On stability of discrete Volterra equations with pattern method

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Abstract: The pattern equation method is used for studying the stability of some discrete Volterra equations. We consider the existence of bounded pattern functions with some asymptotic properties. We illustrate these properties with some examples.

Key–Words: Stability, Discrete Volterra Equations, Asymptotic behavior.

1 Introduction

The Volterra equations arise in the study of viscoelastic models in mechanics [4], economic growth [1], hysteresis models [11], in the difference approximation of some systems with distributed parameters, and in the numerical solution of various type of equations with continuous time [1, 2].

Certain types of functional differential equations with unbounded aftereffects (lags, differences or delays) may be also treated as abstract Volterra equations [5, 6, 11, 12].

In many situations in applications the present state \( z(t) \) or its rate of change \( \dot{z}(t) \) are dependent on the past. For Volterra equations the response \( z(t) \) depends on possibly the total time elapsed since the beginning [11, 12]. The classical Volterra integral equations for \( t \geq 0 \), \( z(t) \in \mathbb{R}^n \), are of the form

\[
 z(t) = \int_0^t K(t,s,z(s)) \, ds + g(t),
\]

and Volterra integro-differential equations are

\[
 \dot{z}(t) = g(t,z(t),\int_0^t K(t,s,z(s)) \, ds).
\]

As a rule, the solutions of the Volterra equations (1) and (2) may be determined only numerically by continuous or discrete methods [1, 2]. Discrete methods are usually associated with the partition of the interval

\[
 T = \{ 0 < t_1 < t_2 < \cdots < t_n < \cdots \} ,
\]

on which a mesh function \( y = y(t_i) = y_i \) is determined by a so called discrete Volterra equation (DVE)

\[
 y(i+1) = F(i,y(0),...,y(i-1),y(i)), \quad i = 0,1,..., k = 0,1,...,i, \quad y(k) \in \mathbb{R}^n ,
\]

The equation (3) can be considered a discrete approximation of Volterra equations (1) or (2). To determine the solution of DVE (3) we only need one initial condition

\[
 y(0) = y_0 .
\]

With the given initial condition in (4), one can iterate all the values \( y(i) \), for \( i \geq 0 \). Then, the existence and the unicity of the solution for the DVE in (3) are valid.

The DVE may also be considered as a natural generalisation of difference equations

\[
 y(i+1) = F(i,y(i-p),...,y(i)), \quad p = const, \quad k = -p,...,0,1,...,i, \quad i \geq p > 0, \quad y(k) \in \mathbb{R}^n .
\]

But the properties of DVEs and difference equations are not identical. Hence, it is correct to study DVEs as a special type of dynamical systems.

We shall study stability and asymptotic properties of DVEs using a new approach named “the pattern equation method” [10].

2 Pattern equation method

Let us consider the following DVE

\[
 y(i+1) = F(i,y(0),...,y(i)), \quad y(k) \in \mathbb{R}^n , i \geq 0, \quad y(0) = y_0 .
\]


The right-hand side of equation (6) is determined on the product of increasing number of euclidean spaces $\mathbb{R}^n$ i.e.,

$$F : \mathbb{Z}^+ \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}^n.$$ 

Here $\mathbb{Z}^+$ is the set of integers $i \geq 0$. Let us choose a norm $\| \cdot \|$ in the euclidean space $\mathbb{R}^n$.

Suppose now that the right-hand side of DVE (6) admits the following linear estimate

$$k \leq \beta_{ij,k} \| y (j_0) \| + \ldots + \beta_{ij,k} \| y (j_i) \|,$$

for any $i \in \mathbb{Z}^+$, $j_k \in S (i)$.

Here $S (i)$ is the set of all indices $j_k$ such that $\beta_{ij,k} > 0$, $1 \leq j_k \leq i$ for any integer $i \in \mathbb{Z}^+$. For the sake of brevity, a DVE in (6) can admit an estimate (7) and it will be denoted simply as DVE (6), (7).

In the sequel, we consider two cases:

1. The estimate (7) holds for any $y (k) \in \mathbb{R}^n$, $k = 1, 2, \ldots, i$;
2. The estimate (7) holds in the product of polydiscs

$$S \alpha_s = s_1 \times \ldots \times s_i \times \mathbb{R}^n,$$

of the neighborhoods

$$s_i = \{ \| y (k) \| \leq a \}, \quad 1 \leq k \leq i.$$

Along with DVE (6), (7), we shall consider a scalar DVE of the same form (i.e. with the same sets of indices $S (i)$)

$$x (i + 1) = \alpha_{ij,k} x (j_0) + \ldots + \alpha_{ij,k} x (j_i),$$

$$i \geq 0, \quad j_k \in S (i), \quad x (0) = x_0, \quad x (k) \in \mathbb{R}. \quad (8)$$

The following comparison theorem is well-known and may be easily proven by induction [2, 3].

**Theorem 1** If $\beta_{ij,k} \leq \alpha_{ij,k}$, $i \in \mathbb{Z}^+$, $j_k \in S (i)$ and $\| y (0) \| \leq \| x (0) \|$, then for all $i \geq 0$ it is satisfied

$$\| y (i) \| \leq | x (i) |.$$

Let us take an arbitrary positive scalar function $p (i)$ of the discrete argument $i \in \mathbb{Z}^+$, such that

$$p (0) = 1, \quad p (i) > 0, \quad i \geq 1. \quad (9)$$

The following theorem is a crucial point of the proposed method.

**Theorem 2** For any scalar function $p (i)$ satisfying the conditions in (9), there exists an infinity of homogeneous scalar DVEs (8) with a solution given by

$$x (i) = p (i) x (0), \quad i \in \mathbb{Z}^+. \quad (10)$$

**Proof:** To prove this theorem, we construct such a scalar DVE in explicit form. Let us introduce positive numbers $\gamma_i (j_k)$ such that

$$\gamma_i (j_k) > 0, \quad \sum_{j_k \in S (i)} \gamma_i (j_k) = 1, \quad i \in \mathbb{Z}^+. \quad (11)$$

Consider now an homogeneous scalar DVE of the form

$$x (i + 1) = p (i + 1) \cdot \left[ \gamma_i (j_k) x (j_0) + \ldots + \gamma_i (j_k) x (j_i) \right], \quad i \in \mathbb{Z}^+, \quad j_k \in S (i). \quad (12)$$

The scalar DVE in (12) will be called a pattern equation and the function $p (i)$ a pattern function.

We can describe the procedure of the pattern equation method:

1. Choose the pattern function (i.e. a positive scalar function $p (i)$, $i \in \mathbb{Z}^+$) in such a way that the solution “$x (i) = p (i) x (0)$” has required stability properties or asymptotic behaviour,
2. Using theorem 2, we can construct a pattern equation for the chosen function $p (i)$,
3. Compare the coefficients $\beta_{ij,k}$ of the initial DVE in (6), (7) with the coefficients of the contracted pattern scalar DVE in (12).

If $\| y (0) \| \leq | x (0) |$ and

$$\beta_{ij,k} \leq \frac{p (i + 1)}{p (j_k)} \gamma_i (j_k), \quad i \geq 0, \quad j_k \in S (i). \quad (13)$$

then, according to theorem 1

$$\| y (i) \| \leq | x (i) | = p (i) x (0).$$

The solutions of initial DVE (6), (7) will have the required properties described by $p (i)$.

### 3 Stability

The trivial solution of the DVE in (6) is $y (i) = 0$, $i \in \mathbb{Z}^+$. We consider the usual definition of stability, see [1, 3, 6, 7, 8, 9].
Definition 3 The trivial solution \( y(i) = 0 \), for \( i \geq 0 \) of DVE (6) is called Lyapunov stable if for any \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) > 0 \) such that \( \|y(i)\| \leq \varepsilon, i > 0 \) if \( \|y(0)\| \leq \delta \).

The following theorem gives sufficient conditions for stability in terms of the corresponding pattern equations and coefficients of DVE [10].

Theorem 4 Assume that the estimate in (7) is defined in the neighborhood \( S_0^i \), with \( a > 0 \) and there exists a bounded pattern function \( p(i) \), \( 0 < p(i) \leq P \) and numbers \( \gamma_i(j_k) \) satisfying conditions in (11) such that
\[
\beta_{i,j_k} \leq \frac{p(i+1)}{p(i)} \gamma_i(j_k), \quad i \in \mathbb{Z}^+, \quad j_k \in S(i).
\] Then the trivial solution of DVE (6), (7) is Lyapunov stable.

Proof: We know that the solution of the pattern equation in (12) is the function \( x(i) = p(i) x(0) \). If we take \( |x(0)| = \|y(0)\| \leq P^{-1}a \), then from conditions in (14) and from the theorem 1 follows the estimate
\[
\|y(i)\| \leq |x(i)| \leq |p(i) x(0)| \\
\leq p(i) \|y(0)\| \leq P \|y(0)\|.
\] The stability of the trivial solution of (6), (7) is a direct consequence of the estimate in (15).

Formally, the theorem 4 for the stability in the pattern equation method seems to be very close to the Theorem on stability in the Lyapunov direct method [6, 7, 8, 9], but does not coincide with it.

As a corollary of theorem 4, we can obtain the following well-known result [1, 2, 10].

Corollary 5 If the pattern function \( p(i) \) in estimate (14) is non-increasing, then
\[
B(i) = \sum_{j_k \in S(i)} \beta_{i,j_k} \leq 1, \quad i \in \mathbb{Z}^+, \tag{16}
\] and the trivial solution of DVE (6), (7) is stable.

Example 6 Consider the DVE with right-hand side depending only on \( y(i), y(0) \) such that
\[
y(i+1) = F(y(0), y(i)), \quad y(i) \in \mathbb{R}^n, \\
\|F(y(0), y(i))\| \leq \beta_{i,0} \|y(0)\| + \beta_{i,i} \|y(i)\|, \quad \beta_{i,j_k} > 0, \quad i \in \mathbb{Z}^+.
\] (17)

We take the following increasing bounded pattern function \( p(i) \) and numbers \( \gamma_i(j_k) \) equal to
\[
p(0) = 1, \quad p(i) = \frac{2i - 1}{i},
\]
\[
\gamma_i(j_k) = 1 - \frac{1}{2i} = \frac{2i - 1}{2i}, \quad \gamma_i(0) = \frac{1}{2i},
\]
\[
\gamma_i(i) + \gamma_i(0) = 1, \quad i \geq 1.
\]

In this case the coefficients of pattern equation (8) are equal to
\[
\alpha_{i,i} = \frac{2i + 1}{2(i + 1)},
\]
\[
\alpha_{i,0} = p(i + 1) \gamma_i(0) = \frac{2i + 1}{2i}.
\]

Hence, for \( \beta_{i,i} \leq \alpha_{i,i}, \beta_{i,0} \leq \alpha_{i,0} \) the trivial solution of DVE (12) is stable, although in case when \( \beta_{i,i} \leq \alpha_{i,i}, \beta_{i,0} \leq \alpha_{i,0} \) for all \( i \) we have
\[
B(i) = \beta_{i,i} + \beta_{i,0} = \frac{2i + 1}{2(i + 1)} \left[ 1 + \frac{1}{i} \right] = 1 + \frac{1}{2i} > 1.
\]

So condition (16) is only sufficient for stability but it is not necessary. More exactly, even in case when condition (16) is wrong for all \( i > 0 \) the trivial solution of (17) may be stable for some special set of coefficients.

Example 7 Consider the following pattern function
\[
p(i) = 2 - \sin \frac{\pi}{2} i, \quad i \geq 1.
\]

This function is periodic with period 4 i.e., \( p(i + 4) = p(i), i \geq 1 \), and \( 1 \leq p(i) \leq 3 \). Consider the initial-present state in DVE (17) and take \( \gamma_i(i) = 0.9 \) and \( \gamma_i(1) = 0.1 \). The coefficients of pattern equation for such a choice of \( p(i) \) and \( \gamma_i(j) \) are given in Table 1.

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_{i,i} )</td>
<td>1</td>
<td>1.35</td>
<td>0.6</td>
<td>0.45</td>
<td>1.8</td>
<td>1.35</td>
<td>0.6</td>
</tr>
<tr>
<td>( \alpha_{i,0} )</td>
<td>1</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>( B(i) )</td>
<td>2</td>
<td>1.65</td>
<td>0.8</td>
<td>0.55</td>
<td>2.0</td>
<td>1.65</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Table 1.
It is interesting to remark that in this case even the average value of $B(i)$ is greater than 1

\[
B_{\text{average}} = 0.25[B(i) + B(i + 1) + B(i + 2) + B(i + 3)] = 1.25 > 1.
\]

Also, the product $B(i)B(i + 1)B(i + 2)B(i + 3) = 1.452 > 1$. Nevertheless, the equation (17) with such a sequence of coefficients is stable.

**Example 8** A discrete pantograph equation with unbounded coefficients

In the theory of functional differential equations the following equation

\[
\dot{y}(t) = \lambda y(t) + \mu y(\alpha t), \quad 0 < \alpha < 1, \quad t \geq 0,
\]

is called the pantograph equation [5, 6]. By analogy, we shall refer to the DVE

\[
y(i + 1) = \lambda_i y(i) + \mu_i y(\lfloor \alpha i \rfloor), \quad i \geq 0,
\]

\[
0 < \alpha < 1, \quad y(i) \in \mathbb{R}, \quad (18)
\]

as the discrete pantograph equation. In (18), $\lfloor z \rfloor$ the integer part of the quantity $z$. Let us now study the stability of discrete pantograph equation (18). Because of $B(i) = \lambda_i + \mu_i \leq 1$, then DVE (18) is stable.

Now, we shall show that DVE (18) may be stable even if $B(i)$ is unbounded. For this purpose, let us consider a pattern function $p(i)$ of the form

\[
p(0) = 1, \quad p(i) = |\sin i|, \quad i \geq 1. \quad (19)
\]

Because of the irrationality of the number $\pi$, it is clear that $p(i) > 0$, $i \geq 1$. The pattern function in (19) is bounded, but not monotone. The coefficients $\alpha_{i,i}$ and $\alpha_{i,\lfloor \alpha i \rfloor}$ of pattern equation (12) corresponding to DVE (18) for pattern function (19) and $\gamma_i(i) = \gamma_i(\lfloor \alpha i \rfloor) = 0.5$ are equal to

\[
\alpha_{i,i} = \frac{|\sin (i + 1)|}{2|\sin i|}, \quad \alpha_{i,\lfloor \alpha i \rfloor} = \frac{|\sin (i + 1)|}{2|\sin \alpha i|}.
\]

The sequences $\alpha_{i,i}$ and $\alpha_{i,\lfloor \alpha i \rfloor}$ are unbounded. In fact, for any given small enough positive number $\delta > 0$, there exist an integer $i$ such that $i = 2k\pi + x$, $0 < x \leq \delta$, and

\[
g(i) = \frac{|\sin (i + 1)|}{|\sin i|} = \left| \frac{\cos (1) + \sin (1) \cos x}{\sin x} \right| > \frac{1}{2\delta}.
\]

Therefore, the sequence $\alpha_{i,i}$ is unbounded.

The direct computation for the function $g(i)$ gives the following values listed in Table 2.

<table>
<thead>
<tr>
<th>$i$</th>
<th>19</th>
<th>355</th>
<th>710</th>
<th>104348</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(i)$</td>
<td>6.09129</td>
<td>27915.3</td>
<td>13957.9</td>
<td>76394.0</td>
</tr>
</tbody>
</table>

Table 2.

The same arguments show that the sequence $\alpha_{i,\lfloor \alpha i \rfloor}$ is unbounded too.

Discuss now a converse theorem concerning stability in the pattern equation method. As we consider only DVEs (6), which satisfy estimates (7), it is natural to suppose that the trivial solution of any inequality

\[
\|y(i + 1)\| \leq \beta_{i,j_1} \|y(j_1)\| + \ldots + \beta_{i,j_i} \|y(j_i)\|, \quad (20)
\]

is stable.

One of the peculiarities of DVEs which makes them differ from difference equations is that the trivial solution of equation (20) is stable, then there exist an integer $i$, such that $B(i) = 0$, $y(i) = 0$ for a long sequence of consecutive integers $i$, but the solution $y(i)$ does not vanish after that. We exclude such cases from consideration.

**Theorem 9** Assume that

\[
B(i) > 0, \quad i \geq 1. \quad (21)
\]

If the trivial solution of equation (20) is stable, then there exists a bounded pattern equation $p(i)$ and numbers $\gamma_i(j_k)$ satisfying (11) such that the coefficients of estimate (7) verify inequalities in (14).

**Proof:** Let us take $\tilde{y}(0)$, such that $\|\tilde{y}(0)\| = 1$. Then, the conditions in (21) imply that the solution $\tilde{y}(i)$ of (20) with such initial conditions satisfy inequalities $\|\tilde{y}(i)\| > 0$, $i \geq 0$. From the stability of the trivial solution of (20) it follows that, the sequence $\|\tilde{y}(i)\|$ is bounded, $\|\tilde{y}(i)\| \leq Y$, $i \geq 0$. Consider $p(i) = \|\tilde{y}(i)\|$, then the function $p(i)$ is strictly positive and bounded, $0 < p(i) \leq Y$, $i \geq 0$. Thus, $p(i)$ may be considered as a pattern function and write the equation (20) as

\[
p(i + 1) = \beta_{i,j_0} p(j_0) + \ldots + \beta_{i,j_i} p(j_i), \quad (22)
\]

Denote by $\gamma_i(j_k)$ the numbers

\[
\gamma_i(j_k) = \frac{\beta_{i,j_k} p(j_k)}{p(i + 1)} > 0.
\]

From (24), it follows

\[
0 \leq \gamma_i(j_k) \leq 1, \quad i \in \mathbb{Z}^+, \quad \sum_{j_k \in S(i)} \gamma_i(j_k) = 1.
\]

Then, the coefficients $\beta_{i,j_k}$ of equation (20) satisfy the inequalities

\[
\beta_{i,j_k} \leq \frac{p(i + 1)}{p(j_k)} \gamma_i(j_k),
\]

where the pattern function $p(i)$ is bounded and the numbers $\gamma_i(j_k)$ satisfy (11). The converse theorem is proven.
4 Asymptotic stability

In this section, some conditions of the asymptotic stability based on pattern equations method are given.

Definition 10 The trivial solution \(y(i) = 0\) of DVE (6) is called asymptotically stable if it is stable and
\[
y(i) \to 0, \quad i \to \infty, \quad y(0) \in Q \subset \mathbb{R}^n.
\]

The set \(Q\) of such \(y(0)\) is called the attraction domain of the trivial solution. If \(Q = \mathbb{R}^n\), then the trivial solution is globally asymptotically stable.

Theorem 11 Suppose that the estimate in (7) is satisfied for any \(y(k) \in \mathbb{R}^n, 1 \leq k \leq i\) and there exist numbers \(\gamma_i(j_k)\) satisfying conditions in (11) and the function \(p(i)\) such that
\[
p(0) = 1, \quad 0 < p(i) \leq P, \quad p(i) \to 0, \quad i \to \infty,
\]
and
\[
\beta_{i,j_k} \leq \frac{P(i+1)}{P(j_k)} \gamma_i(j_k), \quad i \in \mathbb{Z}^+, \quad j_k \in S(i).
\]

Then, the trivial solution of DVE (6), (7) is globally asymptotically stable. If the estimate (7) holds only in \(S_0\), then the trivial solution DVE (6), (7) is asymptotically stable and the attraction domain \(Q\) is defined by the following neighborhood \(\|y(0)\| \leq P^{-1}a\).

Proof: The pattern equation in (12) for such a choice of \(p(i), \gamma_i(j_k)\) has a solution
\[
x(i) = p(i) x(0), \quad i \geq 1,
\]
which is globally asymptotically stable. In this case the proof of global asymptotic stability of the trivial solution of DVE (6), (8) in this case, it follows directly from comparison theorem 1.

If the estimate (7) is fulfilled only in a product \(S_0\), then we take
\[
\|y(0)\| \leq P^{-1}a \leq x(0).
\]

The estimate in (15) is satisfied for such a solution. From comparison theorem 1, it follows that \(\|y(i)\| \leq p(i) \|y(0)\|\), for \(\|y(0)\| \leq P^{-1}a\). The theorem 11 is proven.

The following example shows that if the pattern function is non monotone, then it is possible to construct a globally asymptotically stable DVE (6) with an unbounded sequence \(B(i)\).

Example 12 Asymptotic stability of discrete pantograph equations with unbounded coefficients

We shall show that the trivial solution of equation (18) may be globally asymptotically stable even if the sequences \(\lambda_i\) and \(\mu_i\) are unbounded. Take as a pattern function \(p(i)\) the function of the form
\[
p(0) = 1, \quad p(i) = i^{-1} |\sin i|, \quad i \geq 1.
\]

When the function in (24) satisfy the conditions in (23), the coefficients of the pantograph equation (18) satisfy the estimates
\[
\lambda_i \leq \frac{i |\sin (i+1)|}{2(i+1) |\sin i|}, \quad \mu_i \leq \frac{[\alpha i] |\sin (i+1)|}{2(i+1) |\sin [\alpha i]|}.
\]

With this assumption, the trivial solution of the pantograph equation is globally asymptotically stable. But if
\[
\frac{i}{i+1} \to 1, \quad \frac{[\alpha i]}{i+1} \to \alpha, \quad i \to \infty
\]
and the sequences
\[
\frac{|\sin (i+1)|}{|\sin i|}, \quad \frac{|\sin (i+1)|}{|\sin [\alpha i]|},
\]
are unbounded, then the sequences of \(\lambda_i\) and \(\mu_i\) are unbounded, but the discrete pantograph equation is globally asymptotically stable.

5 Remarks on difference equations

The type of difference equations in (5) may be considered as a particular case of DVE (6) with special set of indeces
\[
S_{ \text{diff}}(i) = \{j_k \in S_{ \text{diff}} : j_k = i, ..., i - p\}.
\]

Suppose that the right-hand side of difference equations
\[
y(i+1) = F(i, y(i-p), ..., y(i)),
\]
where \(p = \text{const}\), admits for \(i \geq p, y(k) \in \mathbb{R}^n\) an estimate similar to (7), such that
\[
\|F(i, y(0), ..., y(i))\| \leq \beta_{i,j_0} \|y(j_0)\| + ... + \beta_{i,j_k} \|y(j_k)\|, \quad \beta_{i,j_k} > 0, \quad j_k \in S_{ \text{diff}}(i), \quad i > 1.
\]

The pattern equation for difference equation (26), (27) has the form
\[
x(i+1) = p(i+1) \cdot \left[ \sum_{j_p \in B(i)} \frac{\gamma(j_p)}{p(j_p)} x(j_p) \right], \quad i \in \mathbb{Z}^+, \quad j_k \in S_{ \text{diff}}(i).
\]

Then, all our results on stability and asymptotically stability are also valid for difference equations (26), (27). A little modifications of the example 1 give a very interesting example of stable or asymptotically stable difference equations.
Example 13 Consider the difference equations
\[ y(i+1) = F(i, y(i-p), y(i)), \]
i ≥ 0, y(i) ∈ \mathbb{R}^n
\[ \|F(i, y(i), y(i-1))\| \leq \beta_{i,i} \|y(i)\| + \beta_{i,i-1} \|y(i-1)\|, \]
(28)
\[ \beta_{i,i} > 0, \beta_{i,i-1} > 0, i > 1. \]
Take pattern function \( p(i), i ≥ 1 \) and numbers \( \gamma_i(j_k) \) equal to
\[ p(i) = \frac{2i - 1}{i}, \gamma_i(i) = \gamma_i(i-1) = \frac{1}{2}. \]
In this case the coefficients of pattern equation (8) are equal to
\[ \alpha_{i,i} = \frac{p(i+1)}{p(i)} \gamma_i(i) = \frac{i(2i+1)}{2(2i-1)(i+1)}. \]
\[ \alpha_{i,i} = \frac{2i^2 + 1}{2(2i^2 + i - 1)} > \frac{1}{2}. \]
\[ \alpha_{i,i-1} = \frac{p(i)}{p(i-1)} \gamma_i(i-1) = \frac{(2i-1)(i-1)}{2i(2i-3)}, \]
\[ \alpha_{i,i-1} = \frac{2i^2 - 3i + 1}{4i^2 - 6i} > \frac{1}{2}. \]
Hence, for \( \beta_{i,i} \leq \alpha_{i,i}, \beta_{i,i-1} \leq \alpha_{i,i-1} \) the trivial solution of difference equation (28) is stable, although in case when \( \beta_{i,i} = \alpha_{i,i}, \beta_{i,0} \leq \alpha_{i,0} \) for all \( i > 0 \) we have
\[ B(i) = \beta_{i,i} + \beta_{i,i-1} > 1. \]
Taking as a pattern function the equation in (19), we obtain some interesting stability conditions for difference equation (28) with unbounded sequence of coefficients.

6 Conclusions

The pattern equation method can be used to obtain the stability and asymptotic behaviour for the discrete and difference equations, but it is required that the estimate in (7) holds for all \( i ∈ \mathbb{Z}^+ \).

References


