An extended Crank-Nicholson method and its Applications in the Solution of Partial Differential Equations:
1-D and 3-D Conduction Equations

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Abstract: - In this paper, an extension of the Crank-Nicholson method for solving parabolic equations is launched. The method uses finite differences. For the derivative of the variable of time, we use central difference at 4 points (instead of 2 points of the classical Crank-Nicholson method), while for the second-order derivatives of the other spatial variables we use lagrangian interpolation at 4 points of the central differences at 5 points (instead of lagrangian interpolation at 2 points of the central differences at 3 points of the classical Crank-Nicholson method). The method is illustrated below.

Key-Words: - Crank-Nicholson, Parabolic Equations, Finite Differences

1 Introduction

The numerical solution of Partial Differential Equations is a topic of great importance in science and engineering because of many applications. Finite Differences, Finite Volumes, Finite Elements, Boundary Elements are among the most valuable numerical tools that we can use in order to approximate the theoretical solution with a numerical one. Error Analysis, Convergence, Stability are only a few topics from the long list of research topics that attract an increasing interest among academic scholars and industrial practitioners working on applied mathematics [1],[2].

Suppose that we have to solve the 1-D (one-dimensional) conduction equation.

\[ \frac{\partial^2 f}{\partial x^2} = k \frac{\partial f}{\partial t} \]  

(1)

We consider the grid of the points \((i_i,i_z)\in\mathbb{Z}^2\) as a discretization of the continuous space of \(x,t\) of \(\mathbb{R}^2\) where we have the function \(f(i_i,i_z)\) that approximates the \(f(x,t)\)

A simple discretization of Eq.(1) is obviously:

\[ f\left(i_i-1,i_z\right) + f\left(i_i+1,i_z\right) - 2f\left(i_i,i_z\right) \]

\[ \frac{h^2}{2} \]

\[ = k \frac{f\left(i_i,i_z+1\right) - f\left(i_i,i_z\right)}{\tau} \]  

(2)

where \(h,\tau\) are the steps of discretization with respect \(x,t\)

This finite difference scheme suffers from convergence problems, instability and errors, so another, better method, the so-called Crank-Nicholson method is applied. This method (Crank-Nicholson) is briefly described as follows:

A point that does not belong to the grid of the points \((i_i,i_z)\) is considered. It is the \(\left(i_i,i_z + \frac{1}{2}\right)\)

At this point we approximate

\[ \frac{\partial^2 f}{\partial x^2} \bigg|_{\left(i_i,i_z + \frac{1}{2}\right)} \approx \]

\[ \frac{1}{h^2} \left( f\left(i_i - 1,i_z\right) + f\left(i_i + 1,i_z\right) - 2f\left(i_i,i_z\right) + \right. \]

\[ \frac{1}{2} \left( f\left(i_i - 1,i_z + 1\right) + f\left(i_i - 1,i_z - 1\right) - 2f\left(i_i,i_z\right) + \right. \]

\[ + f\left(i_i + 1,i_z + 1\right) + f\left(i_i + 1,i_z - 1\right) - 2f\left(i_i,i_z\right) \]

(3.1)

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while
\[
\frac{\partial f}{\partial t}\bigg|_{(i/\tau + \frac{1}{2})} \approx f(i,i_2 + 1) - f(i,i_2) \tag{3.2}
\]

Combining (3.1), and (3.2) we get the following numerical scheme

\[
\begin{align*}
\frac{1}{2}(f(i-1,i_2) + f(i+1,i_2) - 2f(i,i_2)) + \\
+f(i-1,i_2 + 1) + f(i+1,i_2 + 1) - 2f(i,i_2 + 1) = \\
=k(f(i,i_2 + 1) - f(i,i_2)) \tag{4}
\end{align*}
\]

The numerical scheme of Eq.(2) is an explicit numerical scheme because all starting values are directly available from initial and boundary conditions and each new value is obtained from the values that are already known.

The numerical scheme of Eq.(4), i.e. the Crank-Nicholson method, is an implicit numerical scheme because the values to be computed are not just a function of values at the previous time step which are not readily available. This requires us to solve a set of simultaneous linear equations at each time step. However, the Crank-Nicholson method is superior than (2), because of stability, convergence and accuracy [1],[2].

In this paper, we extend the Crank-Nicholson scheme by estimating \( \frac{\partial^2 f}{\partial x^2} \bigg|_{(i,i_2 + \frac{1}{2})} \) by interpolating \( \frac{\partial^2 f}{\partial x^2} \) at the points \((i_1,i_2 - 1), (i_1,i_2), (i_1,i_2 + 1), (i_1,i_2 + 2)\).

In the sequel, we use a central difference at 4 points (instead of 2 points of the classical Crank-Nicholson method) for the derivative of \( t \).

This extended or generalized Crank-Nicholson method is outlined in the following Section 2.

2 Extended (or Generalized) Crank-Nicholson method for the 1-D (one-dimensional) conduction equation

After the preparation of the previous session, we are ready to describe the new proposed numerical method for parabolic PDEs. First of all, using the typical Lagrange interpolation formula we can write for a function \( g \) (with existence and continuity of all its partial derivatives up to 4th order)

\[
g(i_1,i_2 + \frac{1}{2}) = -\frac{1}{16}g(i_1,i_2 - 1) + \frac{9}{16}g(i_1,i_2) + \\
+\frac{9}{16}g(i_1,i_2 + 1) - \frac{1}{16}g(i_1,i_2 + 2) \tag{5}
\]

So, we can write: \( \frac{\partial^2 f}{\partial x^2} \bigg|_{(i,i_2 + \frac{1}{2})} \approx \)
\[
\begin{multline}
- \frac{1}{16} \left( -2 f(i - 2, i_2 - 1) + 16 f(i - 1, i_2 - 1) - 30 f(i, i_2 - 1) + 16 f(i + 1, i_2 - 1) - 2 f(i + 2, i_2 - 1) \right) \\
+ \frac{9}{16} \left( -2 f(i - 2, i_2) + 16 f(i - 1, i_2) - 30 f(i, i_2) + 16 f(i + 1, i_2) - 2 f(i + 2, i_2) \right) \\
+ \frac{9}{16} \left( -2 f(i - 2, i_2 + 1) + 16 f(i - 1, i_2 + 1) - 30 f(i, i_2 + 1) + 16 f(i + 1, i_2 + 1) - 2 f(i + 2, i_2 + 1) \right) \\
- \frac{1}{16} \left( -2 f(i - 2, i_2 + 2) + 16 f(i - 1, i_2 + 2) - 30 f(i, i_2 + 2) + 16 f(i + 1, i_2 + 2) - 2 f(i + 2, i_2 + 2) \right) \\
= k \left( f(i, i_2 - 1) + 15 f(i, i_2) - 15 f(i, i_2 + 1) + f(i, i_2 + 2) \right) \\
\end{multline}
\]

(6.1)

\[
\frac{\partial f}{\partial t} \bigg|_{\hat{t} = \frac{1}{2}} \approx -\frac{f(i, i_2 - 1)}{2} + 8f(i, i_2) - 8f(i, i_2 + 1) + f(i, i_2 + \frac{3}{2})
\]

(6.2)

Replacing \( f\left(i, i_2 - \frac{1}{2}\right) \) by \( \frac{f(i, i_2) + f(i, i_2 - 1)}{2} \)

and

\[
\frac{\partial f}{\partial t} \bigg|_{\hat{t} = \frac{1}{2}} \approx -\frac{f(i, i_2 - 1)}{2} + 15f(i, i_2) - 15f(i, i_2 + 1) + f(i, i_2 + 2)
\]

(6.3)

