Two-Dimensional Queueing Model for a LAN Gateway

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Abstract: In this article, we present a queueing model for a gateway linking two Ethernet-type LANs. Each LAN is assumed to be slotted and have a single-channel allowing for the transmission of only one packet at a time. It turns out that the gateway presents an interesting queueing model not widely seen in the literature. In particular, the model has two interfering, parallel, and back-to-back queues. The model is used to study the gateway occupancy. To this end, a state transition diagram is constructed and then used to extract a two-dimensional, second order difference equation, with eight boundary conditions characterizing the gateway dynamics. A probability generating function (PGF) technique is used to solve the difference equation, resulting in a rather challenging functional equation defining a two-dimensional PGF of the joint gateway occupancy distribution. Pointers are given on the techniques available in the literature to solve the equation and obtain the distribution.

Key–Words: LAN gateway, Ethernet, discrete queueing models, difference equations, generating functions, functional equations

1 Introduction

There are advantages to installing several gatewayed Local Area Networks (LANs) instead of installing a single giant LAN having the same number of nodes. The details and benefits of this practice can be found in many networking texts, e.g. [1].

The gatewaying approach is most fruitful when the nodes can be divided into groups such that a node in a group communicates more with nodes of the same group than with nodes of another group. The nodes of each such group can be integrated in one LAN, with each two such LANs linked by a gateway. Traffic emanating from a certain node can now be classified as either internal or external, according as the receiving node belongs to the same LAN or a different LAN.

The function of a gateway is to handle the external traffic. Specifically, if the gateway links LANs I and II, as shown in Figure 1, it moves to LAN II the traffic destined to it from LAN I, and vice versa. As a consequence, the gateway should be able to distinguish external from internal traffic. Basically, if the gateway recognizes an external packet, it will copy it in a buffer, then transmit it to the other LAN when this transmission is possible. The details of these processes depend on the architectures and protocols of both LANs.

The rest of the article is organized as follows. Section 2 details the proposed model, giving its assumptions and notations. The model is used to analyze the gateway occupancy in Section 3. Finally, Section 4 gives the conclusions and expected future work.

2 Model

To begin with, the two LANs on both sides of the gateway are assumed to have the following two characteristics. First, packets are of fixed size, and time is slotted, with the slot length equal to the transmission time of one packet. Second, a packet may be transmitted onto a LAN only at the start of a slot. Third, only one packet may exist on a LAN at any given time. Many LANs satisfy these characteristics such as the slotted CSMA [2] and Slotted Token Ring [3] networks.
Figure 2: State transition diagram of the gateway. Note that $\omega_1 = \xi_1 \xi_2$, $\omega_2 = \omega_5 = \xi_1 q \bar{p} + \xi_1 \bar{q}$, $\omega_4 = \omega_9 = p \bar{p} \xi_2 + p \xi_2$, $\omega_3 = \omega_6 = \omega_7 = \omega_8 = p \bar{p} q \bar{p} + \xi_1 \bar{q} + \bar{p} \xi_2$. 

given below in caption
Clearly, the gateway should have two buffers, one to queue packets going from LAN I to LAN II, and one in the other direction, as shown in Figure 1. The gateway acts as a normal node when seen from any of the two LANs. If it identifies a packet, say, on LAN I destined to LAN II it will receive it using its LAN I interface and protocol, and later sends it onto LAN II using its LAN II interface and protocol. The two LANs as well as the gateway are synchronized, i.e. have the same slot length and boundaries.

We will call a LAN active in a given slot if it carries in that slot a packet sent by one of its nodes. A LAN that is not active in a slot is called inactive. When a LAN is active, the packet it is carrying is either internal or external, according as the destination node is in the same or the other LAN. Let \( p \) denote the probability that LAN I is active in each slot. Thus, \( \overline{p} = 1 - p \) denotes the probability that LAN I is inactive in each slot. Similarly, let \( q \) denote the probability that LAN II is active in each slot. Thus, \( \overline{q} = 1 - q \) denotes the probability that LAN II is inactive in each slot. Furthermore, let \( r \) denote the probability that if LAN I is active in a certain slot, then the packet is external. Thus, \( \overline{r} = 1 - r \) denotes the probability that if LAN I is active in a certain slot, then the packet is internal. And similarly, let \( s \) denote the probability that if LAN II is active in a certain slot, then the packet is external. Thus, \( \overline{s} = 1 - s \) denotes the probability that if LAN II is active in a certain slot, the packet is internal.

If one of the two LANs is active with an external packet, the gateway copies that packet, queues it in the appropriate buffer, then transmits it to the other LAN in the right slot. This right slot is clearly when a) the destination LAN is inactive and b) there are no packets in the queue ahead of this packet.

In view of these assumptions, the gateway can be modelled by the two back-to-back, parallel, and coupled (interfering, or mutually dependent) queues shown in Figure 1. In this model we label each queue with the same label as the LAN it receives packets from. That is, queue I receives its packets from LAN I and queue II receives its packets from LAN II. Each queue is single server with infinite calling population and infinite waiting room. The coupling of the two queues is due to the fact that one queue cannot have a departure when the other has an arrival. This coupling is signified in the model by the two switch-like mechanisms.

### 3 Gateway Occupancy Analysis

In this section we will analyze the gateway occupancy using the same methodology as in [5]. In slot \( k \), with \( k = 0, 1, 2, \ldots \), there will be \( X^k \) packets in queue I and \( Y^k \) packets in queue II, where \( X^k \) and \( Y^k \) are nonnegative integer-valued random variables. It is convenient then to define the state of the gateway in state \( k \) by the ordered pair \( (X^k, Y^k) \). The process \( \{(X^k, Y^k), k = 0, 1, 2, \ldots \} \) is a two-dimensional, discrete-time Markov chain [4], because, as we will show below, the probability \( \Pr[X^k = i, Y^k = j] \) can be evaluated by knowing the state of the gateway in the previous slot, \( k - 1 \). The state transition diagram of this chain is shown in Figure 2. The basic idea in evaluating the transition probabilities in this diagram is to identify what states in slot \( k \) will yield a particular state in slot \( k + 1 \). It can be seen that there are taboo transitions due to the arrival-departure restriction mentioned above.

In order to find the state probabilities of the chain \( \{(X^k, Y^k), k = 0, 1, 2, \ldots \} \), we will make the following shorthand notations. Let \( \xi_1 \) denote the probability that LAN I is externally active, i.e. active with a packet destined for LAN II. Using our conventions, then \( \overline{\xi}_1 \) denotes the probability that LAN I is not externally active, i.e. either inactive or internally active. In a similar manner let \( \xi_2 \) and \( \overline{\xi}_2 \) denote the probabilities that LAN II is and is not externally active, respectively. Then, we can write

\[
\xi_1 = pr \\
\overline{\xi}_1 = \overline{p} + pr \\
\xi_2 = qs \\
\overline{\xi}_2 = \overline{q} + q\overline{s}
\]

In what follows, we derive a set of subequations which collectively form a difference equation defining the distribution \( p^k_{m,n} \) defined by

\[
p^k_{m,n} = \Pr[X^k = m, Y^k = n]
\]

For region R1, we get

\[
p^k_{0,0} = \overline{\xi}_1 \overline{\xi}_2 p^k_{0,0} + \xi_1 \overline{p} p^k_{1,0} + \overline{p} \overline{q} p^k_{1,1} + \overline{p} \overline{q} \overline{p} p^k_{0,1} \tag{1}
\]

For region R2, we get

\[
p^k_{1,0} = (\overline{\xi}_1 q \overline{s} + \xi_1 q) p^k_{1,0} + \xi_1 \overline{\xi}_2 p^k_{0,0} + \xi_1 \overline{q} p^k_{2,0} + \overline{p} \overline{q} p^k_{2,1} + \overline{p} q s p^k_{1,1} \tag{2}
\]

For region R3, we get

\[
p^k_{1,1} = (p \overline{r} q \overline{s} + \xi_1 q + \overline{\xi}_2) p^k_{1,1} + \xi_1 \overline{\xi}_2 p^k_{0,0} + \overline{\xi}_2 p^k_{1,0} + p \overline{q} p^k_{2,1} + \overline{p} \overline{q} p^k_{2,2} + \overline{p} q s p^k_{1,2} + \xi_1 \overline{\xi}_2 p^k_{0,1} \tag{3}
\]

For region R4, we get

\[
p^k_{1,0} = \overline{\xi}_1 \overline{\xi}_2 p^k_{0,0} + \xi_1 \overline{p} p^k_{1,0} + \overline{p} \overline{q} p^k_{1,1} + \overline{p} \overline{q} \overline{p} p^k_{0,1} \tag{4}
\]
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For region $R4$, we get

$$p_{0,1}^{k+1} = (p\xi_2 + p\xi_1) p_{0,1}^k + \xi_1 \xi_2 p_{0,0}^k + p\xi_2 p_{0,2}^k + \bar{p} \bar{q} p_{1,2}^k + p\bar{q} p_{1,1}^k$$

(4)

For region $R5$, we get, for $m = 2, 3, \ldots$, that

$$p_{m,0}^{k+1} = (\xi_1 \xi_2 + \xi_1 \bar{q}) p_{m,0}^k + \xi_1 \xi_2 p_{m-1,0}^k + \bar{p} \bar{q} p_{m+1,0}^k + p\bar{q} p_{m,1}^k + \bar{p} q p_{m,0}^k$$

(5)

For region $R6$, we get, for $m = 2, 3, \ldots$, that

$$p_{m,1}^{k+1} = (p\bar{q} \xi + \xi_1 \bar{q} + p\xi_2) p_{m,1}^k + \xi_1 \xi_2 p_{m-1,0}^k + \bar{p} \bar{q} p_{m+1,0}^k + p\bar{q} p_{m,1}^k + \bar{p} q p_{m,0}^k$$

(6)

For region $R7$, we get, for $m, n = 2, 3, \ldots$, that

$$p_{m,n}^{k+1} = (p\bar{q} \xi + \xi_1 \bar{q} + p\xi_2) p_{m,n}^k + \xi_1 \xi_2 p_{m-1,n-1}^k + \bar{p} \bar{q} p_{m+1,n-1}^k + p\bar{q} p_{m,n+1}^k + \bar{p} q p_{m,n}^k$$

(7)

For region $R8$, we get, for $n = 2, 3, \ldots$, that

$$p_{1,n}^{k+1} = (p\bar{q} \xi + \xi_1 \bar{q} + p\xi_2) p_{1,n}^k + \xi_1 \xi_2 p_{0,n-1}^k + \bar{p} \bar{q} p_{2,n-1}^k + p\bar{q} p_{1,n+1}^k + \bar{p} q p_{1,n}^k$$

(8)

For region $R9$, we get, for $n = 2, 3, \ldots$, that

$$p_{0,n}^{k+1} = (p\bar{q} \xi + \xi_1 \bar{q} + p\xi_2) p_{0,n}^k + \xi_1 \xi_2 p_{0,n-1}^k + \bar{p} \bar{q} p_{2,n-1}^k + p\bar{q} p_{1,n+1}^k + \bar{p} q p_{1,n}^k$$

(9)

Assuming that the gateway will reach steady state after a sufficiently large number of slots, then the limit

$$\lim_{k \to \infty} p_{m,n}^k = p_{m,n}$$

exists for all $m, n = 0, 1, \ldots$. Thus, in steady state, (1)-(9) become

$$\xi_1 \xi_2 p_{0,0} - \xi_1 \bar{q} p_{0,0} - \bar{p} q p_{1,1} - \bar{p} \bar{q} p_{0,1} = 0$$

(10)

$$\xi_1 \xi_2 p_{0,0} - \xi_1 \bar{q} p_{0,0} - \bar{p} q p_{1,2} - \bar{p} p q p_{1,1} = 0$$

(11)

Clearly, the system (10)-(18) collectively defines the sequence $p_{m,n}, m, n = 0, 1, \ldots$. In this system, (16) is a two dimensional second order difference equation and all other eight equations are its boundary conditions. We will use this equation to find the probability generating function (PGF)

$$P(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n} x^m y^n, \quad |x|, |y| \leq 1.$$  

(19)

which uniquely characterizes the sequence $p_{m,n}$. To find $P(x, y)$, we follow the same methodology as in [5]. Basically, we multiply each equation in the system (10)-(18) by $x^m z^n$, sum each equation over its given domain, and use the normalization condition \(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n} = 1\). Doing that, performing some
rather tough, though straightforward, algebraic manipulations in the process, we get

\[(xy - M(x,y)) P(x,y) =
\]

\[(y-1) (M(x,0) + \overline{\Phi}_2 x y) P(x,0)
\]

\[+ (x-1) (M(0,0) + \overline{\Phi}_1 x y) P(0,y)
\]

\[+ (x-1)(y-1) M(0,0) P(0,0),
\]

(20)

where

\[M(x,y) = (\overline{\Phi} + p\overline{\Phi} y + \xi_1 x y) (\overline{\Phi} + q\overline{\Phi} x + \xi_2 x y),
\]

\[P(x,0) = \sum_{m=0}^{\infty} p_{m,0} x^m,
\]

and

\[P(0,y) = \sum_{n=0}^{\infty} p_{0,n} y^n.
\]

The symmetry of the two queues of the gateway is evidently reflected in the functional equation (20) which attests to the accuracy of that equation. To further verify the equation, we will resort to the following trick. If, say, LAN I is active and LAN II is inactive, then only queue I in the gateway will be active, receiving packets at the rate of \(\xi_1\) packets per slot and serving them at the rate of 1 packet per slot. The occupancy of such discrete queue is known (see, e.g., \([6, 7]\)) to be

\[P(x) = \frac{\xi_1 x + \overline{\xi}_1}{1-\xi_1} P(0,0).
\]

Sure enough, this equation can be obtained immediately from (20) by substituting there \(q = 0\) (LAN II inactive) and noting that \(P(x,y) = P(x,0) = P(x)\) and that \(P(0,y) = P(0,0)\). Alternately, if we substitute \(p = 0\) (LAN I inactive) we would get a similar equation to (21) with \(x\) replaced by \(y\) and \(\xi_1\) replaced by \(\xi_2\), which is more proof to the accuracy of (20).

The solution of the functional equation (20) for the generating function \(P(x,y)\) is nontrivial. Even more challenging is the inversion of the resulting \(P(x,y)\), assuming it is obtained, to get the sequence \(p_{m,n}\). However, methods for the solution of this type of equations and attempts for the inversion abound. Basically, forms close to that of the present equation have been arrived at in various other queueing contexts. Through heavy and lengthy use of complex analysis techniques, solutions for these forms have been given and numerous attempts to invert the resulting generating function or derive asymptotic formulas from them have been made and reported.

One such form is arrived at in [8] in the context of analyzing a priority queue whose arriving customers come from two packet radio nodes, giving rise to two logical interfering queues. The authors do not solve the functional equation, but obtain directly from it expectations for some (simplified) special cases. Also, a whole class of similar functional equations are given in [9], which are handled using Mellin Transform and iteration semigroup techniques. Another form is arrived at in [10] in the context of analyzing a single queue whose arriving customers place two demands handled independently by two servers, giving rise to two logical queues in parallel. The authors manage to solve the equation for the generating function and perform some asymptotic analysis of the behavior of the two queues as either of them grows infinitely high. In a subsequent work [11], the same asymptotic analysis is redone to evaluate the behavior of the two queues as both of them grow infinitely high. One more form of the above functional equation has appeared in [12] in the context of analyzing a multiprogrammed computer. The technique used there is that of analytic continuation. A survey paper [13] gives an array of similar functional equations and techniques for their solution.

\[4\] Conclusion

In this article we have introduced a queueing model for a LAN gateway and used to study the gateway occupancy. The occupancy has been characterized by a two-dimensional second order difference equation extracted from the physics of the gateway. A solution for the difference equation has been attempted resulting in a rather involved functional equation defining the PGF of the gateway occupancy probabilities.

Though the model looks simple at first sight, its analysis is by no means as simple. Nontrivial work is still needed to solve the functional equation (20) for the joint distribution of the gateway occupancy. Pointers to this work have been identified and are listed in the Reference Section for interested researchers.

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References:


