An Interpolative Reasoning Method Based on Hedge Algebras and Its Application to a Problem of Fuzzy Control

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Abstract: The paper aims to show an applicability of the algebraic approach introduced in [7,9,11,6] to solving a problem of fuzzy control. In this approach, linearly ordered term-domains of linguistic variables can be considered as linear hedges algebras (LinHAs). It is interesting that in LinHAs, a formal notion of fuzziness measure of linguistic hedges and terms can be defined in a reasonable way, based on the closeness of term meaning expressed in these algebras. Based on its nature, a notion of semantically quantifying mappings (SQMs) can be introduced to solve the problem of an inverted pendulum. It is shown that the obtained results are much better than that produced by ordinary fuzzy method.

Key-words: fuzzy control, hedge algebras, fuzziness measure, semantically quantifying mapping, interpolative reasoning method, hedge algebras-based controller, inverted pendulum system

1 Introduction

It is well known that human language, which includes vague terms, is a powerful tool for modelling reality in its own way. E.g. in fuzzy control one can use it to describe a dependency between physical variables $X_i$ and $\gamma$ as follows:

If $X_i = A_{i1}$ and ... and $X_m = A_{im}$ then $\gamma = B_i$

(1)

$X_i = A_{i1}$ and ... and $X_m = A_{im}$ then $\gamma = B_n$

where $A_{ji}, j = 1, ..., m,$ and $B_i, i = 1, ..., n,$ are verbal descriptions of variables $X_i$ and $\gamma,$ respectively and the notation $X_i = A_{i1}$ (or $\gamma = B_i$) is an abbreviation of the sentence “$X_i$ is $A_{i1}$” (or, “$\gamma$ is $B_i$”). We may call (1) a fuzzy model representing expert knowledge.

For a given fuzzy model (1), an approximate reasoning problem saying that find an output which is corresponding to a given input $X_i = A_{i0}$ is called a fuzzy multiple conditional reasoning (FMCR) problem [17,13,11,12]. To solve this problem in fuzzy sets approach, many tasks have to be solved:

1) To model the semantics of terms in (1) by constructing appropriate fuzzy sets.

2) To find an aggregation operator [15] to aggregate the constructed fuzzy sets to obtain $A_i$ representing the semantics of each if-part of the sentences in (1) appropriately.

3) To define a fuzzy relation $R_{bij}$ on $U_j \times ... \times U_p \times U_i \times V$, using a suitable implication operator, for each pair $(A_i, B_j)$, where $A_i$ is obtained in 2).

4) To find an aggregation operator to aggregate the obtained fuzzy relations $R_{bij}$, $i = 1, ..., n$, so that the obtained fuzzy relation $R$ can represent the fuzzy model (1), suitably.

5) To determine a composition operator (a composition rule) to compute the output fuzzy set.

6) To find a suitable defuzzification method to transform an output fuzzy set into a real value.

The construction an appropriate FMCR method is a complicated problem, because it depends on several very big problems pointed above. The difficulties may come from the following reasons:

First, in order to simulate a human reasoning process one has to embed a finite term-domain of a linguistic variable into the set of all functions $F(U,[0,1])$ defined on a universe $U$, that has a rich computation structure. On the algebraic viewpoint, the way we use the whole infinite structure of $F(U,[0,1])$ to model finite term-domains of linguistic variables is unreasonable.

Second, it is easy to observe that one can compare meanings of linguistic terms, i.e. one can discover a semantics-based ordering relation on a term-domain, e.g. it is clear that true $\geq$ false, very true $\geq$ more true and approx.false $\geq$ false ... . However, the embedding mentioned above does not preserve this discovered ordering relation.

In [6,7,9,10], an algebraic approach to the structure of term-domains of linguistic variables has been established and it creates a sufficient algebraic basic to study some logics to develop
approximate reasoning methods [5,8]. Then, each term-domain \( X \) of \( \mathcal{X} \) can be considered as a hedge algebra \( \mathcal{A} \mathcal{X} = (X, C, H, \leq) \) which will be presented below. In linearly ordered term-domains of \( X \), we may define formal notions of fuzziness measure of terms and of linguistic hedges and of semantically quantifying mappings (SQMs), which embed term-domains into the corresponding universe of reference. A wide class of SQMs can be defined by an expression with parameters being fuzziness measure \( fm(c^+) \) and \( fm(c^-) \) of primary terms and of hedges \( \mu(h), h \in H \).

Now, (1) can be modelled as follows:

Since terms of linguistic variable \( \mathcal{X} \) and \( Y \) can be considered as elements of a LinHA \( \mathcal{X} \mathcal{Y} = (X, C, H, \leq) \) and \( \mathcal{Y} = (Y, C, H, \leq) \), respectively, each sentence of (1) can be considered as a point and, therefore, (1) defines a linguistic hyper-surface in \( \mathcal{X} \times \ldots \times \mathcal{X} \times Y \). Then, for each linguistic variable \( \mathcal{X} \) and variable \( Y \), we construct SQMs \( u \) and \( \nu \) which assigns to terms of \( \mathcal{X} \) and \( Y \) real values of the reference universes \( U_L \) and \( V \), respectively. Then, each sentence of (1) can be considered as representing a point in \( U_L \times \ldots \times U_L \times V \). So, the expert knowledge (1) now can be represented as a real hyper-surface of \( m+1 \) dimension. For \( m = 1 \), we have a real curve.

2 Hedging Algebras of a Linguistic Variable: A Short Overview

Hedge algebras form a completely new approach to the semantics of term-domains, which many readers are not familiar with. Therefore, in this section we shall give a short overview on it.

2.1 Algebraic Structure of Term-Domains

One of the original reasons to introduce and investigate hedge algebras (HAs) ([7,9,10]) is that term-domains of linguistic variables have their own semantics-based order structure. For example, vager terms of the linguistic variable SPEED can be expressed as follows: fast > slow, very fast > fast, very slow < slow. So, we have a new viewpoint: terms-domains of a linguistic variable can be modelled by a poset (partially ordered set) - a basic mathematical structure. A question now is how we can discover this structure?

In fuzzy control as well as in many scientific fields, one uses verbal descriptions (i.e. linguistic terms) to describe a dependence of a physical variable on another one. We denote by \( X \) a set of terms of a linguistic physical variable \( x \), called a term-domain of \( X \). For example, if \( X \) is the rotation speed of an electrical motor and linguistic hedges used to describe this speed are \( \text{Very}, \text{More}, \text{Possibly}, \text{Little} \), denoted correspondingly for short by \( V, M, P \) and \( L \), then \( X = \{ \text{fast}, V \text{fast}, M \text{fast}, L \text{fast}, P \text{fast}, L \text{fast}, P \text{fast}, L \text{slow}, P \text{slow}, L \text{slow}, V \text{slow}, \ldots \} \cup \{0, W, 1 \} \) is a term-domain of \( X \). It can be considered as an abstract algebra \( \mathcal{X} = (X, C, H, \leq) \), where \( H \) is a set of linguistic hedges, which can be regarded as one-argument operations, \( \leq \) is called a semantic ordering relation on \( X \) and \( C = \{ \text{fast}, \text{slow}, W, 0, 1 \} \) is a set of constants in \( X \) with \( \text{fast} \) and \( \text{slow} \) being primary terms of \( X \) and \( W, 0, 1 \) being interpreted as the neutral, the least and the greatest ones, respectively. Denote by \( hx \) a result of applying an \( h \in H \) to \( x \in X \) and by \( H(x) \) the set of all \( u \in X \) generated algebraically from \( x \) by using hedges in \( H \). So, every element \( u \in X \) can be written in the form \( u = h_0, h_1, h_1' \ldots, h_n \in H \).

Now, we shall discover an inherent mathematical structure of these term-domains. As pointed out in [7], this structure can be axiomatized, based on semantic properties of term-domains that can be expressed in terms of the relation \( \leq \) of \( X \).

A term-domain can be ordered based on the following observations (see in [9,10]):

1) Each primary term has an intuitively semantic tendency which can be recognized by order relationships: fast has a tendency of “going up”, called positive one, and it is denoted by \( c^+ \), while \( \text{slow} \) has a tendency of “going down”, called negative one, denoted by \( c^- \). They can be characterized by \( Vc^+ > c^- \) or \( Vc^- < c^- \).

2) Further, each hedge has an intuitive semantic tendency, which can be expressed in terms of the semantically ordering relation. It can be seen that the one hedges of \( X \) increase the semantic tendency of the primary terms (called positive hedges), while the other ones (called negative hedges) decrease it. For example, \( V \) is positive with respect to (w.r.t.) the semantic tendency of the primary terms, while \( L \) has a reverse effect and hence it is negative. Denote by \( H^- \) the set of all negative hedges and by \( H^+ \) the set of all positive ones of \( X \).
If both hedges $h$ and $k$ do not belong to the same $H^+$ or $H^-$, then they have a reverse effect and hence they are said to be **converse**. Conversely, they are said to be **compatible**. In the latter case it may happen that one hedges change the terms more strongly than the others. For example, $L$ and $P$ are compatible and $L > P$, since $L_{false} > P_{false} > false$. Note that $I < P$ and $I < M$, where $I$ is the identity on $X$, i.e. for any term $x$ in $X$, $I_{x} = x$, and it is considered as an artificial hedge. But, it is obvious that $L$ and $V$ are incompatible, i.e. they are converse!

3) Further, we observe that each hedge has an effect, which either increases or decreases the semantic effect of $h$. For example, consider linguistic hedges $V$ (Very), $M$ (More), $L$ (Little), $P$ (Possibly), $A$ (Approximately) and $ML$ (More or Less) of the variable TRUTH. Since the semantic effect of $L$ is expressed by $L_{true} < true$, it follows from $VL_{true} < L_{true} < PL_{true}$. So, each hedge has a positive w.r.t. $L$. Therefore, we can always build a table of the $PN$-property of hedges, which is read from $V$ to $L$. And, if $x = x_{i}$, then $h_{x}$ inherits the semantic ordering relationship between $hx$ and $kx$. So, as a consequence, we have $H(hx) \leq H(kx)$. For example, from $Ltrue \leq Ptrue$ it follows that $PLtrue \leq Ltrue$, or more generally, $H(Ltrue) \leq H(Ptrue)$.

Based on these observations, the domain of the variable **SPEED** of a motor considered above can be ordered as follows: $V < M < slow < P < slow > L < slow > L < fast < L < P < fast < P < fast < M < fast < V < fast$.

Formally, as proved in [9,10], each linguistic domain $X$ of $x$ can be axiomatized to become a rich mathematical structure for modelling the semantics of $X$. It is called a hedge algebra (HA) and shown to be a complete lattice with unit and zero elements $1$, $0$ under assumption that $H^+ + I$ and $H^+ - I$ are lattices of hedges. In particular, we have:

**Theorem 2.1** ([9]): Let $\mathcal{AX} = (X, C, H, \Sigma, \Phi, \leq)$ be an HA. Then, the following statements hold:

(i) If $x \in X$ is a fixed point of an $h$ in $H$, i.e. $hx = x$, then it is also a fixed point of the other ones.

(ii) If $x = h_{n}...h_{i}u$, then there exists an index $i$ such that the suffix $h_{n}...h_{i}u$ of $x$ is a canonical representation of $x$ w.r.t. $u$ (that is $x = h_{i}h_{j}...h_{i}u$ and $h_{i}h_{j}...h_{i}u \neq h_{j}...h_{i}u$) and $h_{j}x = x$, for all $j > i$.

(iii) If $h \neq k$ and $hx = kx$ then $x$ is a fixed point.

For convenience in the sequel, we recall here the criteria for comparing any two elements in $X$:

**Theorem 2.2** ([9]) Let $x = h_{n}...h_{i}u$ and $y = k_{m}...k_{j}u$ be two canonical representations of $x$ and $y$ w.r.t. $u$, respectively. Then there exists an index $j \leq \min\{m,n\}+1$ (here as a convention it is understood that if $j = \min\{m,n\}+1$, then either $h_{j} = 1$ for $j = n+1 \leq m$ or $k_{j} = 1$ for $j = m+1 \leq n$) such that $h_{j} = k_{j}$, for all $j \leq j$ and

1. $x \equiv y$ iff $m = n$ and $h_{x}y = k_{x}j$;
2. $x \equiv y$ iff $h_{x}y < k_{x}j$;
3. $x$ and $y$ are incomparable iff $h_{x}y$ and $k_{x}j$ are incomparable.

In this paper we restrict ourselves to consider only linear hedge algebras. A hedge algebra $\mathcal{AX}$ is said to be linear if $H^+$ and $H^+$ of $H$ and $C$ are linearly ordered. For studying fuzziness measure of terms, we need a notion of linear complete hedge algebras (LComHAs): an abstract algebra $\mathcal{AX} = (X, C, H, \Sigma, \Phi, \leq)$ is said to be a linear complete HA if $\mathcal{AX} = (X, C, H, \Sigma, \Phi, \leq)$, where $X = H(C)$, is a linear HA and $\Sigma$ and $\Phi$ are additional unary operations, the semantic of which is expressed by $\Phi(x) = \inf H(x)$ and $\Sigma(x) = \sup H(x)$, where infimum and supremum are taken in the poset $X$. Denote by $\text{Lim}(X) = X \setminus X$ and $H_{r} = H \cup \{X, \Sigma, \Phi\}$

For these algebras we have:

**Theorem 2.3.** (Th.4 [9]) Let $\mathcal{AX} = (X, C, H, \Sigma, \Phi, \leq)$ be an LComHA. Then, we have:

(i) For every $u \in X$, $H(u)$ is a linearly ordered set;
(ii) The underlying set $X$ is linearly ordered.

Moreover, if $u \leq v$ and $u$ and $v$ are independent, i.e. $u \not\in H(v)$ and $v \not\in H(u)$, then $H(u) \leq H(v)$.

### 2.2 Fuzziness Measure and Quantification of Linear Hedge Algebras

In this section, it will be shown that HAs provide a formal basis to define a notion of fuzziness
measure of terms and of hedges. Since hedges change meaning of terms a bit only, there is a semantic closeness between terms induced by using hedges. In the same time, it is observed that fuzziness of terms is also a concept of semantic closeness of terms. So, there exists a closed relation between semantic closeness and fuzziness of terms.

Let \( \mathcal{A}_X = (X, C, H, \Sigma, \Phi, \leq) \) be an LComHA which model a linguistic domain \( X \) of a variable \( x \). Semantically, the set \( H(x) \) consists of all terms, each of which still reflects an essential meaning of a term \( x \). For example, consider two terms \( x = \text{PT}rue \) and \( y = \text{AT}rue \). The term \( \text{VP}True \) reflects a definite meaning of \( \text{PT}rue \) but not of \( \text{AT}rue \), while the term \( \text{VAT}rue \) reflects a definite meaning of \( \text{AT}rue \), but not of \( \text{PT}rue \). Therefore, the set \( H(x) \) consisting of all terms, which are semantically closed to \( x \), should express certain characteristics of fuzziness of the term \( x \). Hence, we may use this set to model qualitatively fuzziness of \( x \). It suggests us to examine the family \( \vartheta = \{H(x); x \in H(G)\} \). This observation suggests us to use the “size” of the set \( H(x) \) to measure the fuzziness degree of \( x \). We can realize this idea as follows:

In the framework of fuzzy sets theory, each defuzzification method defines a mapping from a set of terms \( X \) into a real interval \([a,b]\) or \([0,1]\), for normalization. This mapping can be considered as a semantically quantifying mapping (SQM), which will be presented in details in the next section. Now, we take these mappings in mind to determine a notion of fuzziness measure. Let us consider a mapping \( f \) from \( X \) into \([0,1]\), which preserves the ordering relation on \( X \). Then, the “size” of the set \( H(x) \), for \( x \in X \), can be defined by the diameter of \( f(H(x)) \), a subset of \([0,1]\), i.e. this diameter will be considered as fuzziness measure of the term \( x \).

Taking this model of fuzziness measure in mind, we may adopt the following definition:

**Definition 2.1** (see Fig.1) Let \( \mathcal{A}_X = (X, C, H, \Sigma, \Phi, \leq) \) be a linear ComHA. An \( fm: X \rightarrow [0,1] \) is said to be a fuzziness measure of terms in \( X \) if:

- \( fm(\text{c}) + fm(\text{c}') = 1 \)
- \( \sum_{h \in H} fm(hu) = fm(u) \) for \( \forall u \in X \).

For example, consider two terms \( a \) and \( b \) in \( X \). Then, the following properties hold:

1. \( fm(a) + fm(b) = 1 \)
2. \( fm(a) \geq fm(b) \) for \( a \geq b \)
3. \( fm(a) = fm(b) \) if and only if \( a = b \)
4. \( fm(a) = 0 \) if \( a \) is very false
5. \( fm(a) = 1 \) if \( a \) is very true

Especially, \( fm(0) = fm(\text{False}) = fm(1) = 0 \). It is intuitively evident. Theorem 2.4 describes the properties of a fuzziness measure.

The conditions \( fm(1) \) and \( fm(2) \) is intuitively evident. \( fm(3) \) seems also natural: applying a hedge \( h \) to different terms, the relative modification effect of \( h \) is the same, i.e. this proportion does not depend on terms \( h \) applies to.

Fuzziness measure \( H \) on the family \( X \) has the following properties, where \( H = \{h_1, ..., h_p\} \) with \( h_1 < h_2 < ... < h_p \), \( H' = \{h_1, ..., h_p\} \) with \( h_1 < ... < h_p \) and \( h_0 = I \).

**Proposition 2.1** For each fuzziness measure \( fm \) the following statements hold:

1. \( fm(hx) = \mu(h)fm(x) \) for every \( x \in X \);
2. \( fm(hx) = fm(x) \) for \( \forall h \in H \)
3. \( \sum_{q \in C, h \in H} fm(hx) = fm(x) \), where \( \mu(x) = \mu(h) \)
4. \( \sum_{q \in C, h \in H} fm(hx) = fm(x) \)
5. \( \sum_{q \in C, h \in H} \mu(h) = \beta \)

Now, we have the following:

**Theorem 2.4** Let a fuzziness measure \( \mu \) of hedges be given such that it satisfies the inequalities in 5) of Prop.2.1. Let \( fm(c) \) and \( fm(c') \) be such that \( fm(c) > 0, fm(c') > 0 \) and \( fm(c) + fm(c') = 1 \). Then, the mapping \( fm \) defined recursively by the equations \( fm(z) = fm(hx) = \mu(h)fm(x) \) for \( z = h \), and \( fm(z) = 0 \), \( z \in \{0, W, I\} \), is a fuzziness measure on \( X \) which is freely generated from the set of constants \( C \).

### 2.3 Semantically Quantifying Mappings of Linguistic Variables

On account of the above examination, we have a reasonable way to construct SQMs on \( X \).

**Definition 2.2** (Sign function) The function \( \text{Sign}: X \rightarrow \{-1, 0, 1\} \) is a mapping defined recursively as follows, where \( h, h' \in H \) and \( c \in \{c', c''\} \):

a) \( \text{Sign}(c') = -1 \), \( \text{Sign}(c'') = 1 \)

b) \( \text{Sign}(h'x) = \text{Sign}(hx) \) if \( h'x \neq hx \) and \( h' \) is negative w.r.t. \( h \) (or w.r.t. \( c \), if \( h = h' \) w.r.t. \( x = c \))

c) \( \text{Sign}(h'x) = \text{Sign}(hx) \) if \( h'x \neq hx \) and \( h' \) is positive w.r.t. \( h \) (or w.r.t. \( c \), if \( h = h' \) w.r.t. \( x = c \))

d) \( \text{Sign}(h'x) = 0 \) if \( h'x = hx \).

**Proposition 2.2** For any \( h \) and \( x \), if \( \text{Sign}(hx) = +1 \) then \( hx > x \), and if \( \text{Sign}(hx) = -1 \) then \( hx < x \).

**Definition 2.3** Let \( fm \) be a fuzziness measure on \( X \). A semantically quantifying mapping (SQM) \( \nu \) on \( X \) (associated with \( fm \)) is defined as follows:

1. \( \nu(W) = \theta = fm(c'), \nu(c') = \theta - \alpha fm(c') \), \( \nu(c') \)

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Under the mappings mentioned above can be considered as “a linguistic hyper-surface” w.r.t. C. Then, the FMCR problem consists of the following steps:

1) Construct SQMs v_X and v_Y which map X and Y into the interval [0,1], respectively, j = 1, ..., m. They are computed by Def.2.3 and Th.2.4, based on the parameters fm(c⁺), μ(h), H, and θ.

2) Under the mappings v_X and v_Y, the hyper-surface C_L is transformed into a real one C_{m+1}, of (m+1)-dimension defined in [0,1]×⋯×[0,1].

3) Now, we choose an aggregation operator Agg [15] to aggregate the first m components of each point of C_{m+1}, and obtain C_2, in [0,1]×[0,1].

4) Apply a classical interpolative method (INTMd) to the curve C_2, to compute the output.

Our method has some advantages.

(i) For each application, it is able to define suitably fuzziness degree of terms and of hedges. Because SQMs are defined by an expression with parameters, it allows us more easily to find out appropriate SQMs by regulating their parameters;

(ii) The mathematical model of the fuzzy model (1) is very suggestive and it allows solve FMCR problems, using classical INTMs. It is very suggestive, simple and produces accurate results.

(iii) Defuzzification is not necessary and, because SQMs are one-to-one, linguistic approximation problems become simple.

Note that fuzziness measure of hedges and primary terms, and especially the real value θ of the neutral element, which is chosen arbitrarily in [0,1], considered as the parameters will make the method also flexible to adapt specific applications.

One important thing it is worth to emphasize is that the method is rather similar to that of linguistic physical variable C_L, a geographically suggestive mathematical object, allows modelling faithfully the relationship between physical linguistic variables given in (1), in particular, in the case of one input variable, C_L is a fuzzy curve, the shape of the real curve C_{m+1} is rather similar to that of C_L. It will give useful foundation to develop new methods in solving fuzzy control problems as will be shown in the next section.

4 An Algorithm of Control Based on Hedge Algebras and an Application

Let us consider a fuzzy model given in (1) which in fuzzy control is called a Fuzzy Associative Memory (FAM). Since there are m input variables, we call (1) an FAM of m-dimension.

Based on the new interpolative method presented in Section 3, we introduce a general fuzzy control model based on the theory of hedge algebras, called hedge algebra-based controller (HAC). Figure 2 shows a general schema of HAC, where r is a reference, ε is an error and u is a control action and P is a plant.

The control algorithm in an uncertainty environment for HAC consists of the following main steps:

Step1: Quantitative Semantization. Note that the basic knowledge of an application is given in the form of FAM (1) containing vague terms in term-domain X_j of linguistic physical variable X_j, whose reference domain is an interval [s_j,s_{j+1}], j = 1, ..., m. However, the quantitatively semantic values, which are assigned to the terms of each X_j in a terms-domain X_j under an SQM v_j, are taken...
in [0,1]. The values of the mapping \( y_i \) are called semantic values and their variable is called semantic variable denoted by \( x_{ij} \). By a linear transformation, the reference domain \([s_{ij}, s_{i2}]\) of \( x_i \) will be embedded in the semantic domain \([0,1]\). The process, which transforms linguistic values of \( x_i \) to their quantified meaning, is called a quantitative semantization vs. fuzzification process (see Figure 6-8).

![Figure 2: Hedge Algebra-Based Controller](image)

The key-task of the process is to define the parameters, which are fuzziness measures of primary terms and linguistic hedges for each variable \( x_i \) appropriately, based on an analysing the semantics of term-domains. For example, the parameters of the variable SPEED will be changed from cars to trains. In the case of the variable AGE, we can assume that the parameter \( \mu(young) \) for the staffs of a university is greater than that of the citizens. Or, since Very and Little are more specific than More and Possibly, we can assume that \( \mu(\text{More}) > \mu(\text{Very}) \) and/or \( \mu(\text{Possibly}) > \mu(\text{Little}) \). Of course, these constraints are soft and the parameters can still be adjusted in the next step.

**Step 2: Semantics Associative Memory and Reasoning Mechanism.** Using SQMs defined in Step 1, transform FAM into an \( m \)-dimensional table of numerical data, called \( m \)-Semantics Associative Memory (m-SAM). Note that \( n \) cells of the \( m \)-SAM define \( n \) points will model a real supper-surface \( C_{r,n} \) in \( R^{m+1} \). However, the idea is that it is better to choose an aggregation operator \( \text{Agg} \) to integrate \( m \) components of \( m \)-SAM to construct a table called \( 2 \)-SAM. Then, \( n \) cells of the obtained \( 2 \)-SAM define \( n \) points which will model a real curve \( C_{r,2} \) in \( R^2 \). However, these cells may define a multi-valued function and, hence, there two possibilities to solve it: (i) We can take the average of the values of this multi-valued function at each its argument value to transform it into a (single) function; (ii) We can regulate the parameters of SQMs defined in Step 1 so that the obtained \( m \)-SAM will define a function.

Determine a classical interpolative method to solve a classical interpolative problem w.r.t. \( C_{r,2} \) to compute real outputs of the fuzzy model (1).

**Step 3: Desemantisation.**

It is simply a mapping which assigns each semantic value, i.e. a real value in [0,1], to a real value of the control action domain.

So, it can be seen that we have now a basic to believe that the proposed algorithm for fuzzy control is simpler, more effective than the fuzzy one. The reasons are:

1) Instead of solving the membership problem, it is necessary to define only the parameters of SQMs based on Step 1.

2) Reasoning method based on classical interpolative methods w.r.t. the obtained real curve seems to be very simple, suggestive and will produce more exact output results.

3) It is very flexible since one can easily make the method to adapt the application by regulating the parameters of the constructed SQMs.

4) Defuzzification methods are unnecessary.

5) It allows avoid hard problems arising in fuzzy sets based approach such as the membership function, the fuzzy implication, the composition rule, the defuzzification, and so on.

Below, we shall apply our method to solve the inverted pendulum problem examined in [13]. It is one of classical problems, which is considered as an interesting one in the study of nonlinear systems for many years. In [13] a fuzzy sets based fuzzy control method to solve this inverted pendulum problem was presented. The control method is to regulate the pendulum to reach its stable state. The differential equation describing a simplified version of the inverted pendulum system examined in [13] is given below:

\[ -ml^2\frac{d^2\psi}{dt^2} + mlg \sin \psi = u(t) \quad (2) \]

where \( m \) is the mass of the pole located at the tip point of the pendulum; \( l \) is the length of the pendulum; \( \psi \) is the deviation angle from vertical in the clockwise direction; \( u(t) \), the control action, is the torque applied to the pole in the counter clockwise direction; \( t \) is time; \( g \) is the gravitational acceleration constant.

Assuming that \( x_1 = \psi \) and \( x_2 = \frac{d\psi}{dt} \) are state variables. By a simple transformation given in [13], we obtain linearized and discrete state-space equations in the form of matrix difference equation as follows:

\[ x_1(k+1) = x_1(k) + x_2(k) \quad (3) \]
\[ x_2(k+1) = x_1(k) + x_2(k) - u(k) \quad (4) \]
It is assumed that the universes of discourse for the two variables are: 

\[ -2^\circ \leq x_1(k) \leq 2^\circ \; ; \; -5 \text{ dps} \leq x_2(k) \leq 5 \text{ dps} \]

the same as in [13], and the universe of discourse for the control action is 

\[ -16 \text{ mA} \leq u(k) \leq 16 \text{ mA} \]

The goal of the controller design is to seek a control signal \( u \) that will keep the inverted pendulum just in or closely to the vertical stable position (i.e. it is defined by \( e = 0 \) and \( \Delta e = 0 \)).

In [13], a FAM for designing the fuzzy controller is formulated in Table 2. It can be seen that the linguistic labels in the FAM Table are not compatible with linguistic domain modelled by a hedge algebra, since \( P \) and \( N \) are better considered as linguistic hedges, e.g. in FAMs occur often terms like “Positive Big”, “Positive Small”, “Negative Big” and “Negative Small”, but they are suitable like “Positive Big”, “Positive Small”, “Negative Big” and “Negative Small”. Note that in the fuzzy sets approach linguistic labels in the FAM Table are not

\[ \text{Table 2: FAM table} \]

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( P )</th>
<th>( Z )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>( PB )</td>
<td>( P )</td>
<td>( Z )</td>
</tr>
<tr>
<td>( Z )</td>
<td>( P )</td>
<td>( Z )</td>
<td>( N )</td>
</tr>
<tr>
<td>( N )</td>
<td>( Z )</td>
<td>( N )</td>
<td>( NB )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Domain of ( x_1 )</th>
<th>Domain of ( x_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -4 )</td>
<td>( -10 )</td>
</tr>
<tr>
<td>( 0.0 )</td>
<td>( 0.0 )</td>
</tr>
<tr>
<td>( 1.0 )</td>
<td>( 1.0 )</td>
</tr>
</tbody>
</table>

Figure 6: The linear transformation of \( x_1 \)

<table>
<thead>
<tr>
<th>Domain of ( x_2 )</th>
<th>Domain of ( u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -10 )</td>
<td>( -16 )</td>
</tr>
<tr>
<td>( 0.0 )</td>
<td>( 0.0 )</td>
</tr>
<tr>
<td>( +10 )</td>
<td>( +16 )</td>
</tr>
<tr>
<td>( 1.0 )</td>
<td>( 1.0 )</td>
</tr>
</tbody>
</table>

Figure 7: The linear transformation of \( x_2 \)

Figure 8: The linear transformation of \( u \)

Big” and “Negative Small”, but they are suitable only to describe reference domains of the form \([-a, a]\). Note that in the fuzzy sets approach linguistic terms are merely labels of fuzzy sets, i.e. fuzzy sets shape plays an important role.

In the hedge algebra approach, the meaning of terms or the fuzziness measure of terms and hedges, i.e. the parameters of SQMs, are very important. Therefore, before applying the new method to solve the inverted pendulum problem we need define the primary terms, hedges and establish a transformation of linguistic terms in FAM into the new ones. In our case, the transformation is given as in Table 3 and 4.

Note that although the linguistic transformations for \( x_1 \) and \( x_2 \) are the same, however, their quantified semantics are different, since their reference domains are different.

The reason we choose the new terms to represent the labels in FAM is that their fuzziness intervals have positions near to the base sides of the triangles or trapezoids [13] with labels given in FAM. In addition, we may do so because the important thing of a method is how well it can simulate the real process. Hence, we shall try to show that the new method will simulate the process better than the one based on fuzzy sets does.

Apply our method to solve the problem under an assumption that the aggregation in Step 2 is chosen to be the weighted averaging operator. Since our method can easily be implemented, we can compute a lot variants and define its weights of the linguistic state variables \( x_1 \) and \( x_2 \) to be \( w_1 = 0.375 \) and \( w_2 = 0.625 \), respectively. To show the influence of the parameters, we study by cases:

Case 1: The parameters for the linguistic control action variable \( U \) are \( \mu_{fm} = 0.5 \) and \( \mu_{Less} = 0.2 \). Thus, \( \mu_{Possible} = 0.3 \) and for the variables \( x_1 \) and \( x_2 \) they are \( \mu_{fm} = 0.5 \) and the fuzziness measures of hedges ignored, since there are no hedges in the linguistic values of \( x_1 \) and \( x_2 \) in the FAM. With such parameters, we can compute the values of SQM for the term-labels of each linguistic variable in FAM.

Applying the new method described above, we can compute the pendulum states and the control action in each simulation cycle and obtain a simulation curve given in Figure 4. It shows that in the 18th cycle the results repeat those computed in the 9th cycle. In this case, the system cannot reach the equilibrium state.

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Applying the new method described above, we can compute the pendulum states and the control action in each simulation cycle and obtain a simulation curve given in Figure 4. It shows that in the 18th cycle the results repeat those computed in the 9th cycle. In this case, the system cannot reach the equilibrium state.
Case 2: The parameters for $U$ are $\mu(Less) = 0.3$, $\mu(Possible) = 0.2$. The parameters for $X_j$ and $X_k$ are the same as in Case 1. With such parameters, SQMs can be calculated and FAM is transformed into 3-SAM given in the Table 5 and it defines a real curve given in Table 6 and Figure 5.

Denote by $x_j$ and $x_k$ the semantic state variables corresponding to the state variables $X_j$ and $X_k$ which take the real values in $[-4, +4]$ and $[-10, +10]$ and by $u_j$ the semantic control action variable corresponding to $u$ which takes values in $[-16, +16]$. In Figure 6-8, it is shown linear transformations from a real domain of a variable to the domain of the corresponding semantic variable.

Now, we apply the new method above, for which we use the classical linear interpolative method in Step 2, to each simulation cycle with the crisp initial conditions to be $x_j(0) = 1^s$ and $x_k(0) = -4^s$. For example, in the first simulation cycle we obtain the following results:

For $x_j = 1.00000$, $x_k = -4.00000$, we have $x_{j1} = 0.625000$, $x_{k2} = 0.30000$ and their aggregation is $w_1^* x_{j1} + w_2^* x_{k2} = 0.421875$. So, using the classical linear interpolative method we obtain $u_j = 0.40000$ and by the linear transformation given in Figure 8 we obtain the control action $u = -3.20000$. In the second simulation cycle, whose input values are the outputs of the last cycle, the corresponding values will be: $x_j = -3.00000$, $x_k = -0.20000$; $x_{j2} = 0.125000$, $x_{k2} = 0.510000$; $w_1^* x_{j2} + w_2^* x_{k2} = 0.365625$ and hence $u_j = 0.30000$ or $u = -6.40000$.

Table 5: 3-SAM for Case 1

<table>
<thead>
<tr>
<th>$X_j$</th>
<th>$X_k$</th>
<th>$L$</th>
<th>$S$</th>
<th>$W$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.75</td>
<td>0.25</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 6: Points of $C_{r,2}$ defined by 3-SAM

<table>
<thead>
<tr>
<th>$w_1^* x_{j1} + w_2^* x_{k2}$</th>
<th>$u_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3750000</td>
<td>0.800000</td>
</tr>
<tr>
<td>0.5937500</td>
<td>0.700000</td>
</tr>
<tr>
<td>0.4375000</td>
<td>0.500000</td>
</tr>
<tr>
<td>0.6562500</td>
<td>0.700000</td>
</tr>
<tr>
<td>0.5000000</td>
<td>0.500000</td>
</tr>
<tr>
<td>0.3437500</td>
<td>0.300000</td>
</tr>
<tr>
<td>0.5625000</td>
<td>0.300000</td>
</tr>
<tr>
<td>0.4062500</td>
<td>0.300000</td>
</tr>
<tr>
<td>0.2350000</td>
<td>0.200000</td>
</tr>
</tbody>
</table>

Table 7: Simulation results

<table>
<thead>
<tr>
<th>HAC Method</th>
<th>FC Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_j(k)$</td>
<td>$x_k(k)$</td>
</tr>
<tr>
<td>1.0</td>
<td>-4.0</td>
</tr>
<tr>
<td>-3.0</td>
<td>0.50</td>
</tr>
<tr>
<td>-1.0</td>
<td>-4.0</td>
</tr>
<tr>
<td>-2.8</td>
<td>0.50</td>
</tr>
<tr>
<td>-0.8</td>
<td>0.50</td>
</tr>
<tr>
<td>1.6</td>
<td>0.50</td>
</tr>
<tr>
<td>3.2</td>
<td>-2.0</td>
</tr>
<tr>
<td>0.6</td>
<td>0.50</td>
</tr>
<tr>
<td>0.0</td>
<td>0.50</td>
</tr>
<tr>
<td>0.2</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Table 8: Simulation results of HAC

<table>
<thead>
<tr>
<th>Cycle No</th>
<th>$x_j$</th>
<th>$x_k$</th>
<th>$x_{j1}$</th>
<th>$x_{k2}$</th>
<th>$w_1^* x_{j1} + w_2^* x_{k2}$</th>
<th>$u_j$</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>-4.0</td>
<td>0.625</td>
<td>0.30</td>
<td>0.625</td>
<td>0.30</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-3.0</td>
<td>-0.2</td>
<td>0.125</td>
<td>0.51</td>
<td>0.421875</td>
<td>0.4</td>
<td>-3.2</td>
</tr>
<tr>
<td>2</td>
<td>-2.8</td>
<td>3.6</td>
<td>0.150</td>
<td>0.68</td>
<td>0.365625</td>
<td>0.3</td>
<td>-6.4</td>
</tr>
<tr>
<td>3</td>
<td>0.8</td>
<td>0.8</td>
<td>0.600</td>
<td>0.54</td>
<td>0.481250</td>
<td>0.5</td>
<td>0.0</td>
</tr>
<tr>
<td>4</td>
<td>1.6</td>
<td>1.6</td>
<td>0.700</td>
<td>0.58</td>
<td>0.562500</td>
<td>0.5</td>
<td>0.0</td>
</tr>
<tr>
<td>5</td>
<td>3.2</td>
<td>-3.2</td>
<td>0.900</td>
<td>0.34</td>
<td>0.625000</td>
<td>0.7</td>
<td>6.4</td>
</tr>
<tr>
<td>6</td>
<td>0.0</td>
<td>0.0</td>
<td>0.500</td>
<td>0.50</td>
<td>0.550000</td>
<td>0.5</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Similarly, we can compute the values of the state variables and the control action variable for the next cycles. The results are given in Table 7, where the values in the cells of each $i^{th}$-row corresponding to $x_i(k)$-columns are the outputs of the cycle $i-1$ and inputs of the cycle $i$, as well. It shows that the system reaches the equilibrium state in cycle 6.

The simulation results of the HAC and of the fuzzy controller (FC) presented in [13] are given in the Table 8. To compare the new method with the fuzzy sets based method presented in [13], we use the error criterion defined by

$$e(k) = \sqrt{x_j^2(k) + x_k^2(k)}.$$  

The comparison of the two methods based on this criterion is presented in Figure 9 and it shows that the simulation of the HAC is much better than the simulation of the FC examined in [13].
5 Conclusions

We have overviewed an algebraic approach to term-domains of linguistic variables, in which fuzziness measures of linguistic values and of hedges can be reasonably defined and a method of quantification of hedge algebras by defining SQMs has been examined. Taking the advantages of the fuzziness-based semantics of linguistic values, a new method for designing fuzzy controller based on hedge algebras approach has been introduced and studied. We pointed out that the new method has many potential benefits. In order to show the effectiveness of the new method, we have re-examined the classical fuzzy control method in solving the inverted pendulum problem. The results of the simulation show that the new method based on hedge algebras is much simpler, easier to simulate a real process. It is worth to emphasize that our method allows control the inverted pendulum to reach its stable position, while fuzzy control method does not. With these results, we believe that the new method can find several other applications in fuzzy control.

The effectiveness of the new method suggest us to set up an investigation of some problems to enhance the effectiveness of the HAC-controller, for example, problems of finding optimal parameters of SQMs and of finding optimal weights of the average aggregation used to solve the inverted pendulum problem and so on in designing HAC-controllers.

References

[6] Nguyen Cat Ho, Huynh Van Nam, Ordered Structure-Based Semantics of Linguistic Terms of Linguistic Variables and Approximate Reasoning,