Finally our numerical scheme becomes

\[
\begin{multline}
- \frac{1}{16} \left( -2 f(i - 2, i_2 - 1) + 16 f(i - 1, i_2 - 1) - 30 f(i, i_2 - 1) + 16 f(i + 1, i_2 - 1) - 2 f(i + 2, i_2 - 1) \right) \\
+ \frac{9}{16} \left( -2 f(i - 2, i_2) + 16 f(i - 1, i_2) - 30 f(i, i_2) + 16 f(i + 1, i_2) - 2 f(i + 2, i_2) \right) \\
+ \frac{9}{16} \left( -2 f(i - 2, i_2 + 1) + 16 f(i - 1, i_2 + 1) - 30 f(i, i_2 + 1) + 16 f(i + 1, i_2 + 1) - 2 f(i + 2, i_2 + 1) \right) \\
- \frac{1}{16} \left( -2 f(i - 2, i_2 + 2) + 16 f(i - 1, i_2 + 2) - 30 f(i, i_2 + 2) + 16 f(i + 1, i_2 + 2) - 2 f(i + 2, i_2 + 2) \right) \\
= k \left( f(i, i_2 - 1) + 15 f(i, i_2) - 15 f(i, i_2 + 1) + f(i, i_2 + 2) \right)
\end{multline}
\]

(7)

Eq.(7) can be solved by a Gauss-Seidel iterative method as one can see at the end of Section 3.
3. Extended (or Generalized) Crank-Nicholson method for the general case of the 3-D (three-dimensional) conduction equation

Suppose that we have to solve the parabolic equation:

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = k \frac{\partial f}{\partial t}
\]  

(8)

We consider the grid of the points \((i_1, i_2, i_3, i_4) \in \mathbb{Z}^4\) as a discretization of the continuous space of \(x, y, z, t\) of \(\mathbb{R}^4\) where we have the function \(f (i_1, i_2, i_3, i_4)\) that approximates the \(f(x, y, z, t)\)

A simple discretization of Eq. (1) is obviously:

\[
\begin{align*}
\frac{f(i_1-1, i_2, i_3, i_4) + f(i_1+1, i_2, i_3, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_x^2} &+ \\
\frac{f(i_1, i_2-1, i_3, i_4) + f(i_1, i_2+1, i_3, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_y^2} &+ \\
\frac{f(i_1, i_2, i_3-1, i_4) + f(i_1, i_2, i_3+1, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_z^2}
\end{align*}
\]

(9)

where \(h_x, h_y, h_z, \tau\) are the steps of discretization with respect to \(x, y, z, t\)

This finite difference scheme suffers from convergence problems, instability and errors, so another, better method, the so-called Crank-Nicholson method is applied. This method (Crank-Nicholson) is briefly described as follows:

A point that does not belong to the grid of the points \((i_1, i_2, i_3, i_4)\) is considered. It is the \((i_1, i_2, i_3, i_4 + \frac{1}{2})\)

At this point we approximate

\[
\frac{\partial^2 f}{\partial x^2} \bigg|_{(i_1, i_2, i_3, i_4 + \frac{1}{2})} = \frac{f(i_1-1, i_2, i_3, i_4) + f(i_1+1, i_2, i_3, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_x^2}
\]

\[
\frac{f(i_1, i_2-1, i_3, i_4) + f(i_1, i_2+1, i_3, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_y^2}
\]

\[
\frac{f(i_1, i_2, i_3-1, i_4) + f(i_1, i_2, i_3+1, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_z^2}
\]

\[
\frac{f(i_1, i_2, i_3, i_4+1) + f(i_1, i_2, i_3, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_t^2}
\]

(10.1)

Similarly

\[
\frac{\partial^2 f}{\partial y^2} \bigg|_{(i_1, i_2, i_3, i_4 + \frac{1}{2})} = \frac{f(i_1, i_2-1, i_3, i_4) + f(i_1, i_2+1, i_3, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_y^2}
\]

\[
\frac{f(i_1, i_2, i_3-1, i_4) + f(i_1, i_2, i_3+1, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_z^2}
\]

\[
\frac{f(i_1, i_2, i_3, i_4+1) + f(i_1, i_2, i_3, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_t^2}
\]

(10.2)

and

\[
\frac{\partial^2 f}{\partial z^2} \bigg|_{(i_1, i_2, i_3, i_4 + \frac{1}{2})} = \frac{f(i_1, i_2, i_3-1, i_4) + f(i_1, i_2, i_3+1, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_z^2}
\]

\[
\frac{f(i_1, i_2, i_3, i_4+1) + f(i_1, i_2, i_3, i_4) - 2f(i_1, i_2, i_3, i_4)}{h_t^2}
\]

(10.3)

while

\[
\frac{\partial f}{\partial t} \bigg|_{(i_1, i_2, i_3, i_4 + \frac{1}{2})} = \frac{f(i_1, i_2, i_3, i_4+1) - f(i_1, i_2, i_3, i_4)}{\tau}
\]
As in the 1-D conduction equation, the numerical scheme of Eq.(9) is an explicit numerical scheme because all starting values are directly available from initial and boundary conditions and each new value is obtained from the values that are already known. The numerical scheme of Eq.(11), i.e. the Crank-Nicholson method of the 3-D conduction equation, is an implicit numerical scheme because the values to be computed are not just a function of values at the previous time step which are not readily available. This requires us to solve a set of simultaneous linear equations at each time step. (but the Crank-Nicholson method is superior than (2), because of stability, convergence and accuracy)

Combining (10.1), (10.2), (10.3) and (10.4) we get the following numerical scheme

\[
\begin{align*}
12 & \frac{1}{2} \left( f(i-1,1,i_3,i_4) + f(i+1,1,i_3,i_4) - 2f(i_1,i_2,i_3,i_4) \right) + \\
& + \frac{f(i-1,1,i_3,i_4+1) + f(i+1,1,i_3,i_4+1) - 2f(i_1,i_2,i_3,i_4+1)}{h_1^2} \\
& + \frac{1}{2} \left( f(i_1,i_2-1,i_3,i_4) + f(i_1,i_2+1,i_3,i_4) - 2f(i_1,i_2,i_3,i_4) \right) + \\
& + \frac{f(i_1,i_2-1,i_3,i_4+1) + f(i_1,i_2+1,i_3,i_4+1) - 2f(i_1,i_2,i_3,i_4+1)}{h_1^2} \\
& + \frac{1}{2} \left( f(i_1,i_2,i_3-1,i_4) + f(i_1,i_2,i_3+1,i_4) - 2f(i_1,i_2,i_3,i_4) \right) + \\
& + \frac{f(i_1,i_2,i_3-1,i_4+1) + f(i_1,i_2,i_3+1,i_4+1) - 2f(i_1,i_2,i_3,i_4+1)}{h_1^2} \\
= & \frac{k}{\tau} \left[ f(i_1,i_2,i_3,i_4+1) - f(i_1,i_2,i_3,i_4) \right]
\end{align*}
\]

As in the 1-D conduction equation, the numerical scheme of Eq.(9) is an explicit numerical scheme because all starting values are directly available from initial and boundary conditions and each new value is obtained from the values that are already known. The numerical scheme of Eq.(11), i.e. the Crank-Nicholson method of the 3-D conduction equation, is an implicit numerical scheme because the values to be computed are not just a function of values at the previous time step which are not readily available. This requires us to solve a set of simultaneous linear equations at each time step. (but the Crank-Nicholson method is superior than (2), because of stability, convergence and accuracy)

In this paper, we extend also the Crank-Nicholson scheme for the 3-D conduction equation by estimating \( \frac{\partial^2 f}{\partial x^2} \) at the points \((i_1,i_2,i_3,i_4 -1), (i_1,i_2,i_3,i_4), (i_1,i_2,i_3,i_4 +1), (i_1,i_2,i_3,i_4 +2)\).
First of all, using again the typical Lagrange interpolation formula we can write for a function $g$ (with existence and continuity of all its partial derivatives up to 4th order)

\[
g(i_1, i_2, i_3, i_4 + \frac{1}{2}) = -\frac{1}{16}g(i_1, i_2, i_3, i_4 - 1) + \frac{9}{16}g(i_1, i_2, i_3, i_4) + \frac{9}{16}g(i_1, i_2, i_3, i_4 + 1) - \frac{1}{16}g(i_1, i_2, i_3, i_4 + 2)
\]

(12)

Therefore, we can write:

\[
\frac{\partial^2 f}{\partial x^2} \approx \frac{1}{16} \left( -2f(i_1 - 2, i_2, i_3, i_4 - 1) + 16f(i_1 - 1, i_2, i_3, i_4 - 1) - 30f(i_1, i_2, i_3, i_4 - 1) + 16f(i_1 + 1, i_2, i_3, i_4 - 1) - 2f(i_1 + 2, i_2, i_3, i_4 - 1) \right) \\

+ \frac{9}{16} \left( -2f(i_1, i_2, i_3, i_4) + 16f(i_1 - 1, i_2, i_3, i_4) - 30f(i_1, i_2, i_3, i_4) + 16f(i_1 + 1, i_2, i_3, i_4) - 2f(i_1 + 2, i_2, i_3, i_4) \right) \\

+ \frac{9}{16} \left( -2f(i_1 - 2, i_2, i_3, i_4 + 1) + 16f(i_1 - 1, i_2, i_3, i_4 + 1) - 30f(i_1, i_2, i_3, i_4 + 1) + 16f(i_1 + 1, i_2, i_3, i_4 + 1) - 2f(i_1 + 2, i_2, i_3, i_4 + 1) \right) \\

+ \frac{1}{16} \left( -2f(i_1 - 2, i_2, i_3, i_4 + 2) + 16f(i_1 - 1, i_2, i_3, i_4 + 2) - 30f(i_1, i_2, i_3, i_4 + 2) + 16f(i_1 + 1, i_2, i_3, i_4 + 2) - 2f(i_1 + 2, i_2, i_3, i_4 + 2) \right)

(13.1)

as well as

\[
\frac{\partial^2 f}{\partial y^2} \approx \frac{1}{16} \left( -2f(i_1, i_2 - 2, i_3, i_4 - 1) + 16f(i_1, i_2 - 1, i_3, i_4 - 1) - 30f(i_1, i_2, i_3, i_4 - 1) + 16f(i_1, i_2 + 1, i_3, i_4 - 1) - 2f(i_1, i_2 + 2, i_3, i_4 - 1) \right) \\

+ \frac{9}{16} \left( -2f(i_1, i_2 - 2, i_3, i_4) + 16f(i_1, i_2 - 1, i_3, i_4) - 30f(i_1, i_2, i_3, i_4) + 16f(i_1, i_2 + 1, i_3, i_4) - 2f(i_1, i_2 + 2, i_3, i_4) \right) \\

+ \frac{9}{16} \left( -2f(i_1, i_2 - 2, i_3, i_4 + 1) + 16f(i_1, i_2 - 1, i_3, i_4 + 1) - 30f(i_1, i_2, i_3, i_4 + 1) + 16f(i_1, i_2 + 1, i_3, i_4 + 1) - 2f(i_1, i_2 + 2, i_3, i_4 + 1) \right) \\

+ \frac{1}{16} \left( -2f(i_1, i_2 - 2, i_3, i_4 + 2) + 16f(i_1, i_2 - 1, i_3, i_4 + 2) - 30f(i_1, i_2, i_3, i_4 + 2) + 16f(i_1, i_2 + 1, i_3, i_4 + 2) - 2f(i_1, i_2 + 2, i_3, i_4 + 2) \right)

(13.2)

and
\[
\frac{\partial^2 f}{\partial z^2}
\bigg|_{(x_1, i_2, i_3, i_4 + \frac{1}{2})} \approx
\frac{1}{16} \left( -2 f(i_1, i_2, i_3 - 2, i_4) + 16 f(i_1, i_2, i_3, i_4 - 1) - 30 f(i_1, i_2, i_3, i_4) + 16 f(i_1, i_2, i_3 + 1, i_4 - 1) - 2 f(i_1, i_2, i_3 + 2, i_4 - 1) \right)
\]

\[
+ \frac{9}{16} \left( -2 f(i_1, i_2, i_3, i_4 - 1) + 16 f(i_1, i_2, i_3, i_4) - 30 f(i_1, i_2, i_3, i_4 + 1) + 16 f(i_1, i_2, i_3 + 1, i_4) - 2 f(i_1, i_2, i_3 + 2, i_4) \right)
\]

\[
+ \frac{9}{16} \left( -2 f(i_1, i_2, i_3, i_4 + 1) + 16 f(i_1, i_2, i_3, i_4 + 2) - 30 f(i_1, i_2, i_3 + 1, i_4 + 1) + 16 f(i_1, i_2, i_3 + 2, i_4 + 1) \right)
\]

\[
- \frac{1}{16} \left( -2 f(i_1, i_2, i_3 - 2, i_4 + 2) + 16 f(i_1, i_2, i_3, i_4 + 2) - 30 f(i_1, i_2, i_3 + 1, i_4 + 2) + 16 f(i_1, i_2, i_3 + 2, i_4 + 2) \right)
\]

\[
\left(13.3\right)
\]

\[
\frac{\partial f}{\partial t}
\bigg|_{(x_1, i_2, i_3, i_4 + \frac{1}{2})} \approx \frac{-f(i_1, i_2, i_3, i_4 - \frac{1}{2}) + 8 f(i_1, i_2, i_3, i_4) - 8 f(i_1, i_2, i_3, i_4 + 1) + f(i_1, i_2, i_3, i_4 + \frac{3}{2})}{6 \tau}
\]

\[
\left(13.4\right)
\]

Replacing \( f(i_1, i_2, i_3, i_4 - \frac{1}{2}) \) by \( \frac{f(i_1, i_2, i_3, i_4 - 1) + f(i_1, i_2, i_3, i_4)}{2} \)

and

\[
f(i_1, i_2, i_3, i_4 + \frac{3}{2}) \text{ by } \frac{f(i_1, i_2, i_3, i_4 + 1) + f(i_1, i_2, i_3, i_4 + 2)}{2}
\]

one finds:

\[
\frac{\partial f}{\partial t}
\bigg|_{(x_1, i_2, i_3, i_4 + \frac{1}{2})} \approx \frac{-f(i_1, i_2, i_3, i_4 - 1) + 15 f(i_1, i_2, i_3, i_4) - 15 f(i_1, i_2, i_3, i_4 + 1) + f(i_1, i_2, i_3, i_4 + 2)}{12 \tau}
\]

\[
\left(13.5\right)
\]

Finally our numerical scheme becomes
The solution of Equations (7) and (14) can be achieved by a Gauss-Seidel iterative scheme as you can see below for (14). The same Gauss-Seidel iterative scheme can be produced for (7) very easily, since (7) can be dealt as a partial case of (14).

So, for the iterative solution of (14), we obtain:

\[
\begin{align*}
&\frac{1}{16} \left[ -2 f(i_1 - 2, i_2, i_3, i_4 - 1) + 16 f(i_1 - 1, i_2, i_3, i_4 - 1) - 30 f(i_1, i_2, i_3, i_4 - 1) + 16 f(i_1 + 1, i_2, i_3, i_4 - 1) - 2 f(i_1 + 2, i_2, i_3, i_4 - 1) \right] \\
&\quad + \frac{9}{16} \left[ -2 f(i_1 - 2, i_2, i_3, i_4) + 16 f(i_1 - 1, i_2, i_3, i_4) - 30 f(i_1, i_2, i_3, i_4) + 16 f(i_1 + 1, i_2, i_3, i_4) - 2 f(i_1 + 2, i_2, i_3, i_4) \right] \\
&\quad + \frac{9}{16} \left[ -2 f(i_1 - 2, i_2, i_3, i_4 + 1) + 16 f(i_1 - 1, i_2, i_3, i_4 + 1) - 30 f(i_1, i_2, i_3, i_4 + 1) + 16 f(i_1 + 1, i_2, i_3, i_4 + 1) - 2 f(i_1 + 2, i_2, i_3, i_4 + 1) \right] \\
&\quad + \frac{9}{16} \left[ -2 f(i_1 - 2, i_2, i_3, i_4 + 2) + 16 f(i_1 - 1, i_2, i_3, i_4 + 2) - 30 f(i_1, i_2, i_3, i_4 + 2) + 16 f(i_1 + 1, i_2, i_3, i_4 + 2) - 2 f(i_1 + 2, i_2, i_3, i_4 + 2) \right] \\
&\quad + \frac{1}{16} \left[ -2 f(i_1, i_2, i_3 - 2, i_4) + 16 f(i_1, i_2, i_3 - 1, i_4) - 30 f(i_1, i_2, i_3, i_4) + 16 f(i_1, i_2, i_3 + 1, i_4) - 2 f(i_1, i_2, i_3 + 2, i_4) \right] \\
&\quad + \frac{9}{16} \left[ -2 f(i_1, i_2, i_3, i_4 - 1) + 16 f(i_1, i_2, i_3, i_4) - 30 f(i_1, i_2, i_3, i_4 + 1) + 16 f(i_1, i_2, i_3, i_4 + 2) - 2 f(i_1, i_2, i_3, i_4 + 3) \right] \\
&\quad + \frac{9}{16} \left[ -2 f(i_1, i_2, i_3, i_4 + 1) + 16 f(i_1, i_2, i_3, i_4 + 1) - 30 f(i_1, i_2, i_3, i_4 + 2) + 16 f(i_1, i_2, i_3, i_4 + 3) - 2 f(i_1, i_2, i_3, i_4 + 4) \right] \\
&\quad + \frac{1}{16} \left[ -2 f(i_1, i_2, i_3, i_4 + 2) + 16 f(i_1, i_2, i_3, i_4 + 2) - 30 f(i_1, i_2, i_3, i_4 + 3) + 16 f(i_1, i_2, i_3, i_4 + 4) - 2 f(i_1, i_2, i_3, i_4 + 5) \right] \\
&\quad + \frac{9}{16} \left[ -2 f(i_1, i_2, i_3, i_4 + 3) + 16 f(i_1, i_2, i_3, i_4 + 3) - 30 f(i_1, i_2, i_3, i_4 + 4) + 16 f(i_1, i_2, i_3, i_4 + 5) - 2 f(i_1, i_2, i_3, i_4 + 6) \right] \\
&\quad + \frac{9}{16} \left[ -2 f(i_1, i_2, i_3, i_4 + 4) + 16 f(i_1, i_2, i_3, i_4 + 4) - 30 f(i_1, i_2, i_3, i_4 + 5) + 16 f(i_1, i_2, i_3, i_4 + 6) - 2 f(i_1, i_2, i_3, i_4 + 7) \right] \\
&\quad + \frac{1}{16} \left[ -2 f(i_1, i_2, i_3, i_4 + 5) + 16 f(i_1, i_2, i_3, i_4 + 5) - 30 f(i_1, i_2, i_3, i_4 + 6) + 16 f(i_1, i_2, i_3, i_4 + 7) - 2 f(i_1, i_2, i_3, i_4 + 8) \right] \\
&\quad + \frac{9}{16} \left[ -2 f(i_1, i_2, i_3, i_4 + 6) + 16 f(i_1, i_2, i_3, i_4 + 6) - 30 f(i_1, i_2, i_3, i_4 + 7) + 16 f(i_1, i_2, i_3, i_4 + 8) - 2 f(i_1, i_2, i_3, i_4 + 9) \right] \\
&\quad + \frac{1}{16} \left[ -2 f(i_1, i_2, i_3, i_4 + 7) + 16 f(i_1, i_2, i_3, i_4 + 7) - 30 f(i_1, i_2, i_3, i_4 + 8) + 16 f(i_1, i_2, i_3, i_4 + 9) - 2 f(i_1, i_2, i_3, i_4 + 10) \right] \\
&\quad + \frac{9}{16} \left[ -2 f(i_1, i_2, i_3, i_4 + 8) + 16 f(i_1, i_2, i_3, i_4 + 8) - 30 f(i_1, i_2, i_3, i_4 + 9) + 16 f(i_1, i_2, i_3, i_4 + 10) - 2 f(i_1, i_2, i_3, i_4 + 11) \right] \\
&\quad + \frac{1}{16} \left[ -2 f(i_1, i_2, i_3, i_4 + 9) + 16 f(i_1, i_2, i_3, i_4 + 9) - 30 f(i_1, i_2, i_3, i_4 + 10) + 16 f(i_1, i_2, i_3, i_4 + 11) - 2 f(i_1, i_2, i_3, i_4 + 12) \right] \\
&= k \left[ f(i_1, i_2, i_3, i_4 - 1) + 15 f(i_1, i_2, i_3, i_4 + 1) + f(i_1, i_2, i_3, i_4 + 2) \right] - \frac{15 f(i_1, i_2, i_3, i_4 + 1) + f(i_1, i_2, i_3, i_4 + 2)}{12r} \quad (14)
\end{align*}
\]
\[ f(t_i, t_2, t_3, t_4)\rceil_{n+1}^{n} = \frac{-4\tau}{5k} B^n \]

Where

\[ B = -\frac{1}{16} \left( -2f(t_i - 2, t_2, t_3, t_4 - 1) + 16f(t_i - 1, t_2, t_3, t_4 - 1) - 30f(t_i, t_2, t_3, t_4 - 1) + 16f(t_i + 1, t_2, t_3, t_4 - 1) - 2f(t_i + 2, t_2, t_3, t_4 - 1) \right) \]

\[ + \frac{9}{16} \left( -2f(t_i - 2, t_2, t_3, t_4) + 16f(t_i - 1, t_2, t_3, t_4) - 30f(t_i, t_2, t_3, t_4) + 16f(t_i + 1, t_2, t_3, t_4) - 2f(t_i + 2, t_2, t_3, t_4) \right) \]

\[ + \frac{9}{16} \left( -2f(t_i - 2, t_2, t_3, t_4 + 1) + 16f(t_i - 1, t_2, t_3, t_4 + 1) - 30f(t_i, t_2, t_3, t_4 + 1) + 16f(t_i + 1, t_2, t_3, t_4 + 1) - 2f(t_i + 2, t_2, t_3, t_4 + 1) \right) \]

\[ + \frac{1}{16} \left( -2f(t_i - 1, t_2, t_3, t_4 + 2) + 16f(t_i, t_2, t_3, t_4 + 2) - 30f(t_i, t_2, t_3, t_4 + 2) + 16f(t_i + 1, t_2, t_3, t_4 + 2) - 2f(t_i + 2, t_2, t_3, t_4 + 2) \right) \]
4 Conclusion
In this paper, we presented an extension of the Crank-Nicholson method for solving parabolic equations (1-D and 3-D conduction equation). The method uses finite differences. For the derivative of the variable of time, we use central difference at 4 points, while for the second-order derivatives of the other spatial variables we use Lagrangian interpolation at 4 points of the central differences at 5 points. Some other relevant studies can be found in [3], [4], [5] and [6]. Finally, the solution of the Equations (7) and (14) can be achieved by a Gauss-Seidel iterative scheme.

References