[
\prod_{i=1}^{n} a_{ii} \geq \det(A),
\]

which is (17) with \( i = 0 \).
Corollary 18. Let $A = [a_{ij}]$ be an $n \times n$ positive definite symmetric matrix. Then we obtain the well known formula for trace of $A$:

$$\text{tr } A = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i,$$

which is (17) with $i = n - 1$. □

Acknowledgements: The first author dedicates this paper to his mother, Poolsuk Pranayanuntana, who was his first teacher of Mathematics and who always concerned of his well being, and to Associate Professor Chandni Shah, Sep 13, 1959 - Apr 9, 2005, who was like a mother to him.

He wishes to thank Professor Erwin Lutwak, Professor Xingzhi Zhan and Assistant Professor Franziska Berger for some very useful conversations and suggestions.

The second author dedicates this paper to his parents and teachers.

The “one of the best” book by Professor Xingzhi Zhan [10] and such a wonderful publication by Professor Tsuyoshi Ando [2] have inspired both authors so much.

We wish to thank Associate Professor Kobchai Dejhan, Dean of the Faculty of Engineering, KMITL, for allocating funds for this research. We also thanks their Faculty of Engineering and Faculty of Science for the resources provided.

References:

[9] Poramate (Tom) Pranayanuntana, Patrawut Chansangiam, “A Proof of $S_{n-i}(a_{11}, \ldots, a_{nm}) \geq S_{n-i}(\lambda_1, \ldots, \lambda_n), 0 \leq i \leq n - 1$ for an $n \times n$ Positive Definite Symmetric Matrix $A = [a_{ij}]_{n \times n}$, Using Mixed Determinants,” to be appeared in WSEAS TRANSACTION ON MATHEMATICS.