Inverse Initial Heat Flux and Relaxation Time
Finding Problems for Hyperbolic Heat Equation

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Abstract: - In this paper the one and two-dimensional initial-boundary value problems for hyperbolic heat equation as mathematical model of intensive steel quenching is studied. In the statement of problem, instead of second initial condition, the temperature distribution and its time derivative at the end of time interval are given. Firstly, we construct the solution in closed form by reducing the solution to Fredholm type integral equation of the first kind according to the second time derivative of the temperature. Secondly, the closed explicit representation for the relaxation time parameter is obtained.

Key-Words: - hyperbolic heat equation, inverse problems, initial flux, relaxation time, Green function.

1 Introduction
Real processes taking place in intensive steel quenching move very quickly, especially at very beginning of this important industrial technology process [1], [2]. This is the reason why we decided to use hyperbolic heat equation as mathematical model for the technology of intensive steel quenching [3], [4]. However, the utilization of this hyperbolic type partial differential equation brings two serious difficulties. Firstly, it is practically impossible to determine experimentally the initial heat fluxes \( V_0(x) \) (see equation (5)). Secondly, is not clear what sense has the relaxation time \( \tau_r \) in the hyperbolic heat equation (1). The first question was answered in our papers [3], [4] by solving time inverse (reverse) problem; second question will be answered in this paper. Here we construct (in a bit different form compared to our previous papers) the exact solution for hyperbolic heat equation. We obtain the solution for initial heat flux in the form of the first kind Fredholm integral equation. This solution was constructed by the usage of Green’s function method. Then we give in explicit closed form the expression for relaxation time parameter. Finally, we solve correspondent inverse two-dimensional problem for finite cylinder.

2 Mathematical Formulation of Model for Intensive Steel Quenching for Semi-infinite Interval
We will start with the formulation of the one-dimensional mathematical model for intensive steel quenching as in our paper [3]:

\[
\begin{align*}
\tau_r \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} &= a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t), \\
x \in (0, \infty), t \in (0, T),
\end{align*}
\]

(1)

\[
\begin{align*}
-k \frac{\partial u}{\partial x} + hu &= h\Theta(t), x = 0, t \in [0, T],
\end{align*}
\]

(2)

\[
\lim_{x \to +\infty} u(x, t) < \infty, t \in [0, T],
\]

(3)

\[
\begin{align*}
&u = u^0(x), x \in [0, \infty], t = 0, \\
&\frac{\partial u}{\partial t} = V_0(x), x \in [0, \infty], t = 0.
\end{align*}
\]

(4)

(5)

Here in formulae (2), (3) \( k \) is heat conductivity coefficient, \( c \) - specific heat, \( h \) - heat exchange coefficient, \( a^2 = k / (c \rho) \), where \( \rho \) is the density of steel.

From the point of view of practical steel quenching technology, the condition (5) is unrealistic. The initial heat flux can’t be measured experimentally and must be calculated. As additional condition for the determination of right hand side of equation (5) we assume experimentally realizable condition – the temperature distribution at the end of process is given (known):

\[
u(x, T) = U_f(x), x \in [0, l].
\]

(6)
As it was shown in our papers [3], [4] the condition (6) is sufficient for the calculation of the initial heat flux $V_0(x)$. In this paper, we will prove that it is possible to determine the relaxation time too, if additionally is given the temperature flux of the end of process:

$$\frac{\partial u}{\partial t} = V_f(x), x \in [0,l], t = T.$$ (7)

Moreover, the additional condition (7) can be substantially reduced: it is enough to know the final heat flux in only one point $x_0$, comfortable from experimental point of view:

$$\frac{\partial u(x_0,T)}{\partial t} = V_f(x_0), x_0 \in [0,l].$$ (8)

### 3 Analytical Approach for Calculation of the Initial Temperature Flux

In our paper [3] this time inverse problem was solved by using the Green function method. We rewrite the main equation (1) in form of non-homogeneous parabolic heat equation (see also [5]):

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + F(x,t,\tau, u),$$ (9)

$x \in (0,l), t \in (0,T), l \leq \infty$.

