Sampling-Reconstruction Procedure of Gaussian Fields with Jitter

VLADIMIR A. KAZAKOV AND DANIEL RODRIGUEZ S.
Department of Telecommunications of the School of Mechanical and Electrical Engineering
National Polytechnical Institute of Mexico
U. Zacatenco, C.P. 07738, Mexico City
MEXICO

Abstract: - The description of the optimal Sampling – Reconstruction Procedure of Gaussian fields with jitter is given on basis of the conditional mean rule when the quantity of samples is limited. We analyse the case when the jitter is only presented in some samples. The jitter of each sample is independent and can have the same or different distribution law. We apply the statistical average to the conditional mean and conditional variance with respect to the random position of the samples. The surfaces of the optimal error reconstruction functions are calculated in the whole space domain. The results of calculations have clear interpretations.

Key-Words: - Sampling, Jitter, Gaussian fields and Error reconstruction function.

1 Introduction

In the present paper, the statistical description of the Sampling-Reconstruction Procedure (SRP) with jitter of Gaussian fields is described by means of the multidimensional conditional mean rule [1].

The description of the Sampling-Reconstruction Procedure (SRP) of both deterministic images and stochastic fields has been discussed in many publications [2]-[9]. On the other hand, the reconstruction problem with jitter of random processes began to be investigated 50 years ago [10]-[11]. Nowadays the interest in jitter has increased; see [12]-[15]. The differences between the present paper and the previous works are the following: i) we apply the conditional mean rule; ii) we analyse the optimal SRP algorithms with an arbitrary and limited number of samples; iii) we can consider the same or different jitter distribution for each sample; iv) we concretise the nature of the multidimensional probability density function (pdf) as the Gaussian pdf; v) we can evaluate the minimum reconstruction error surfaces on the whole space domain.

The present paper is a generalization of the application of the conditional mean rule for the multidimensional case - the Gaussian random fields. This approach has been applied in the statistical description of the optimal SRP with jitter of some stochastic processes [16]-[18].

In order to describe the Gaussian fields completely it is sufficient to use the usual statistical characteristics - the spatial covariance function and the mathematical expectation. In this work, the Gaussian random fields have the spatial exponential covariance function. The fields can be isotropic and un-isotropic. This feature is analytically reflected by equal or different radius of the covariance function.

As it is well known, the conditional mean rule [19] provides the minimum of the mean square error.

2 General Expressions

We restrict our analysis to the case when the field does not depend on time. It depends on two coordinate \( \xi(x,y) \). We study the case of the Cartesian coordinates \( x \) and \( y \). Hence, the field is represented by the infinity of surfaces as separate realizations. In the Gaussian case the field can be completely determined by its mathematical expectation \( \left< \xi(x,y) \right> = m(x,y) \) and the spatial covariance function \( K(x,x+\Delta x;y,y+\Delta y) \).

We require to know every two-dimensional random variable with the current coordinate \( \xi(x,y) \). If we fix an arbitrary set of \( N \) samples of the Gaussian field \( \Xi = \{\xi(x_1,y_1), \xi(x_2,y_2),...,\xi(x_N,y_N)\} \), we obtain a conditional field where all its realizations pass through all fixed points of the set \( \Xi \). The general expressions for the conditional mean matrix and the conditional covariance matrix of Gaussian multidimensional random variable are known [19].

We have the following expressions when we apply the conditional mean rule to Gaussian fields:

\[
\tilde{\mathbf{m}}(x,y) = m(x,y) + \sum_{i=1}^{N} \sum_{j=1}^{N} K(x,y;x_i,y_j) \times \\
\times a_y \left[ \xi(x_i,y_j) - m(x_i,y_j) \right],
\]

(1)
\[
\tilde{\sigma}^2(x, y) = \sigma^2(x, y) - \sum_{i=1}^{N} \sum_{j=1}^{N} K(x, y; x_i, y_j) \times a_{ij} K(x_j, y_j; x, y),
\]

(2)

where \(\sigma^2(x, y)\) is the unconditional variance of the initial field and \(a_{ij}\) represents to each element of the inverse covariance matrix.

We consider the general case of a stationary Gaussian fields with the mathematical expectation of the field \(m(x, y) = \langle \xi(x, y) \rangle = 0\) and the variance \(\sigma^2 = 1\). Then the formulas (1) and (2) will be:

\[
\tilde{m}(x, y) = \sum_{i=1}^{N} \sum_{j=1}^{N} K(x, y; x_i, y_j) a_{ij} \xi(x_j, y_j),
\]

(3)

\[
\tilde{\sigma}^2(x, y) = 1 - \sum_{i=1}^{N} \sum_{j=1}^{N} K(x, y; x_i, y_j) a_{ij} K(x_j, y_j; x, y).
\]

(4)

With the expression (3)-(4), we can get the surfaces of the optimal reconstruction function and the minimum reconstruction error function. These expressions are only valid when the samples locations of the field are known and have fixed positions. When the samples have jitter, it is necessary to consider the new position of these:

\[
\xi(\tilde{x}_i, \tilde{y}_j) = \xi(x_i + \eta_i, y_j + \phi_j),
\]

(5)

where \(\eta\) and \(\phi\) correspond to the random variables of the current coordinates, \(x\) and \(y\), of each sample. Thus, these random variable, \(\eta\) and \(\phi\), represent the jitter with known two-dimensional pdf. We consider the case of independent jitter. Therefore, each sample can have its own jitter distribution law.

Knowing the distribution of each random variable, we can obtain the general expressions of the average reconstruction function and the average error reconstruction function:

\[
\langle \tilde{m}(x, y) \rangle = \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{\xi}(\tilde{x}_i, \tilde{y}_j) a_{ij} \times K(x, y; \tilde{x}_i, \tilde{y}_j) \right\}_{\tilde{x}_i, \tilde{y}_i, \tilde{x}_j, \tilde{y}_j},
\]

(6)

\[
\langle \tilde{\sigma}^2(x, y) \rangle = 1 - \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} K(x, y; \tilde{x}_i, \tilde{y}_j) \times a_{ij} K(\tilde{x}_j, \tilde{y}_j; x, y) \right\}_{\tilde{x}_i, \tilde{y}_i, \tilde{x}_j, \tilde{y}_j},
\]

(7)

where the angle parentheses with the indices indicate the statistical average operation with respect to the random variables \(\tilde{x}\) and \(\tilde{y}\) of each sample.

The calculation of the statistical average in (6) and (7) is the principal and generally non-trivial operation in the description of the SRP with jitter. Below we give some examples of these calculations.

### 3 Examples

Now we apply the general expressions for some examples. The covariance function, the quantity of samples, their Cartesian coordinates, the jitter distribution and the values of the samples must be known.

**Fig. 1.** The average error reconstruction function for the isotropic Gaussian field in the presence of jitter.

**Fig. 2.** The contour line of the surface of the error reconstruction function for the isotropic case.
The average error reconstruction function for the un-isotropic Gaussian field in the presence of jitter.

In this case, we use the exponential covariance function which describes the Markov Gaussian field.

\[
K(\Delta x, \Delta y) = \sigma^2 \exp[-\alpha_x |\Delta x| + \alpha_y |\Delta y|],
\]

where \(\alpha_x\) and \(\alpha_y\) are the inverse of the covariance radius on \(x\)-axis and \(y\)-axis, respectively.

If \(\alpha_x = \alpha_y\), then the Gaussian field is isotropic. In other cases the fields are un-isotropic.

In this paper, the jitter distribution is described by McFadden pdf. The two-dimensional McFadden pdf has restricted limits in \(\pm 1\) [20]. For our goals, we adapt the original expression in order to have arbitrary limits. Omitting some derivations, we give here the result:

\[
w(\eta, \varphi) = \left(\frac{\lambda}{a^2\pi}\right)(1 - \rho^2)^{\frac{d-1}{2}} \times
\left[1 - \rho^2 - \frac{2\rho\varphi - (\eta^2 + \varphi^2)}{a^2}\right]^{\frac{d-1}{2}},
\]

where \(|\eta| < a\) and \(|\varphi| < a\); \(\rho\) is the covariance between \(\eta\) and \(\varphi\), and \(\lambda\) is the parameter of the distribution shape. In all calculations the value of \(\lambda = 1\). With this value, the McFadden’s pdf gives the uniform distribution.

In order to illustrate the above-mentioned procedure, we consider four samples which have the following coordinates: \(x_1 = y_1 = -0.5\); \(x_2 = -0.5\), \(y_2 = 0.5\); \(x_3 = y_3 = 0.5\); \(x_4 = 0.5\), \(y_4 = -0.5\). The covariance radius is the same, that is \(\alpha_x = \alpha_y = 1\). As it is above-mentioned, the jitter distribution is uniform and independent. Only the first sample does not have jitter. The deterministic position \(\xi(x_i, y_i)\) of each sample is the centre of the jitter distribution which is limited by a radius equal to: 0.25 for the second and third sample; and 0.1 for the fourth sample.

The surface of the average error reconstruction function \(\langle \tilde{\sigma}^2(x, y) \rangle\) and the contour line of this surface are shown in Fig. 1 and Fig. 2, respectively. We can observe that the error reconstruction function is zero when the samples do not have jitter, just as the first sample, since in this location we know the exact value of the realization. On the other hand, in the presence of jitter the reconstruction error in the environs of the other samples is different to zero. The reconstruction error with jitter tends to be smaller when the variance of the jitter distribution decreases, just as it is observed in the possible position of the fourth sample.

In the centre of the samples \(\xi(0,0)\), there is a maximum error. This maximum increases if the coordinates of the current samples are put at a bigger distance.

Let us change the covariance radius. In this case \(\alpha_x = 2\) and \(\alpha_y = 1\). The other parameters stay equal. The new surface of the average error reconstruction function and their corresponding contour lines are shown in Fig. 3 and Fig. 4.

In Fig. 4, we can notice that the contours are narrower than in Fig. 2 and the error levels are bigger. The process becomes more chaotic on \(x\)-axis because the covariance radius decreases in \(1/\alpha_x = 1/2\).
4 Conclusion

Based on the conditional mean rule the statistical description of reconstruction with jitter of Gaussian fields is described. The case of independent jitters is considered with the two-dimensional McFadden’s distribution when the number of samples is arbitrary. The surfaces of the error reconstruction function are calculated when one of the samples is fixed and the other have jitter, and when some samples have the same or different jitter distribution. The applied method of the investigation provides the possibility to describe the jitter in Gaussian fields with high precision.

References:


