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Algorithms for computing moments of the length of busy periods of single-server systems

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Abstract: We compute moments of the length of the busy period of $M^X/G/1/n$ systems starting with an arbitrary number of customers in the system using a recursive algorithm that is based on the fact that the length of a busy period initiated by multiple customers in a system of some fixed capacity can be expressed in terms of the lengths of busy periods of systems with smaller or equal capacities and initiated by a single customer. The computational complexity of the algorithm proposed in the paper is derived and numerical examples provided.

Keywords: Batch arrivals, busy period, $M^X/G/1/n$ queue, moments, recursions

1 Introduction

The $M/G/1/n$ system with finite capacity has been extensively studied and the analysis of its busy period has been addressed by many authors; see, e.g., [1, 2, 3, 4]. The characterization of busy periods is important in the performance evaluation of queueing systems as, e.g., some check up tasks of computer systems may be started only when the system is idle.

As moments are important descriptors of probability distributions, we address in the paper the computation of moments of the length of busy periods in $M^X/G/1/n$ systems, which have the properties of $M/G/1/n$ systems with the addition of allowing customer batch arrivals.

Traditionally, busy periods have been characterized through their Laplace-Stieltjes transforms [1, 2, 5, 6], which tend to lead to simple relations for an infinite capacity $M/G/1$ system since the number of customers served in its busy-period has the structure of a Galton-Waltson branching process. However, this nice property does not hold for finite capacity $M/G/1/n$ systems.

By busy period it is (usually) meant the period of time that starts when a customer arrives to an empty system and ends at the first subsequent time at which the system becomes empty. In this paper we work with an extended definition of busy period (that may be) initiated by multiple customers. More precisely, we consider i-busy periods, where an i-busy period is a period that starts at an instant at which i customers are present in the system, with a customer initiating service at that time, and ends at the next time at which the system becomes empty.

This definition is richer and more natural than the usual definition when addressing systems with batch arrivals. The definition of i-busy period which we propose coincides with that of *remaining busy period from state* i given in Harris [1]. Note that the 1-busy period corresponds to the busy period as usually defined in the literature (see, e.g., [7]).

In [8] we characterize the length of busy periods of $M^X/G/1/n$ systems by conditioning on the number of customers that arrive to the system during the first service time offered during the busy period while, concurrently, taking full advantage of the Markov-regenerative structure of the number of customers in $M^X/G/1/n$ systems (see, e.g., [9] for the definition and properties of Markov regenerative processes).

In addition, in [8] we derive important properties satisfied by moments of the length of busy periods and put those into use with the proposal of a recursive algorithm that leads to the computation of the moments of order k, $1 \le k \le K$, of the length of busy periods of $M^X/G/1/n$ systems, $1 \le n \le N$, in the order of $N^3 + N^2K^2$ flops in general, following [10] for the definition of *flop*.

We end this introduction with an outline of the paper. In Section 2 we briefly introduce the $M^X/G/1/n$ model, along with some relevant notation. In Section 3 we present a recursion derived in [8] to compute moments of the length of busy periods of $M^X/G/1/n$ systems and propose an efficient algorithm to implement it. We discuss the computational complexity of the proposed algorithm in Section 4, and in Section 5 we provide numbers and figures relative to moments of the length of busy periods of $M^X/G/1/n$ systems. Finally, in Section 6 we state the main conclusions and accomplishments of the paper.

2 The model

In the paper we address the implementation of the computation of moments of the length of the i-busy period of an $M^X/G/1/n$ system, i.e., a single-server queue with: compound Poisson customer arrival process; general (independent) customer service times; and finite capacity n, including the customer in service – if any. As the system has finite capacity, customers may be blocked.

As regards the customer acceptance policy, we consider what is known as *partial blocking* (see, e.g. [11]) in which, if at arrival of a batch of l customers there are only m, $m < l$, free positions available in the system, then the first m customers of the batch enter the system and the remaining $l - m$ customers of the batch are blocked. Moreover, we consider the standard assumption that the customer arrival process is independent of the sequence of customer service times.

For future use, it is convenient to introduce the following parameters of an $M^X/G/1/n$ system:

- λ: (customer) batch arrival rate;
- $(f_j)_{j\in\mathbb{N}_+}$: batch size probability function, i.e., f_j is the probability that a customer batch contains exactly j customers;
- $\bar{f} = \sum_{j\in\mathbb{N}_+} jf_j$: mean batch size;
- $A(\cdot)$: customer service time distribution function; and
- μ: reciprocal of the mean customer service time (or customer service rate), i.e.,

$$\mu^{-1} = \int_0^\infty (1 - A(t))\,\mathrm{d}t.$$

Note, in particular, that the offered traffic intensity of the system is equal to $\rho = \lambda\bar{f}/\mu$.

We let $X(t)$ denote the number of customers in the $M^X/G/1/n$ system at time t and B_{in} denote the length of an associated generic i-busy period. According to our definition of i-busy period, we have

$$B_{in} \overset{\mathrm{d}}{=} \inf\{t \geq 0 : X(t) = 0\} \,|\, [X(0) = i, X(0^-) \neq i]$$

where "$\overset{\mathrm{d}}{=}$" denotes equality in distribution and "$|$" denotes conditioning of random variables.

In addition, we let $\overline{S}_l$ denote a random variable with the distribution of the service time of a customer during which exactly l customers arrive to the system, i.e.,

$$\overline{S}_l \overset{\mathrm{d}}{=} S|M(S) = l$$

where $\{M(t),\, t \geq 0\}$ denotes the compound Poisson (counting) customer arrival process, and S a generic service time. Moreover, we let $(p_j)_{j\in\mathbb{N}}$ denote the probability function of the number of customer arrivals during a service time.

3 The main recursion

In this section we present the recursive scheme derived in [8] to compute moments of the length of busy periods of $M^X/G/1/n$ systems.

Proposition 1 *The integer moments of i-busy periods in $M^X/G/1/n$ systems are such that $\mathbf{E}[B_{11}^k] = \mathbf{E}[S^k]$, $k \in \mathbb{N}$, and, for $n \geq 2$:*

$$\mathbf{E}[B_{1n}^k] = \Bigg[\mathbf{E}[S^k] + \sum_{l=1}^{n-2} p_l\psi_{ln}^{(k)} + \sum_{l\geq n-1} p_l\psi_{n-1,n}^{(k)} + \sum_{j=1}^{k-1}\binom{k}{j}\sum_{l=1}^{n-2} p_l\,\mathbf{E}[\overline{S}_l^j]\,\bar{\psi}_{ln}^{(j)} + \sum_{j=1}^{k-1}\binom{k}{j}\sum_{l\geq n-1} p_l\,\mathbf{E}[\overline{S}_l^j]\,\bar{\psi}_{n-1,n}^{(j)}\Bigg]\Bigg/ p_0 \qquad (1)$$

and

$$\mathbf{E}[B_{in}^k] = \sum_{j=0}^{k}\binom{k}{j}\mathbf{E}[B_{i-1,n-1}^j]\,\mathbf{E}[B_{1n}^{k-j}] \qquad (2)$$

for $2 \leq i \leq n$, where the random variables B_{0m} are null with probability one and

$$\bar{\psi}_{im}^{(j)} = \sum_{l=0}^{j}\binom{j}{l}\mathbf{E}[B_{i-1,m-1}^l]\,\mathbf{E}[B_{1m}^{j-l}] \qquad (3)$$

$$\psi_{im}^{(j)} = \sum_{l=1}^{j}\binom{j}{l}\mathbf{E}[B_{i-1,m-1}^l]\,\mathbf{E}[B_{1m}^{j-l}] \qquad (4)$$

so that $\bar{\psi}_{im}^{(j)} = \mathbf{E}[B_{im}^j]$ and $\psi_{im}^{(j)} = \mathbf{E}[B_{im}^j] - \mathbf{E}[B_{1m}^j]$. ■

The previous recursion is based on the fact that the length of a busy period initiated by multiple customers in a finite capacity system can be expressed in terms of the lengths of busy periods of systems with smaller or equal capacities and initiated by a single customer.

We note that the most immediate application of the proposition is for the recursive computation of the expected value of the length of the classical busy period (the 1-busy period) of $M/G/1/n$ systems, in which case we conclude that

$$\mathbf{E}[B_{1n}] = \frac{1}{\alpha_0}\left[\mathbf{E}[S] + \sum_{i=2}^{n-1}\mathbf{E}[B_{1i}]\sum_{l\geq n-i+1}\alpha_l\right] \qquad (5)$$

for $n \geq 2$. This leads to the recursive scheme derived by Miller [2] to compute the length of a classical busy period of an $M/G/1/N$ system – starting from $\mathbf{E}[B_{11}^k] = \mathbf{E}[S^k]$ and recursively computing $\mathbf{E}[B_{1n}]$, $n = 2, 3, \ldots, N$, via (5).

As regards the computation of moments of integer orders strictly larger than one, we can easily conclude that the moments $\mathbf{E}[B_{iN}^K]$, $1 \leq i \leq N$, may be computed using (1)-(2) provided one has available the set of analogous moments of lengths of busy periods

$$\left(\mathbf{E}[B_{in}^k]\right)_{1\leq k\leq K-1, 1\leq i\leq n\leq N}$$

by first computing $\mathbf{E}[B_{1N}^K]$ using (1) followed by the recursive computation of $\mathbf{E}[B_{iN}^K]$, for $i = 2, 3, \ldots, N$, using (2). In other words, we may use the following generic algorithm

ALGORITHM
Input: $(N, K, \lambda, f, \alpha, m)$
[Step 1] Compute c
[Step 2] Compute g
[Step 3] Compute $(p, \bar{p})$
[Step 4] Compute $(b, \bar{b})$
[Step 5] Make $\mathbf{E}[B_{in}^0] \equiv 1$, $\mathbf{E}[B_{0n}^k] \equiv 0$, $\psi_{in}^{(0)} \equiv 0$
[Step 6]
  for $k = 1, 2, \ldots, K$ do
    for $n = 1, 2, \ldots, N$ do
      [Step 6.1] Compute $(\psi_{ln}^{(k)}, \bar{\psi}_{ln}^{(k-1)})_{1\leq l\leq n-1}$
      [Step 6.2] Compute $\mathbf{E}[B_{1n}^k]$ using (1)
      [Step 6.3] Compute $(\mathbf{E}[B_{in}^k])_{2\leq i\leq n}$ using (2)
    end for
  end for
Output: $\left(\mathbf{E}[B_{in}^k]\right)_{1\leq k\leq K, 1\leq i\leq n\leq N}$

to compute the moments $\left(\mathbf{E}[B_{in}^k]\right)_{1\leq k\leq K, 1\leq i\leq n\leq N}$, where:

- $f = (f_l)_{0\leq l\leq N-2}$;
- $\alpha = (\alpha_l)_{0\leq l\leq N-2}$, where the quantities α_l are mixed-Poisson probabilities (see, e.g., [12] and [13]) with rate λ associated to the structural distribution $A(\cdot)$, i.e.,

$$\alpha_l = \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^l}{l!} A(\mathrm{d}u), \quad l \in \mathbb{N}.$$

- $m = (m_k)_{0\leq k\leq K}$, with $m_k = \mathbf{E}[S^k]$;
- $c = (c_{kj})_{0\leq j\leq k\leq K}$, with $c_{kj} = \binom{k}{j}$;
- $g = (g_{lj})_{0\leq j\leq l\leq N-2}$, with g_{lj} denoting the probability that the total number of customers in j customer batches is equal to l;
- $p = (p_l)_{0\leq l\leq N-2}$;
- $\bar{p} = (\bar{p}_n)_{0\leq n\leq N-2}$, with $\bar{p}_n = \sum_{l>n} p_l$;
- $b = (b_{lk})_{0\leq l\leq N-2, 0\leq k\leq K}$, with $b_{lk} = p_l\, \mathbf{E}[\overline{S}_l^k]$; and
- $\bar{b} = (\bar{b}_{lk})_{0\leq n\leq N-2, 0\leq k\leq K}$, with $\bar{b}_{lk} = \sum_{l>n} b_{lk}$.

The algorithm consists of six steps. The computation of the moments of the length of busy periods in a cycle is done in the last step (Step 6) in a way that incorporates the findings of Proposition 1, with Step 5 serving as initialization for the recursive procedure of Step 6. The first four steps of the algorithm include the computation of auxiliary quantities that are used in Step 6.

Note that in Step 5 the indices vary in their natural ranges, namely: we make $\mathbf{E}[B_{in}^0] = 1$ and $\psi_{in}^{(0)} = 0$ for $n = 1, 2, \ldots, N$ and $i = 0, 1, \ldots, n$, and we make $\mathbf{E}[B_{0n}^k] = 0$ for $n = 1, 2, \ldots, N$ and $k = 0, 1, \ldots, K$.

4 Computational complexity

We next present some remarks on the computational complexity of the steps of the Algorithm, expressed in terms of the number of flops they require.

In this analysis when we state the order of the number of flops required in a procedure, the word "order" refers to the number of flops as function of N (maximum capacity of the considered systems), and K (maximum order of the moments of the lengths of busy periods that need to be computed). In order to simplify the writing, $\mathrm{O}(x)$ means "in order of x". We will used the following well known combinatorial identities:

$$\sum_{i=1}^m i = \frac{m(m+1)}{2} = \mathrm{O}\left(m^2\right)$$
$$\sum_{i=1}^m i^2 = \frac{m(m+1)(2m+1)}{6} = \mathrm{O}\left(m^3\right)$$

The Algorithm requires as input mixed-Poisson probabilities $(\alpha_l)_{0\leq l\leq N-2}$; these may be computed in linear time (on the capacity of the system, N) for a large class of service time distributions that includes the distributions most commonly used in practice [12], by means of simple recursive schemes.

Step 1 of the Algorithm consists of the computation of the combinations $\binom{k}{j}$, $0 \leq j \leq k \leq K$. By using the well-known identities: $\binom{k}{j} = \binom{k-1}{j-1} + \binom{k-1}{j}$, $1 \leq j \leq k$, and $\binom{k}{j} = \binom{k}{k-j}$, along with $\binom{k}{0} = 1$, we conclude that Step 1 can be carried out in $\mathrm{O}(K^2)$ flops.

Step 2 of the Algorithm addresses the computation of the set of convolution probabilities $g = (g_{lj})_{0\leq j\leq l\leq N-2}$. For the implementation of Step 2 the following facts are relevant: $g_{00} = 1$ and $g_{l0} = 0$ for $l \neq 0$; and,

$$g_{lj} = \sum_{m=j-1}^{l-1} g_{m,j-1} f_{l-m} \tag{6}$$

for $j \geq 1$, and, in particular, $g_{l1} \equiv f_l$. This implies that Step 2 can be carried out in $\mathrm{O}(N^3)$ flops.

Step 3 of the Algorithm consists of the computation of the vectors $p = (p_l)_{0 \leq l \leq N-2}$ and $\bar{p} = (\bar{p}_l)_{0 \leq l \leq N-2}$. As $\bar{p}_l = \sum_{j>l} p_j = 1 - \sum_{j=0}^{l} p_j$, the main issue on this step is the computation of the probabilities p_l, of l customers arriving to the system during the service time of a customer. By conditioning on the service time of a customer and the number of customer batches arriving in a fixed interval of time, we arrive at the equality

$$p_l = \sum_{j=0}^{l} \alpha_j g_{lj}.$$

This equality explains why the mixed-Poisson probabilities α_l constitute input to the Algorithm and implies that Step 3 can be carried out in $\mathrm{O}(N^2)$ flops.

Step 4 of the Algorithm consists of the computation of $b = (b_{lk})_{0 \leq l \leq N-2, 0 \leq k \leq K}$, with $b_{lk} = p_l \, \mathbf{E}[\overline{S}_l^k]$, as well as $\bar{b} = (\bar{b}_{lk})_{0 \leq n \leq N-2, 0 \leq k \leq K}$, with $\bar{b}_{lk} = \sum_{l>n} b_{lk}$. The next result, stated in [8], allows for the computation of the coefficients $(b_{lk})_{0 \leq l \leq N-2, 0 \leq k \leq K}$.

Proposition 2 *The absolute moment of order k, $k \in \mathbb{N}_+$, of the conditional random variable $\overline{S}_l$, verifies:*

$$p_l \, \mathbf{E}[\overline{S}_l^k] = \sum_{j=0}^{l} \frac{(k+j)!}{\lambda^k j!} \, \alpha_{k+j} \, g_{lj}$$

for $l \in \mathbb{N}$, and, moreover,

$$\sum_{l \geq n-1} p_l \mathbf{E}[\overline{S}_l^k] = \mathbf{E}[S^k] - \sum_{l=0}^{n-2} \sum_{j=0}^{l} \frac{(k+j)!}{\lambda^k j!} \alpha_{k+j} g_{lj}. \quad \blacksquare$$

From the previous proposition, we can conclude that Step 4 of the Algorithm can be carried out in $\mathrm{O}(K(N^2 + K))$ flops.

Note that Step 5 of the Algorithm requires no flops and that Step 6 of the same algorithm possesses a (double) cycle whose steps contain three sub-steps such that in each of them summations of products of the form $\sum_{i=1}^{m} \prod_{j=1}^{p} d_{ij}$ have to be carried out. By using the fact that the computation of such an expression requires $\mathrm{O}(mp)$ flops, we can conclude that the Step 6 of the Algorithm can be carried out in $\mathrm{O}(K^2 N^2)$ flops.

By adding up the previous findings, we conclude the following result.

Theorem 3 *For fixed arrival rate λ, customer batch size distribution probability function, and service time distribution, and aside from the computation of: the customer batch size probability function, $(f_l)_{l \in \mathbb{N}_+}$; the mixed-Poisson probabilities associated to the service time distribution, $(\alpha_l)_{0 \leq l \leq N+K-1}$, and the moments of the service time distribution, the computation of the moments*

$$\left(\mathbf{E}\left[B_{in}^k\right]\right)_{1 \leq k \leq K, 1 \leq i \leq n \leq N}$$

of the lengths of busy periods of $M^X/G/1/n$ systems can be carried out using

$$O(N^2(N + K^2))$$

flops. $\blacksquare$

Note that steps 3 and 6 of the Algorithm, i.e., the computation of the convolution probabilities associated to the customer batch size distribution and the double-cycle that appears in Step 6, contribute to form the general upper bound $N^2(N + K^2)$ for the order of the number of flops required to compute the moments $\mathbf{E}\left[B_{in}^k\right]$, $1 \leq k \leq K, 1 \leq i \leq n \leq N$, of the lengths of busy periods of $M^X/G/1/n$ systems.

5 Numerical examples

In this section, we illustrate the power of the approach proposed in the previous section to compute moments of the length of busy periods of $M^X/G/1/n$ systems using several service time distributions.

The customer service time distributions have been chosen from the following types of distributions, always with mean $1/\mu$: deterministic (D), uniform on the interval (a, b) (Uniform(a, b)), negative exponential with rate μ ($M(\mu)$), Erlang with three phases E_3 ($E_3(3\mu)$), and Pareto with parameters $(\beta, \kappa = (\beta - 1)/\beta\mu)$, $\beta > 1$, (P$(\beta, (\beta - 1)/\beta\mu)$) distribution, whose moment of order k exists only if $\beta > k$ – in which case it is equal to $\beta\kappa^k/(\beta - k)$ (see, e.g., [12]).

Figure 1 is relative to $M/G/1/n$ systems with unit arrival and service rates and gives expected values and coefficients of variation of the length of busy periods starting with a single customer (i.e, 1-busy periods) as functions of the capacity of the system. Figure 1 shows that the expected value of the length of 1-busy periods is roughly an exponential function (with constant rate) of the capacity of the system. This rate depends on the customer service time distribution and increases as this distribution becomes less variable and more concentrated around its mean. The figure also shows that there exist again differences between the coefficient of variation of the length of busy periods associated to the various customer service time distributions considered, with the highest values being observed for exponential service times and the lowest with deterministic service times.

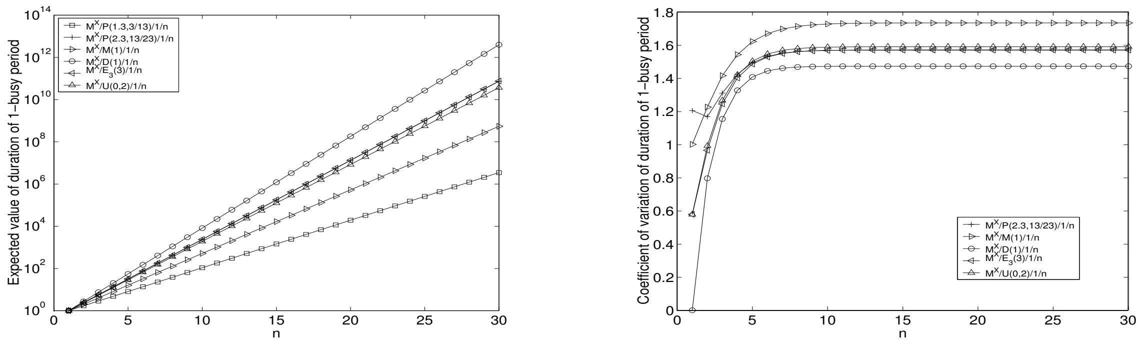

Figure 1: Expected values and coefficients of variation of the length of busy periods initiated by single customers in $M/G/1/n$ systems, with unit arrival and service rates, as a function of the capacity of the system, n.

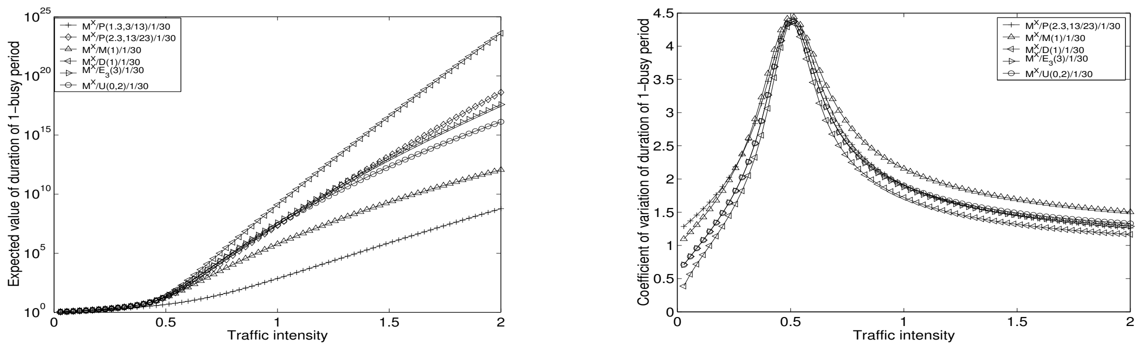

Figure 2: Expected values and coefficients of variation of the length of busy periods initiated by single customers in $M/G/1/n$ systems, with unit arrival and service rates, as a function of the capacity of the system, n.

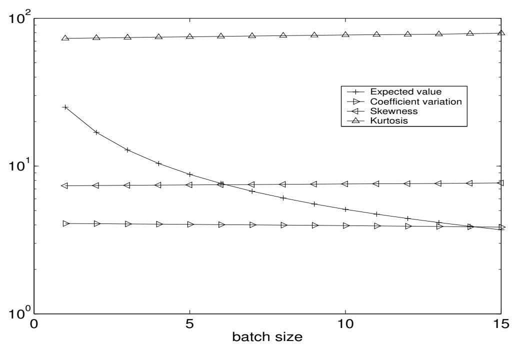

Figure 3: Expected values and coefficients of variation, skewness, and kurtosis of the length of 1-busy periods of $M^X/M/1/25$ systems, with unit arrival and service rates, as a function of the (deterministic) batch size.

Figure 2 is relative to $M/G/1/30$ systems with unit arrival rate and shows how the expected value and coefficient of variation of the length of 1-busy periods evolve as the traffic intensity (i.e., the service rate) increases. Figure 2 shows that the expected value of the duration of 1-busy periods tends to increase slowly for small traffic intensities (essentially for $\rho < 0.5$) but it is roughly an exponential function of the traffic intensity of the system (with constant rate) for higher traffic intensities. The rate of increase of the expected value depends on the customer service time distribution, being smaller for the Pareto distribution (which may be explained in part by the higher losses associated to systems with Pareto service time) and also for deterministic service times. It also shows that although there are differences among the coefficient of variation of the lengths of busy periods associated to the various customer service time distributions considered, qualitatively these coefficients of variation change in a similar way as functions of the traffic intensity, with their highest values being attained at traffic intensities slightly larger than 0.5.

Finally, Figure 3 shows the effect of varying the (deterministic) batch size of $M^X/M/1/25$ queues on the expected value and the coefficients of variation, skewness, and kurtosis of the length of 1-busy periods as the customer batch size varies. We can see that non negligible variations occur as the batch size varies, with this effect being specially salient for the expected value, which exhibits a strong decrease as the batch size increases.

6 Conclusions

The characterization of busy periods is important in performance evaluation of queueing systems as, e.g., some check up tasks of computer systems may be started only when the system is idle. Moreover, moments are important descriptors of probability distributions. In this line, we have addressed in the paper the computation of moments of the length of busy periods in $M^X/G/1/n$ systems.

Resorting to recursions derived in [8], we have proposed an algorithm that is able to compute the moments of order k, $1 \leq k \leq K$, of the length of busy periods of $M^X/G/1/n$ systems, $1 \leq n \leq N$, in $\mathrm{O}(N^2(N + K^2))$ flops. Note that $\mathrm{O}(N^3)$ flops are needed, in general, to compute the convolution probabilities of orders $2, 3, \ldots, N-2$ of the customer batch size distribution, which are required to obtain the probability function of the number of customer arrivals during the service of a customer, needed to use the recursions derived in [8].

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