Here

$$F(x,t,\tau, u(x,t)) = f(x,t) - U(x,t),$$

$$U(x,t) = \tau \frac{\partial u}{\partial t}.$$ (10)

It must be underlined that here in contradiction to our previous papers, we add the relaxation time parameter $\tau$ in the second derivative term.

The solution of equation (9) can be immediately represented in closed form by usage of Green’s function for classic heat equation:

$$u(x,t) = \int_0^x u^0(\xi)G(x,\xi,t)d\xi + \frac{h}{c\rho} \int_0^t \int_0^\infty \Theta(\tau)G(x,0,t-\tau)d\tau d\xi + \int_0^t \int_0^\infty F(x,\tau,\tau, u(x,\tau))G(x,\xi,t-\tau)d\xi d\tau.$$ (11)

Here the Green function $G(x,\xi,t)$ has the well known form:

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi a^2 t}} \left[ e^{\frac{(x+\xi)^2}{4a^2t}} + e^{\frac{(x-\xi)^2}{4a^2t}} \right] - \frac{4h|d|e^{-\frac{h^2}{4c\rho}}}{k(\frac{h^2}{4c\rho})^{\frac{1}{2}} \text{erfc} \left( \frac{x+\xi}{\sqrt{2a^2t}} + \frac{\sqrt{t}}{c\rho|d|} \right),$$ (12)

where

$$\text{erfc}(z) = \int z e^{-z^2} dz.$$ (13)

The solution (11) can be rewritten in shorter form:

$$u(x,t) = G_1(x,t) + \int_0^t \int_0^\infty F(x,\tau,\tau, u(x,\tau))G(x,\xi,t-\tau)d\xi d\tau,$$ (14)

where we denote the known integrals in the representation (11) by $G_1(x,t)$:

$$G_1(x,t) = \int_0^\infty u^0(\xi)G(x,\xi,t)d\xi +$$

$$\frac{h}{c\rho} \int_0^t \int_0^\infty \Theta(\tau)G(x,0,t-\tau)d\tau d\xi +$$

$$\int_0^t \int_0^\infty F(x,\tau,\tau, u(x,\tau))G(x,\xi,t-\tau)d\xi d\tau.$$ (15)

Here

$$G_2(x,t) = G_1(x,t) +$$

$$\int_0^t \int_0^\infty f(\xi,\tau)G(x,\xi,t-\tau)d\xi d\tau.$$ (16)

The solution at the final time $t = T$ is known from our additional condition (6); we get Fredholm type integral equation of the first kind regarding the function $U(x,t)$ from (14):

$$T \int_0^\infty \int_0^\infty G(x,\xi,T-t)d\xi d\tau = \Psi(x,T),$$ (17)

$$\Psi(x,T) = G_2(x,T) - U_\tau(x).$$

Unfortunately, this problem is ill-posed. Nevertheless the integral equation (15) can be solved, e.g. by Tykhonov regularization method. Let us denote the regularized solution of integral equation (15) by $\tilde{U}(x,t)$.

We can write out the first time derivative of acquired solution (14):
\[ \frac{\partial u(x,t)}{\partial t} = \frac{\partial G_2(x,t)}{\partial t} - \int_0^t d\tau \int_0^\infty \tilde{U}(\xi,\tau) \frac{\partial}{\partial \xi} G(x,\xi,t-\tau) d\xi. \]  

(15)

It is the right time to make the last step for the determination of initial heat flux:

\[ V_0(x) = \frac{\partial G_2(x,+0)}{\partial t} - \int_0^\infty \tilde{U}(\xi,+0) G(x,\xi,+0) d\xi. \]  

(16)

The well known filtration property of Green function gives (see, e.g. [6]):

\[ V_0(x) = \frac{\partial G_2(x,+0)}{\partial t} - \tilde{U}(x,0). \]  

(17)

Now it is elementary to find other characteristics of intensive steel quenching process, e.g. the heat flux

\[ Q(x,t) = \frac{\partial u(x,t)}{\partial x} \] regarding the space argument

\[ x \text{ for arbitrary time point (moment) } t : \]

\[ Q(x,t) = \frac{\partial G_2(x,t)}{\partial x} - \int_0^\infty \tilde{U}(\xi,t) \frac{\partial}{\partial \xi} G(x,\xi,t-\tau) d\xi. \]

(18)

### 4 Determination of the Relaxation Time Parameter

#### Time Parameter

As it was mentioned earlier, in our paper [3] time inverse problem was solved by using the Green function method. We assumed that there the relaxation time parameter \( \tau_r \) is given. In reality its determination is the second most important task in mathematical model for the intensive steel quenching technology. The modification presented here in comparison with our previous paper [3] allows us to achieve this goal very easy. We start with the integration of the complex (10) with second time derivative over given time interval:

\[ \int_0^\tau \tilde{U}(x,t) dt \]

\[ = \left[ V_r(x) - V_0(x) \right] = \tau_r \left[ V_r(x) - \frac{\partial G_2(x,+0)}{\partial t} - \tilde{U}(x,0) \right]. \]  

(19)

In case of additionally given condition (8) it remains to fix \( x = x_0 \), express the relaxation time parameter from equality (18) and we have explicit representation for the \( \tau_r \):

\[ \tau_r = \frac{1}{\int_0^\tau \tilde{U}(x_0,t) dt} \left[ V_r(x_0) - \frac{\partial G_2(x_0,+0)}{\partial t} - \tilde{U}(x_0,0) \right]. \]  

(19)

In case of more general additional condition (9) it is necessary to realize one additional step – integration over space argument \( x \) definitions interval. We have finally:

\[ \tau_r = \frac{1}{\int_0^\tau \int_0^\infty \tilde{U}(x,t) dt} \left[ V_r(x) \int_0^\infty \frac{\partial G_2(x,+0)}{\partial t} dx - \int_0^\tau \tilde{U}(x,0) dx \right]. \]  

### 5 Two-Dimensional Model for Intensive Steel Quenching and its Solution for Finite Cylinder

In our paper [4] two-dimensional time inverse problem for finite cylinder was solved. Here we extend this result by adding the given second additional condition at final time moment and by solving the coefficient inverse problem for determination of the relaxation time parameter \( \tau_r \).

The mathematical model looks as follow:

\[ \tau_r \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = a^2 \left[ -r^{-1} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} \right] + f(r,z,t), \]

\[ r \in (0,R), z \in (0,H), t \in (0,T], \]

\[ \frac{\partial u}{\partial r} + k_1 u = \gamma_1(z,t), \]

\[ \frac{\partial u}{\partial z} - k_2 u = \gamma_2(r,t), \]

\[ \frac{\partial u}{\partial r} + k_3 u = -\gamma_3(r,t), \]

\[ \frac{\partial u}{\partial z} \rightarrow 0, r \rightarrow 0, z \in [0,Z], t \in [0,T], \]

\[ u = u^0(r,z), t = 0, r \in [0,R], z \in [0,H]. \]

Instead of the second needed initial condition:
\[ \frac{\partial u}{\partial t} = V_0(r,z), \quad t = 0, r \in [0, R], z \in [0, H] \]

we assume as given the temperature and its time derivative at the final moment \( t = T \):

\[ u(r, z, T) = U_T(r, z), \quad r \in [0, R], z \in [0, H], \]

\[ \frac{\partial}{\partial t} u(r_0, z_0, T) = V_1(r_0, z_0), \]

\[ r_0 \in [0, R], z_0 \in [0, H]. \]

The solution – the determination of initial heat flux repeat all the substantial steps of sections 3 and 4.

\[ u(r, z, t) = \Gamma(r, z, t) = \]

\[ 2\pi \int_{0}^{t} d\tau \int_{R}^{H} d\zeta \times \]

\[ \int_{\mathbb{R}} \rho G(r, \rho, z, \zeta, t-\tau) U(\rho, \zeta, \tau) d\rho. \]

This formula gives the solution of direct problem, in which the first term of right hand side contains all known integrals, but second term contains the unknown multiplier \( U(\rho, \zeta, \tau) \):

\[ \Gamma(r, z, t) = 2\pi \int_{H}^{0} d\zeta \int_{0}^{R} \rho G(r, \rho, z, \zeta, 0) G(r, \rho, z, \zeta, t) d\rho + \]

\[ 2\pi a^2 R \int_{0}^{t} d\tau \int_{H}^{0} \gamma_1(\zeta, \tau) G(r, R, \zeta, \tau, t-\tau) d\zeta - \]

\[ 2\pi a^2 \int_{0}^{t} d\tau \int_{0}^{R} \rho \gamma_2(\rho, \tau) G(r, \rho, z, 0, t-\tau) d\rho - \]

\[ 2\pi a^2 \int_{0}^{t} d\tau \int_{0}^{R} \rho \gamma_3(\rho, \tau) G(r, \rho, z, H, t-\tau) d\rho + \]

\[ 2\pi \int_{0}^{t} d\tau \int_{0}^{R} \rho f(\rho, \zeta, \tau) G(r, \rho, z, H, t-\tau) d\rho. \]

The Green function for finite cylinder can be represented as multiplication of two one space dimensional Green functions:

\[ G(r, \rho, z, \zeta, t) = G_1(r, \rho, t) G_2(z, \zeta, t), \]

where

\[ G_1(r, \rho, t) = \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 e^{-\lambda_n^2 R^2 t}}{R^2 + \mu_n^2} J_0(\frac{\mu_n^2 R}{R}) J_0(\frac{\mu_n^2 R}{R}), \]

\[ G_2(z, \zeta, t) = \sum_{\nu=1}^{\infty} \frac{\cos(\lambda_n^\nu z) + k_n^2 \sin(\lambda_n^\nu z)}{2 \lambda_n^\nu + k_n^2 + \frac{k_n^2 + H}{2} \left( 1 + \frac{k_n^2}{\lambda_n^\nu} \right)} \times \left[ \cos(\lambda_n^\nu \zeta) + \frac{k_n^2}{\lambda_n^\nu} \sin(\lambda_n^\nu \zeta) \right] e^{-\nu^2 \lambda_n^\nu t}. \]

\[ G_1(r, \rho, t) = \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 e^{-\lambda_n^2 R^2 t}}{R^2 + \mu_n^2} J_0(\frac{\mu_n^2 R}{R}) J_0(\frac{\mu_n^2 R}{R}), \]

In last formulæ \( \mu_n > 0 \) and \( \lambda_n > 0 \) are roots of following transcendental equations:

\[ \mu J_1(\mu) - k_i R J_0(\mu) = 0, \]

\[ \left( \lambda - \frac{k_r k_i}{\lambda} \right) \tan(\lambda R) = \frac{k_r + k_i}{\lambda^2 - k_r k_i}. \]

The unknown multiplier \( U(\rho, \zeta, \tau) \) can be found from following the first kind Fredholm type integral equation:

\[ \int_{0}^{t} d\tau \int_{0}^{R} \rho G(r, \rho, z, \zeta, T-\tau) U(\rho, \zeta, \tau) d\rho = \Gamma(r, z, T) - U_T(r, z). \]

The time derivative of the solution takes the form:

\[ \frac{\partial u(r, z, t)}{\partial t} = \frac{\partial \Gamma(r, z, t)}{\partial t} - 2\pi U(r, z, t) - \]

\[ 2\pi \int_{0}^{t} d\tau \int_{0}^{R} \rho \frac{\partial G(r, \rho, z, \zeta, t-\tau)}{\partial t} U(\rho, \zeta, \tau) d\rho. \]

From this expression we obtain easy the closed representation for our first goal – the initial heat flux:

\[ V_0(r, z) = \frac{\partial \Gamma(r, z, 0)}{\partial t} - U(r, z, 0). \]

Similarly as in section 4 we obtain explicit expression for relaxation time parameter:

\[ \tau_r = \left[ \frac{\int_{0}^{T} U(r_0, z_0, t) dt}{V_T(r_0, z_0) - \frac{\partial \Gamma(r_0, z_0, 0)}{\partial t} - U(r_0, z_0, 0)} \right]. \]
6 Conclusion
The closed explicit representation for the initial heat flux (initial temperature velocity) and relaxation time parameter in cases of the semi-infinite interval and the finite cylinder is obtained.

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References: