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A Parallel Algorithm for Nonlinear Multi-commodity Network Flow Problem

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Abstract: - In this paper, we propose a method to solve the Nonlinear Multicommodity Network Flow (NMNF) Problem. We have combined this method with a projected-Jacobi (PJ) method and a duality based method possessing decomposition effects. With the decomposition, our method can be parallel processed and is computationally efficient. We have tested our method on several examples of NMNF problem and obtained some successful results.
Key-Words: - NMNF problem, projected-Jacobi method, duality based method, decomposition.

1 Introduction

There are many large practical systems formed by the network-like mesh-interconnected buses or nodes through tier-lines. For example, the Network system is formed by a number of nodes interconnected with each other through tier-lines.

Nonlinear multicommodity network flow (NMNF) problem is computationally difficult because of their large dimension and nonlinearity and has important applications to traffic assignment [1]-[4] and data network routing [5]-[8]. Such problems are typical convex programming problems, and the NMNF problem solution techniques mostly originate from nonlinear programming algorithm that are especially to exploit the linear constraint structure with various approaches [9]-[17]. Recently, an efficient method developed in [18], Dual Projected Pseudo Quasi-Newton method takes advantage of the special structure of inequality constraints and network sparsity. The method in [18] abbreviated DPPQN have achieved a dramatic speed-up ratio over a typical method for NMNF problem. Considering the trend about the number of commodities in the networks is increasing in NMNF problem. In this paper, we use the framework of the DPPQN method and propose a parallel algorithm to solve the NMNF with many commodities problem. Furthermore, we implement the proposed method in two real Processor-Network systems and demonstrate the computational efficiency through the simulation results.

The paper is organized in the following manner. Section 2 states the problem of the NMNF problem. Section 3 presents the method combining the projected-Jacobi and the parallel dual-type methods for solving NMNF problems. The simulation and conclusion are given in Section 4.

2 Statement of the NMNF Problem

We first introduce the notation for the K-commodity NMNF problem in the following:

k : denotes the index of the commodity of the network system.

K : denotes the total number of the commodity of the network system.

i : denotes the index of the node of the network system.

I : denotes the total number of the node of the network system.

$L_{i\bar{j}}$: denotes the set of the node j which connected with node i, from node i to node j.

$L_{\bar{h}i}$: denotes the set of node h which connected with node i, from node h to node i.

f_{ij}^{k} : denotes the flow over the branch (i,j) with the destination node (commodity) k.

r_i^k : denotes the flow requirement at node i with the destination node k.

(i, j) : denotes the branch from node i to node j.

(h,i) : denotes the branch from node h to node i.

B : denotes the set of all network branches in the network system.

$|\cdot|$: denotes the cardinality of the set $\cdot$.

$F_k(\sum_{i=1}^{I}\sum_{j\in L_{\vec{i}j}} f_{ij}^k)$: denotes the total branch cost associated with total branch flows $\sum_{i=1}^{I}\sum_{j\in L_{\vec{i}j}} f_{ij}^k$ for the commodity k, and $F_k(\cdot)$ is a convex function in $(\cdot)$.

$\sum_{k=1}^{K} F_k(\sum_{i=1}^{I}\sum_{j\in L_{\vec{i}j}} f_{ij}^k)$: represents the sum of all the commodity of all the branch costs of the network system.

$\Delta(\cdot)$: the increment of the vector $(\cdot)$.

$(\cdot)^{\mathrm{T}}$: denotes the transpose of the vector $(\cdot)$.

diag[□]: a diagonal matrix formed by the diagonal terms of the matrix [□].

$\nabla^2_{f_{ij}^k} F_k(t)$: denotes the Hessian of F_k with respect to f_{ij}^k evaluated at $f_{ij}^k(t)$.

$\frac{\partial^2 F_k(t)}{\partial f_{ij}^{k^2}}$: denotes the diagonal entry corresponding to the branch (i,j) of the matrix $\nabla^2_{f_{ij}^k} F_k(t)$.

$\nabla_{f_{ij}^k} F_k(t)$: denotes the gradient of F_k with respect to f_{ij}^k evaluated at $f_{ij}^k(t)$.

$\frac{\partial F_k(t)}{\partial f_{ij}^k}$: denotes the component corresponding to the branch (i,j) of the vector $\nabla_{f_{ij}^k} F_k(t)$.

f: denotes the vector of all $f_{ij}^k(t), \forall (i,j)\in B, k=1,2,...,K$ and $f=[f_1^1, f_2^1,..., f_{|B|}^1, f_1^2, f_2^2,..., f_{|B|}^2,..., f_1^K, f_2^K,..., f_{|B|}^K]^T$.

$\Delta f(t)$: denotes the vector of all $\Delta f_{ij}^k(t), \forall (i,j)\in B, k=1,2,...,K$ and $\Delta f=[\Delta f_1^1,\Delta f_2^1,...,\Delta f_{|B|}^1,\Delta f_1^2,\Delta f_2^2,..., \Delta f_{|B|}^2,...,\Delta f_1^K,\Delta f_2^K,...,\Delta f_{|B|}^K]^T$

QP: quadratic programming.

$\Omega(f)$: denotes the set of f satisfying the inequality constraints of flow.

t, w: iteration index.

α, β: step size.

$\varepsilon_1, \varepsilon_2$: predetermined positive real values.

$\phi(\lambda)$: the dual function of constrained QP sub-problem.

$\phi^u(\lambda)$: the dual function of unconstrained QP sub-problem.

λ: denotes the vector of the Lagrange multipliers, $\lambda_i^k, i=1,2,...,I, k=1,2,...,K$ and $\lambda=[\lambda_1^1,\lambda_2^1,...,\lambda_I^1,\lambda_1^2,\lambda_2^2,...,\lambda_I^2,...,\lambda_1^K,\lambda_2^K,...,\lambda_I^K]^T$.

$\|(\cdot)\|_\infty$: denotes the infinite norm of the vector $(\cdot)$.

The NMNF with *K*-commodity problem can be stated as follows:

$$\min F(f_{ij}^k)=\sum_{k=1}^{K} F_k(\sum_{i=1}^{I}\sum_{j\in L_{\vec{ij}}} f_{ij}^k) \tag{1a}$$

subject to

$$\sum_{j\in L_{\vec{ij}}} f_{ij}^k - \sum_{h\in L_{\vec{hi}}} f_{hi}^k - r_i^k = 0, i=1,2,...,I, k=1,2,...,K \tag{1b}$$

$$0\le f_{ij}^k, \forall (i,j)\in B, k=1,2,...,K. \tag{1c}$$

The object of NMNF with many commodities problem is to find an optimal flow solution that satisfies the flow balance constraints (1b), and the nonnegative constraints (1c), while minimizing the objective function (1a).

3 Solution Method

3.1 The Projected-Jacobi (PJ) Method

The projected-Jacobi method uses the following iterations to solve the NMNF problem given in Eqs.(1a)-(1c),

$$f(t+1)=f(t)+\alpha(t)\Delta f^*(t), \tag{2}$$

where $f(t)$ denotes the vector of all $f_{ij}^k(t), \forall (i,j)\in B, k=1,2,...,K$, at iteration t and the $|B|\cdot K\times 1$ column vector f is described in the following:
$f=[f_1^1, f_2^1,..., f_{|B|}^1, f_1^2, f_2^2,..., f_{|B|}^2,..., f_1^K, f_2^K,..., f_{|B|}^K]^T$; $\alpha(t)>0$ is a step-size determined by the centralized Amijo rule [19], and $\Delta f^*(t)$ denotes the vector of all $\Delta f_{ij}^{k^*}, \forall (i,j)\in B, k=1,2,...,K,$ the $|B|\cdot K\times 1$ column increment vector Δf is of the following form,

$\Delta f = [\Delta f_1^1, \Delta f_2^1, ..., \Delta f_{|B|}^1, \Delta f_1^2, \Delta f_2^2, ..., \Delta f_{|B|}^2, ...,.$
$\Delta f_1^K, \Delta f_2^K, ..., \Delta f_{|B|}^K]^T$, that solves the following quadratic programming (QP) sub-problems:

$$\min_{\Delta f_{ij}} \sum_{i=1}^{I} \sum_{j \in L_{i\bar{j}}} [\frac{1}{2} \Delta f_{ij}(t)^T D_{ij}(t) \Delta f_{ij}(t) + \nabla_{f_{ij}} F(t)^T \Delta f_{ij}(t)] \tag{3a}$$

subject to

$$\sum_{j \in L_{i\bar{j}}} (f_{ij}^k(t) + \Delta f_{ij}^k(t)) - \sum_{h \in L_{\bar{h}i}} (f_{hi}^k(t) + \Delta f_{hi}^k(t)) - r_i^k = 0, \; i = 1,2,...,I, k = 1,2,...,K \tag{3b}$$

$$0 \le f_{ij}^k(t) + \Delta f_{ij}^k(t), \forall (i,j) \in B, k = 1,2,...,K \tag{3c}$$

where $D_{ij}(t), \forall (i,j) \in B$ is a $|B| \cdot K \times |B| \cdot K$ block diagonal matrix and $D_{ij}(t) = diag[\nabla_{f_{ij}}^2 F(t)] + \frac{1}{2}\delta I$, and
$\nabla_{f_{ij}} F(t)^T = [\nabla_{f_{ij}^1} F_1(t)^T, \nabla_{f_{ij}^2} F_2(t)^T, ..., \nabla_{f_{ij}^K} F_K(t)^T]$, $\nabla_{f_{ij}}^2 F$ and $\nabla_{f_{ij}} F^T$ denote the Hessian and the gradient of F with respect to f_{ij} for the branch (i,j), $\forall (i,j) \in B$, respectively, and $\Delta f_{ij}(t)$ denotes the vector of all $\Delta f_{ij}^k(t), \forall (i,j) \in B$, k =1,2,…,K. And δ is a small real number but large enough to make D_{ij} positive definite.

Let D_{ij}^k be the block diagonal term of the block diagonal matrix D_{ij}, and

$$D_{ij}(t) = \begin{bmatrix} D_{ij}^1(t) & 0 & \cdots & 0 \\ 0 & D_{ij}^2(t) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & D_{ij}^K(t) \end{bmatrix} + \frac{1}{2}\delta I \tag{4}$$

where $D_{ij}^k(t)$ is a $|B| \times |B|$ diagonal matrix corresponding to commodity k, and $D_{ij}^k(t) = diag[\nabla_{f_{ij}^k}^2 F_k(t)] + \frac{1}{2}\delta I$, and the diagonal entry corresponding to the branch (i, j) is $\frac{\partial^2 F_k(t)}{\partial f_{ij}^{k^2}}$. From Eqs.(1a)-(1c) and since D_{ij} is a block diagonal matrix, we can rewrite Eqs.(3a)-(3c) as

$$\min_{\Delta f_{ij}} \sum_{k=1}^{K} \sum_{i=1}^{I} \sum_{j \in L_{i\bar{j}}} [\frac{1}{2} \Delta f_{ij}^k(t)^T D_{ij}^k(t) \Delta f_{ij}^k(t) + \nabla_{f_{ij}^k} F_k(t)^T \Delta f_{ij}^k(t)] \tag{5a}$$

subject to

$$\sum_{j \in L_{i\bar{j}}} (f_{ij}^k(t) + \Delta f_{ij}^k(t)) - \sum_{h \in L_{\bar{h}i}} (f_{hi}^k(t) + \Delta f_{hi}^k(t)) - r_i^k = 0 i = 1,2,...,I, k = 1,2,...,K \tag{5b}$$

$$0 \le f_{ij}^k(t) + \Delta f_{ij}^k(t) \;, \forall (i,j) \in B, k = 1,2,\ldots,K \tag{5c}$$

3.2 The Duality based Parallel Method

Let Ω denote the set of Δf satisfying the inequality constraints of flows, that is, $f + \Delta f \in \Omega$ represents $f + \Delta f$ satisfying (3c). Therefore, we see that if $f + \Delta f \in \Omega$, then $f + \alpha \Delta f \in \Omega$ for $\forall \alpha$, $0 < \alpha \le 1$. Hence, we can rewrite (5a)-(5c) as

$$\min_{(f(t)+\Delta f(t)) \in \Omega} \sum_{k=1}^{K} \sum_{i=1}^{I} \sum_{j \in L_{i\bar{j}}} [\frac{1}{2} \Delta f_{ij}^k(t)^T D_{ij}^k(t) \Delta f_{ij}^k(t) + \nabla_{f_{ij}^k} F_k(t)^T \Delta f_{ij}^k(t)] \tag{6a}$$

subject to

$$b_i^k(t) + \sum_{j \in L_{i\bar{j}}} \Delta f_{ij}^k(t) - \sum_{h \in L_{\bar{h}i}} \Delta f_{hi}^k(t) = 0, \quad i = 1,2,...,I, k = 1,2,...,K \tag{6b}$$

where

$$b_i^k(t) = \sum_{j \in L_{i\bar{j}}} f_{ij}^k(t) - \sum_{h \in L_{\bar{h}i}} f_{hi}^k(t) - r_i^k, \quad i = 1,2,...,I, k = 1,2,...,K,$$

$$\Omega \equiv \{\Delta f_{ij}^k \,|\, 0 \le f_{ij}^k(t) + \Delta f_{ij}^k(t), \forall (i,j) \in B, k = 1,2,...,K\}.$$

The dual problem of the QP sub-problems Eqs.(6a),(6b) is

$$\max_{\lambda} \phi(\lambda) \tag{7}$$

where the dual function,

$$\phi(\lambda) = \min_{f(t)+\Delta f(t)\in\Omega} \sum_{k=1}^{K}\sum_{i=1}^{I}\sum_{j\in L_{ij}}[\frac{1}{2}\Delta f_{ij}^{k\,T} D_{ij}^{k}(t)\Delta f_{ij}^{k}$$
$$+\nabla_{f_{ij}^{k}} F_k(t)^T \Delta f_{ij}^{k}] + \sum_{k=1}^{K}\sum_{i=1}^{I}\lambda_i^k[b_i^k(t) \tag{8}$$
$$+\sum_{j\in L_{ij}}\Delta f_{ij}^{k} - \sum_{h\in L_{\bar{h}i}}\Delta f_{hi}^{k}]$$

is a function of λ, which is the vector of Lagrange multipliers $\lambda_i^k, i=1,2,...,I, k=1,2,...,K$, and $\lambda=[\lambda_1^1,\lambda_2^1,...,\lambda_I^1,\lambda_1^2,\lambda_2^2,...,\lambda_I^2,...,\lambda_1^K,\lambda_2^K,...,\lambda_I^K]^T$. It should be noticed that for the sake of clarity, we ignore the index t from the vector of the variable $\Delta f(t)$ in (8).

We define g_i^k be the vector of equality constraint function on the LHS of (6b). Therefore,

$$g_i^k = b_i^k(t) + \sum_{j\in L_{ij}}\Delta f_{ij}^{k} - \sum_{h\in L_{\bar{h}i}}\Delta f_{hi}^{k},$$

$i=1,2,...,I, k=1,2,...,K$, and the $I\cdot K\times 1$ column vector g can be described in the following:

$$g=[g_1^1,g_2^1,...,g_I^1,g_1^2,g_2^2,...,g_I^2,...,g_1^K,g_2^K,...,g_I^K]^T$$

Since

$$f=[f_1^1,f_2^1,...,f_{|B|}^1,f_1^2,f_2^2,...,f_{|B|}^2,...,f_1^K,f_2^K,...,f_{|B|}^K]^T$$

and

$$\Delta f=[\Delta f_1^1,\Delta f_2^1,...,\Delta f_{|B|}^1,\Delta f_1^2,\Delta f_2^2,...,\Delta f_{|B|}^2,...,$$
$$\Delta f_1^K,\Delta f_2^K,...,\Delta f_{|B|}^K]^T,$$

we see that ∇g, the gradient of g with respect to Δf is an $I\cdot K\times|B|\cdot K$ block diagonal matrix shown in the following:

$$\nabla g = \begin{bmatrix} \nabla g^1 & 0 & \cdots & 0 \\ 0 & \nabla g^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nabla g^K \end{bmatrix} \tag{9}$$

Each ∇g^k is an $I\times|B|$ matrix corresponding to commodity k in the network system.

The duality based method uses the following iteration to solve (7):

$$\lambda^k(w+1) = \lambda^k(w) + \beta(w)\Delta\lambda^k(w), k=1,2,...,K \tag{10}$$

where w denotes the iteration index, $\beta(w)>0$ is a step-size determined according to the centralized Armijo's rule [19] and $\Delta\lambda^k=[\Delta\lambda_1^k,\Delta\lambda_2^k,...,\Delta\lambda_I^k]^T$. Furthermore,

$$\Delta\lambda=[\Delta\lambda_1^1,\Delta\lambda_2^1,...,\Delta\lambda_I^1,\Delta\lambda_1^2,\Delta\lambda_2^2,...,\Delta\lambda_I^2,...,$$
$$\Delta\lambda_1^K,\Delta\lambda_2^K,...,\Delta\lambda_I^K]^T$$

can be obtained by solving the following linear equations:

$$\nabla^2\phi^u(\lambda(w))\Delta\lambda(w) + \nabla\phi(\lambda(w)) = 0 \tag{11}$$

where the column vector $\nabla\phi(\lambda(w))$ with dimension $I\cdot K\times 1$ is the gradient of $\phi(\lambda)$ with respect to λ at $\lambda(w)$, and $\nabla\phi(\lambda(w))$ is also the vector of all $\nabla_{\lambda_i^k}\phi(\lambda(w))$, $i=1,2,\ldots,I$, $k=1,2,\ldots,K$, and the matrix $\nabla^2\phi^u(\lambda(w))$ with dimension $I\cdot K\times I\cdot K$ denotes the Hessian of the unconstrained dual function $\phi(\lambda)$.

From the Duality theorem [19], we see that the $\nabla\phi(\lambda(w))$ and $\nabla^2\phi^u(\lambda(w))$ can be computed by

$$\nabla_{\lambda_i^k}\phi(\lambda(w)) = b_i^k(t) + \sum_{j\in L_{ij}}\Delta\hat{f}_{ij}^{k} - \sum_{h\in L_{\bar{h}i}}\Delta\hat{f}_{hi}^{k},$$
$$i=1,2,\ldots,I,\ k=1,2,\ldots,K, \tag{12}$$

$$\nabla^2\phi^u(\lambda(w)) = -\nabla g D(t)^{-1}\nabla g^T. \tag{13}$$

The $\Delta\hat{f}$ in (12) is the optimal solution of the constrained minimization problem on the RHS of (8).

To compute $\nabla_{\lambda_i^k}\phi(\lambda(\mathrm{w}))$, we need to solve the minimization problem on the RHS of Eq.(8) to obtain $\Delta\hat{f}_{ij}^{k}$.

This can be achieved by the following two-stage algorithm:

Stage 1: Solve the following unconstrained minimization problem:

$$\phi^u(\lambda) = \min \sum_{k=1}^{K} \sum_{i=1}^{I} \sum_{j \in L_{ij}} [\frac{1}{2} \Delta f_{ij}^{k\,T} D_{ij}^k(t) \Delta f_{ij}^k + \nabla_{f_{ij}^k} F_k(t)^T \Delta f_{ij}^k] + \sum_{k=1}^{K} \sum_{i=1}^{I} \lambda_i^k [b_i^k(t) + \sum_{j \in L_{ij}} \Delta f_{ij}^k - \sum_{h \in L_{hi}^-} \Delta f_{hi}^k] \tag{16}$$

By using a gradient method [20] to obtain an approximately solution for the entry of branch (i, j) in each commodity k, and each $\Delta \tilde{f}_{ij}^k$ can be computed in parallel described in the following:

$$\Delta \tilde{f}_{ij}^k(\lambda(w)) = -\frac{\partial^2 F_k(t)}{\partial f_{ij}^{k\,2}} (\frac{\partial F_k(t)}{\partial f_{ij}^k} + \lambda_i^k(w) - \lambda_j^k(w)), \quad \forall (i,j) \in B, k = 1,2,...,K \tag{17}$$

Stage 2: Project $\Delta \tilde{f}_{ij}^k, \forall (i,j) \in B, k = 1,2,...,K,$ the solution obtained from Stage 1, onto Ω. The resulting projection is $\Delta \hat{f}(\lambda(w))$. It is the solution of the minimization problem on RHS of Eq.(8). Due to the decomposition effect, the resulting projection $\Delta \hat{f}_{ij}^k, \forall (i,j) \in B, k = 1,2,...,K$, can be parallel computed by the following analytical formula,

$$\Delta \hat{f}_{ij}^k(\lambda(w)) = \begin{cases} -f_{ij}^k(\lambda(w)) & if \;\; f_{ij}^k(\lambda(w)) + \Delta \tilde{f}_{ij}^k(\lambda(w)) \le 0 \\ \Delta \tilde{f}_{ij}^k(\lambda(w)) & otherwise \end{cases} \tag{18}$$

It should be noticed that the computations in Stage 2 are the comparison check shown in Eg.(18).

3.3 The Complete method for solving Nonlinear Multicommodity Network Flow problem

Our method for solving Nonlinear Multicommodity Network Flow problem is using PJ method Eq.(2) where $\Delta f^*(t)$ is the solution of the QP sub-problem Eqs.(3a)-(3c). The proposed parallel dual-type method uses Eq.(10) to solve Eq.(7), the dual problem of QP sub-problem, instead of solving Eqs.(3a)-(3c) directly. The $\Delta\lambda^k(w)$ in Eq.(10) is obtained from solving Eq.(14) using linear programming technique. And can be parallel processed. The $\Delta \hat{f}_{ij}^k, \forall (i,j) \in B, k = 1,2,...,K$ is needed to set up $\nabla_{\lambda^k} \phi(\lambda(w))$ and can be computed using the two-stage method. Consequently, the duality based method converges to optimal solution λ^* and the solution $\Delta \hat{f}$ of Eq.(8) with $\lambda = \lambda^*$ is Δf, the solution of Eqs.(3a)-(3c).

4 Simulation and Conclusion

We have developed a parallel algorithm for solving NMNF problem. The method combines with the projected-Jacobi method and a duality based method possessing commodity decomposition effects. With the decomposition, the proposed method can be parallel processed. We have tested our method on several examples of NMNF problem and obtained some successful results.

References:

[1] S. C. Dafermos and F. T. Sparrow, " The traffic assignment problem for a general network, " *J. Res. Nat. Bur. Stand. B,* 1969, pp. 395-412.

[2] T. Leventhal, G. Nemhauser, and L. Trotter, Jr., " A column generation algorithm for optional traffic assignment, " *Trans. Sci.,* Vol.7, No.2 1973, pp. 168-176.

[3] S.Nguyen, " An algorithm for the traffic assignment problem, " *Trans. Sci.,* Vol.8, No.3, 1974, pp. 203-216.

[4] R. B. Potts and R. M. O liver, *Flows in transportation network.* Academic Preaa, New York, London, 1972.

[5] D. G. Cantor and M. Gerla, " Optimal routing in a packet switched computer network, " *IEEE Trans. Comput.* Vol.*C23*, No.10, 1974, pp. 1062-1068.

[6] H. Frank and W. Chou, "Routing in computer networks, " *Networ*ks, Vol.1, No.2, 1971, pp. 99-112.

[7] L. Fratta, M. Gerla, and L. Kleinrock, " The flow deviation method: An approach to store-and-forward communication network design, " *Networks* Vol.3, No.2, 1973, pp. 97-113.

[8] B. Yaged, " Minimum cost routing for static network models, " *Networks* Vol.1, No.2, 1971, pp. 139-172.

[9] S. C. Dafermos, " An extended traffic assignment model with applications to two-way traffic, "

*Trans. Sci.*Vol.*5* , 1971, pp. 336-389.

[10] B. Golden, " A minimum-cost multicommodity network flow problem concerning imports and exports, " *Networks,* Vol.5, No.4, 1975, pp. 331-356.

[11] A. M. Geoffrion and G. W.Graves, " Multi-commodity distribution system design by benders decomposition, " *Mgmt. Sci.* Vol.20, No. 5, 1974, pp. 822-884.

[12] M. A. Hall and E. L. Peterson, " Highway traffic equilibrium analyzed via geometric programming, " *Traffic Equilibrium Methods* (M. A. Florian, Ed.). Springer-Verlag, New York, 1976, pp. 53-105.

[13] J. L. Kennington and R. V. Helgason, *Algorithms for network programming*. Wiley, New York, 1980.

[14] R. W. Klessing, " An algorithm for nonlinear multicommodity flow problems, " *Networks*, Vol.4, No. 4, 1974, pp. 343-355.

[15] L. J. Leblanc, E. K. Morlok, and W. P. Pierskalla, " An accurate and efficient approach to equilibrium traffic assignment on congested networks. Interactive graphics & trans. systems planning, " *Trans. Res. Rec,.* Vol.491, 1974, pp. 12-33.

[16] M. Schwartz and C. K. Cheung, *The gradient projection algorithm for multiple routing in message-switched networks. Preprin* , 1975.

[17] J. G. Wardrop, " Some theoretical aspects of road traffic research, " *Proc. Inst. Civ. Eng.* Vol.2, No.1, 1952, pp. 325-378.

[18] S.-Y. Lin and C.-H. Lin, " A Computational efficient method for nonlinear multicommodity network problems, " *Networks*, Vol.29, 1997, pp. 255-244.

[19] D. Luenberger, *Linear and nonlinear programming, 2nd ed*. Addison-Wesley Reading, MA,1984.

[20] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Prentice-Hall, London. 1989.