Hierarchical Games and Fair Systems of Taxation
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Abstract: Mathematical model of income and corporative taxes is considered as an hierarchical game of different social groups. Existence of equilibrium solution is proven, and constructive method of determining this equilibrium is discussed.

Key Words: Hierarchical, Games, Equilibrium, Taxes,

1. Introduction
A hierarchical game consists of \( n + 1 \) players. One player is said to be the Center (0th player) with the space of strategies:

\[ U = U_1 \times \cdots \times U_n \]

where each \( U_i \) is a subset of some metric space, and the \( i \)th player has space of strategies \( X_i \).

Let us denote the payoff functions by

\[ G(x_1, \ldots, x_n, u_1, \ldots, u_n) \]

for the Center, and \( f(x_i, u_i) \) for \( i \)th player.

2 Subsidiary Hierarchical Games
We can see that the Center's payoff depends on the strategies used by each player of the hierarchical game, while the payoff of the \( i \)th player is a function of its own strategies and moves of the Center. In such setup of the hierarchical game the Center has full control.

By using subsidiary payoffs \( z_1, \ldots, z_n \geq 0 \), the Center can achieve the best payoff and can force this game to an effective equilibrium.

Indeed, let us introduce

\[ L_i = \left( \max_{x_i} \min_{u_i} \right) f(x_i, u_i) \]

We have the following statement:

**Theorem 1:** If \( z_i \geq 0, \ i = 1, ..., n \) and

\[ \sum_{i=1}^{n} Z_i \leq Z_0, \]

then for any \( Z_0 > 0 \), there is an effective equilibrium solution, such that the optimal payoff for the Center is

\[ \left( \max_{x, u, z} \right) \left( G(x_1, \ldots, x_n, u_1, \ldots, u_n) - \sum_{i=1}^{n} Z_i \right) \]

under the conditions

\[ f(x_i, u_i) + Z_i > L_i \]

In such a way, the centralized system may be in the stable equilibrium under control of the Center.

Other possible situations in hierarchical games consist of the Center without control. This means that all sets \( U_1, U_2, \ldots, U_n \) are empty, and at the same time all players have the common goal \( G(x_1, \ldots, x_n) \) (payoff function of the Center) in addition to their self-interests. This approach may be applied...
to the problem of taxation.

In general, we can consider a centralized system of taxation as an hierarchical game, in which players (different social groups and groups of interests) have two goals. One of them is self-interest of the social group, and the other is common interest of a society as a whole.

3 Social Groups as Players

Suppose there are \( n \) different social groups and the total income of \( i \)th social group is \( a_i \), and assume, we already know the self-interest utility functions \( f_1(x_1), \ldots, f_n(x_n) \) of these groups. Determination of such functions and determinations of the social groups are a very important part which will be considered later. Assume also that \( G(y_1, \ldots, y_n) \) is the utility function for the common goal of the society. It is reasonable assumption that all utility functions are supposed to be monotonic increasing functions of their arguments.

Here \( x_i \) is the amount of money \( i \)th social group keeps for itself and \( y_i \) is what it gives for achieving a common goal. We understand that \( x_i + y_i = a_i \).

It is well known in social science that different groups may have different attitudes toward common goals. Assuming that \( \lambda_i \) is the weight of common interest compared with the self-interest of the \( i \)th social group, we can present the payoff function of this group in the following forms:

\[
F_i = \min \left\{ \lambda_i f_i(x_i), G(y_1, \ldots, y_n) \right\}
\]

Or

\[
F_i = f_i(x_i) + \lambda_i G(y_1, \ldots, y_n)
\]

Suppose we have the order of the values:

\[
\lambda_1 f_1(a_1) \geq \lambda_2 f_2(a_2) \geq \ldots \geq \lambda_n f_n(a_n).
\]

The equilibrium theorem states the following:

**Theorem 2:** There is such a number \( k \), that contribution to the common interest is equal to zero for any social group in this order with the number greater than \( k \):

\[
y_{k+1} = y_{k+2} = \ldots = y_n = 0,
\]

and contributions of other social groups (players) may be found as the only solution of the system of equations:

\[
\lambda_i f_i(y_i) = G(y_1, \ldots, y_k, 0, \ldots, 0), i = 1, \ldots, k
\]

The social interpretation of this fact is very clear. If the social group has low income or has no great interest in achieving common goals then contribution of this group is equal to zero.

3 Sketch of Proof

Let us consider as an illustration of the prove a game with two social groups A and B (for \( n \)-person game the prove of this theorem is identical). In this case, their objective functions are:

\[
F_1 = \min \left\{ G(x_1, x_2), \lambda_1 f_1(a_1 - x_1) \right\} \to \max
\]

\[
F_2 = \min \left\{ G(x_1, x_2), \lambda_2 f_2(a_2 - x_2) \right\} \to \max
\]

Suppose that \((x_1^*, x_2^*)\) is the equilibrium point:

\[
F_1(x_1^*, x_2^*) = \max_{x_1} F_1(x_1, x_2^*)
\]

\[
F_2(x_1^*, x_2^*) = \max_{x_2} F_2(x_1^*, x_2)
\]

**Lemma.** Previous conditions are fulfilled if and only if

\[
G(x_1^*, x_2^*) = \lambda_1 f_1(a_1 - x_1^*) \quad \text{and}
\]

\[
G(x_1^*, x_2^*) = \lambda_2 f_2(a_2 - x_2^*)
\]

Suppose that system (2) has a solution for \( 0 < x_1 < a_1 \) and \( 0 < x_2 < a_2 \), and assume that

\[
x_1^* = \frac{x_1}{x_1} + \delta
\]
where \( x_1^* \) and \( x_2^* \) are the solutions of (2). If \( \delta > 0 \), then due to the monotonic properties of functions, we obtain:

\[
G^* = G(x_1^*, x_2^*) > \overline{G} = G(\overline{x_1}, \overline{x_2})
\]

and

\[
\lambda_i f_1^* = \lambda_i f_1(x_1^*) < \lambda_i \overline{f_1} = \lambda_i f_1(a_i - \overline{x_1}).
\]

Because \( \overline{G} = \lambda_i f_1 \) there are the inequalities

\[
G^* > \lambda_i \overline{f_1} \quad \text{and} \quad G^* > \lambda_i f_1^*.
\]

Hence,

\[
F_i^* = \min \{ G(x_1^*, \overline{x_2}), \lambda_i f_1(x_1^*) \} = \lambda_i f_1^*
\]

On the other hand,

\[
\overline{F}_i = \overline{G} = \lambda_i \overline{f_1} > \lambda_i f_1^* = F_i^*
\]

which means that \( F_i^* \) is not the maximum value of \( F_i \). Here we have a contradiction. The case for \( \delta < 0 \) may be considered symmetrically.

Suppose now that \((x_1, x_2)\) is the solution of the system (2), so

\[
\overline{F}_i = \overline{G} = \lambda_i \overline{f_1}
\]

and

\[
\overline{F}_2 = \overline{G} = \lambda_2 \overline{f_2},
\]

if \( x_i = \overline{x_i} + \delta \), then, as previously discussed,

\[
F_i(\overline{x_1} + \delta, \overline{x_2}) < F_i(\overline{x_1}, \overline{x_2}).
\]

This means that \( x_1, x_2 \) are satisfied conditions of (1).

Let us show that such an equilibrium is effective, or that there is no other equilibrium with greater values of the objective functions. Such effectiveness is the conclusion from previous discussion, in which, by the way, we showed that point \((x_1^*, x_2^*)\) belongs to Pareto set.

One moment is interesting. It may happen that \( \lambda_2 \) and \( a_2 \) are such that

\[
G(x_i, 0) > \lambda_2 f_2(a_2),
\]

and \( x_i \) is found from the condition

\[
G(x_i, 0) = \lambda_i f_1(a_i - x_i).
\]

In this case of \( F_2 = \lambda_2 f_2(a_2) \), and these last two equalities define the equilibrium solution.

The last situation may happen if resources of the player B are very small, or if \( \kappa_2 \) is very little, which means that this player almost has no interest in the common goal.

Prove of the Theorem 2 for two players may be extended without any changes for a general game with \( n \) players (social groups), and it gives so-called cut-index \( k \), after which contributions of the social groups are zero.

4 Conclusion

Usage of this game-theoretical background can bring—together with the determination of the utility functions and weight coefficients for different social groups—foundation for reasonable and fair taxation in the modern American society. Moreover, this approach may be extended for situations in which the common goal is some integrated function of possible spending on different programs (health, defense, crime prevention, etc.). It means, that we can find fair, stable equilibrium distribution of common funds, such as budget, while avoiding confrontation of social groups and parties.

The next step in research of hierarchical games is investigation of dynamical situations. Preliminary results show existence of effective equilibrium and optimal stationary strategies.

Another part of analytical analysis consists of applications of mathematical and statistical methods to obtain and to transform needed information from answers given in the questionnaires.

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References: