On the homogenization of a nonlinear problem arising in elasticity

CLAUDIA TIMOFTE
Faculty of Physics
University of Bucharest
P.O. Box MG-11, Bucharest-Magurele
ROMANIA

Abstract: The effective behavior of the solutions of Signorini's nonlinear problems in periodically perforated regular materials is analyzed. Such boundary-value problems involve the existence of two distinct sources of oscillations, one coming from the geometrical structure of the domain and the other one from the fact that the medium is also an heterogeneous one. In the case of a critical size of the perforations, the effective problem is a Dirichlet one, given by a new operator, which is the sum of a standard homogenized operator and of two extra zero-order terms generated by the periodic geometrical structure of the heterogeneous domain and the nonlinearity of this problem.

Key-Words: Signorini's problem, homogenization, regular material, variational inequality.

1 Introduction

The goal of this paper is to get the effective behavior of Signorini's type-like problems in periodically perforated domains. Such boundary-value problems are relevant in nonlinear elasticity.

Let Ω be an open bounded set in $\mathbb{R}^n$ $(n \geq 3)$ and let us perforate it by holes. As a result, we get an open set Ω^ε, called the *perforated domain,* ε representing a small parameter related to the characteristic size of the perforations. Let us consider a family of inhomogeneous media occupying the region Ω, parameterized by ε and represented by $n \times n$ matrices $A^\varepsilon(x)$ of real-valued coefficients defined on Ω.

With Ω^ε we associate the nonempty closed convex subset of $H^1(\Omega^\varepsilon)$:

$$K^\varepsilon = \left\{ v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \partial\Omega,\ v \geq 0 \text{ on } S^\varepsilon \right\},$$

where S^ε is the boundary of the holes and $\partial\Omega$ is the external boundary of Ω. Our main motivation is to study the asymptotic behavior of the solution of the following variational problem in Ω^ε :

$$\begin{cases} \text{Find } u^\varepsilon \in K^\varepsilon \text{ such that} \\ \int\limits_{\Omega^\varepsilon} g(u^\varepsilon)(v^\varepsilon - u^\varepsilon)dx + \int\limits_{\Omega^\varepsilon} A^\varepsilon D u^\varepsilon D(v^\varepsilon - u^\varepsilon)dx \\ \geq \int\limits_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)dx, \quad \forall v^\varepsilon \in K^\varepsilon, \end{cases} \tag{1}$$

where f is a given function in $L^2(\Omega)$ and g is a continuously differentiable function, monotonously non-decreasing and such that $g(0) = 0$.

We shall consider *periodic structures* defined by $A^\varepsilon(x) = A(\frac{x}{\varepsilon})$. Here, $A = A(y)$ is a continuous matrix-valued function on $\mathbb{R}^n$ which is Y-periodic and $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ is the basic cell. From a geometrical point of view, we consider periodic perforated structures obtained by removing periodically from Ω, with period ε, an elementary hole T which has been appropriated rescaled. We use the symbol # to denote periodicity properties. We also suppose that $A \in L^{\infty}_{\#}(\Omega)^{n\times n}$, A is a symmetric matrix and for some $0 < \alpha < \beta$, $\alpha |\xi|^2 \leq A(y)\xi \cdot \xi \leq \beta |\xi|^2$, $\forall \xi$, $y \in \mathbb{R}^n$. Moreover, we shall assume that the material represented by the matrix A is a regular one, i.e. $\mathrm{div} A = 0$ in $\mathbb{R}^n$.

Under the above hypotheses and the conditions fulfilled by K^ε, it is well-known by a classical existence and uniqueness result of J.L. Lions and G. Stampacchia (see [6]) that (1) is a well-posed problem.

We will just focus on the only case which gives a real interaction between the above mentioned sources of oscillation, i.e. the so-called *critical case*, for which the size of the perforations is of order $\varepsilon^{n/(n-2)}$. In this critical case, the limit of u^ε is the solution of a Dirichlet problem in Ω associated with a new operator which is the sum of the standard homogenized one and two extra term coming from the geometry and the nonlinearity of the problem. More precisely, the solution u^ε converges to the unique solution of the following variational problem:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ \int\limits_\Omega g(u)v dx + \int\limits_\Omega \overline{A} Du Dv dx - \\ \langle \mu_0 u^-, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int\limits_\Omega f v dx \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{2}$$

Here, $\overline{A}$ is the mean value of the matrix A and $\mu_0 = \inf\limits_{\zeta \in H^1(\mathbb{R}^n)} \left\{ \int\limits_{\mathbb{R}^n} A(0) D\zeta D\zeta dx \mid \zeta \geq 1 \text{ q.e. on } T \right\}$.

In this limit problem the oscillations coming from the periodic heterogeneous structure of the medium are reflected by the presence of the homogenized matrix $\overline{A}$ and those due to the critical size of the holes are reflected by the appearance of the zero order term $\mu_0 u^-$. The other ingredient contained in this limit problem is the spreading effect of the unilateral condition $u^\varepsilon \geq 0$ on S^ε, which can be seen by the fact that μ_0 only charges the negative part of u.

The method we used is the so-called energy method introduced by L. Tartar (see [7]-[8]) for studying homogenization problems, coupled with monotonicity methods and results from the theory of semilinear problems .

This paper is a generalization of the well-known work of D. Cioranescu and F. Murat [3]. In their article, the authors deal with the asymptotic behavior of solutions of Dirichlet problems in perforated domains, showing the appearance of a "strange" extra-term as the period of the perforations tends to zero and the holes are of critical size. In [5], this framework was generalized to a class of Signorini's problem in heterogeneous media, involving just a positivity condition imposed on the boundary of the holes. In the present paper, we generalize some of the results obtained in [5], by considering also the nonlinear term given by the function g, which gives rise in the limit to a new zero-order extra-term.

The structure of our paper is as follows: in Chapter 2, after some necessary preliminaries, we formulate the main convergence result, the proof of which is given in Chapter 3.

2 Preliminaries and the main result

Let Ω be a bounded connected open subset of $\mathbb{R}^n$ ($n \geq 3$), with $\partial\Omega$ of class C^2 and let T be another open bounded subset of $\mathbb{R}^n$, with a smooth boundary ∂T (of class C^2). We shall refer to T as being *the elementary hole*. We assume that 0 belongs to T and that T is star-shaped with respect to 0. Since T is bounded, to simplify matters, without loss of generality, we shall assume that $\overline{T} \subset Y$, where $Y = (-\frac{1}{2}, \frac{1}{2})^n$ is the representative cell in $\mathbb{R}^n$.

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero and let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous map, which will represent the size of the holes. We shall just focus on the case in which the size $r(\varepsilon)$ of the holes is exactly of the order of $\varepsilon^{n/(n-2)}$. We shall refer to this case as being the "critical" one.

For each ε and for any integer vector $\mathbf{i} \in \mathbb{Z}^n$, we shall denote by $T^\varepsilon_{\mathbf{i}}$ the translated image of $r(\varepsilon)T$ by the vector $\varepsilon\mathbf{i}$, $\mathbf{i} \in \mathbb{Z}^n$. Also, let us denote by T^ε the set of all the holes contained in Ω, i.e. $T^\varepsilon = \bigcup \left\{ T^\varepsilon_{\mathbf{i}} \mid \overline{T^\varepsilon_{\mathbf{i}}} \subset \Omega,\ \mathbf{i} \in \mathbb{Z}^n \right\}$. Set $\Omega^\varepsilon = \Omega \setminus \overline{T^\varepsilon}$. Hence, Ω^ε is a periodically perforated domain with holes of the size $r(\varepsilon)$. All of them have the same shape, the distance between two adjacent holes is of order ε and they do not overlap. Let $S^\varepsilon = \cup\{\partial T^\varepsilon_{\mathbf{i}} \mid \overline{T^\varepsilon_{\mathbf{i}}} \subset \Omega, \mathbf{i} \in \mathbb{Z}^n\}$. So, $\partial\Omega^\varepsilon = \partial\Omega \cup S^\varepsilon$. Moreover, we denote by χ_ω the characteristic function of a set ω and we set $Y^* = Y \setminus \overline{T},\ \theta = \dfrac{|Y^*|}{|Y|}$.

As already mentioned in Introduction, we are interested in studying the behavior of solutions, in such perforated domains, of variational inequalities with highly oscillating obstacles constraints of the form (1).

The domain Ω^ε will be filled in by a regular material in the sense of [5], i.e we assume that $\text{div}A = 0$ in $\mathbb{R}^n$.

We shall consider that the function g in (1) is a continuously differentiable function, monotonously non-decreasing and such that $g(0) = 0$ and we shall take $G(v) = \int\limits_0^v g(s)ds$. Finally, we assume that there exist $C \geq 0$ and an exponent q, with $0 \leq q < n/(n-2)$, such that

$$\left|\frac{dg}{dv}\right| \leq C(1 + |v|^q).$$

For example, we can take g to be a linear function or we can consider nonlinearities of the Langmuir type (see [4]).

By classical results (see [1] and [6]), since K^ε is a nonempty convex set, for any given $f \in L^2(\Omega)$, we know that there exists a unique weak solution $u^\varepsilon \in V^\varepsilon \bigcap H^2(\Omega^\varepsilon)$. We shall be interested in getting the asymptotic behavior of this solution, when $\varepsilon \to 0$ and the holes are of critical size.

For obtaining the limit behavior of our homogenization problem, let us recall a result from [4]. Let F be a continuously differentiable function, monotonously non-decreasing and such that $F(v) = 0$ iff $v = 0$. We shall suppose that there exist a positive constant C and an exponent q, with $0 \leq q < n/(n-2)$, such that $\left|\dfrac{\partial F}{\partial v}\right| \leq C(1 + |v|^q)$. It is not difficult to prove that for any $z^\varepsilon \rightharpoonup z$ weakly in $H^1_0(\Omega)$, we get

$$F(z^\varepsilon) \rightharpoonup F(z) \quad \text{weakly in } W^{1,\overline{q}}_0(\Omega), \tag{3}$$

where $\overline{q} = \dfrac{2n}{q(n-2)+n}$.

The main result of this paper is the following one:

Theorem 2.1. *One can construct an extension $\widehat{u^\varepsilon}$ of the solution u^ε of the variational inequality (1), positive inside the holes, such that*

$$\widehat{u^\varepsilon} \rightharpoonup u \quad \textit{weakly in } H^1_0(\Omega),$$

where u is the unique solution of

$$\begin{cases} \textit{Find } u \in H^1_0(\Omega) \textit{ such that} \\ \int\limits_\Omega g(u)(v-u)dx + \int\limits_\Omega \overline{A}DuD(v-u)dx \\ -\langle \mu_0 u^-, v-u\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \geq \\ \geq \int\limits_\Omega f(v-u)dx, \quad \forall v \in H^1_0(\Omega). \end{cases} \tag{4}$$

Here, $\overline{A}$ is the mean value of the matrix A and it coincides, in this case, with the homogenized matrix associated to A^ε and $\mu_0 = \inf\limits_{w \in H^1(\mathbb{R}^n)} \left\{ \int\limits_{\mathbb{R}^n} A(0)DwDwdx \mid w \geq 1 \textit{ q.e on } T \right\}$.

3 Proof of the main result

Let $Y_\varepsilon = \varepsilon Y$ denote the periodicity cell and let $N \in \mathbb{N}$, $N > 1$ and w^ε_N be the unique solution of the following minimization problem on Y_ε:

$$\min_{w \in H^1(Y_\varepsilon)} \{ \int\limits_{Y_\varepsilon} A^\varepsilon DwDwdx \mid w = 0 \text{ in } T^\varepsilon,$$

$$w = 1 \text{ on } Y_\varepsilon \setminus B_{Nr(\varepsilon)}\},$$

where $B_{Nr(\varepsilon)}$ is the ball of radius $Nr(\varepsilon)$ and center 0 included in Y_ε. We extend w^ε_N by ε-periodicity to all of $\mathbb{R}^n$.

$$\int\limits_\Omega A^\varepsilon Dw^\varepsilon_N Dw^\varepsilon_N dx \simeq \frac{|\Omega|}{\varepsilon^n} \int\limits_{Y_\varepsilon} A^\varepsilon Dw^\varepsilon_N Dw^\varepsilon_N dx,$$

since the number of cells Y^i_ε included in Ω is equivalent to $|\Omega|/\varepsilon^n$. Hence,

$$\int\limits_\Omega A^\varepsilon Dw^\varepsilon_N Dw^\varepsilon_N dx \simeq$$

$$\frac{|\Omega|}{\varepsilon^n} \min_{w \in H^1(Y_\varepsilon)} \{ \int\limits_{B_{Nr(\varepsilon)}} A^\varepsilon DwDwdx \mid w = 0 \text{ in } T^\varepsilon,$$

$$w = 1 \text{ on } \partial B_{Nr(\varepsilon)}\}.$$

By changing the scale,

$$x = r(\varepsilon)y,$$

we get

$$\int_\Omega A^\varepsilon Dw_N^\varepsilon Dw_N^\varepsilon dx \simeq$$

$$r(\varepsilon)^{n-2}\frac{|\Omega|}{\varepsilon^n}\min_{w\in H^1(B_N(0))}\{\int_{B_N} A(\frac{r(\varepsilon)}{\varepsilon}y)DwDwdy \mid$$

$$w = 0 \text{ in } T,\ w = 1 \text{ on } \partial B_N\}.$$

Consider the sequence $\overline{\mu}_N^\varepsilon$ of positive Radon measures on Ω defined by

$$\langle \overline{\mu}_N^\varepsilon, \varphi\rangle = \int_\Omega A^\varepsilon Dw_N^\varepsilon Dw_N^\varepsilon \varphi dx, \tag{5}$$

for any $\varphi \in C_0^0(\Omega)$. It is easy to see that if we assume that the matrix A is continuous, the sequence $\overline{\mu}_N^\varepsilon$ converges weakly * in the space of Radon's measures on Ω to a limit measure $\overline{\mu}_N$ defined, for any $\varphi \in C_0^0(\Omega)$, by

$$\langle \overline{\mu}_N, \varphi\rangle = \mu_N \int_\Omega \varphi dx, \tag{6}$$

where

$$\mu_N = \inf_{w\in H^1(B_N(0))}\{\int_{B_N(0)} A(0)DwDwdx \mid$$

$$w = 0 \text{ in } T,\ w = 1 \text{ on } \partial B_N\}.$$

If we let $N \to +\infty$, we get

$$\lim_{N\to\infty} \overline{\mu}_N = \overline{\mu}_0,$$

where, for any $\varphi \in C_0^0(\Omega)$, $\langle \overline{\mu}_0, \varphi\rangle = \mu_0 \int_\Omega \varphi dx$ and

$$\mu_0 = \inf_{w\in H^1(\mathbb{R}^n)}\{\int_{\mathbb{R}^n} A(0)DwDwdx \mid$$

$$w \geq 1 \text{ q.e on } T\}.$$

The sequence w_N^ε is bounded in $H^1(\Omega)$, and hence relatively compact in $L^2(\Omega)$. Introducing χ_ε equal to one on $\bigcup_i(Y_\varepsilon^i \backslash B_{Nr(\varepsilon)}^i)$ and zero elsewhere, we have $(w_N^\varepsilon - 1)\chi_\varepsilon = 0$ on $\mathbb{R}^n$ and it is not difficult to see that $w_N^\varepsilon \to 1$ strongly in $L^2(\Omega)$.

Now, we can pass to the proof of Theorem 2.1. We shall divide it into three steps.

First step. The main convergence result of this paper, given by the above mentioned theorem, involves any extension $\widehat{u^\varepsilon}$ of the solution u^ε of the variational inequality (1) inside the holes such that it depends continuously on u^ε and it is positive in T^ε. In fact, for example, one could just decide to use the positive part of any classical continuous extension of u^ε (see [2]). Clearly,

$$\|\widehat{u^\varepsilon}\|_{H_0^1(\Omega)} \leq C.$$

Consequently, by passing to a subsequence, still denoted by $\widehat{u^\varepsilon}$, we can assume that there exists $u \in H_0^1(\Omega)$ such that

$$\widehat{u^\varepsilon} \rightharpoonup u \text{ weakly in } H_0^1(\Omega).$$

It remains to identify the limit variational inequality satisfied by u.

Second step. Let us introduce the functionals:

$$J^\varepsilon(v) = \int_{\Omega^\varepsilon} A^\varepsilon DvDvdx + 2\int_{\Omega^\varepsilon} G(v)dx - 2\int_{\Omega^\varepsilon} fvdx$$

and

$$J^0(v) = \int_\Omega \overline{A}DvDvdx + \langle \mu_0, (v^-)^2\rangle +$$

$$2\int_\Omega G(v)dx - 2\int_\Omega fvdx.$$

Let $v \in \mathcal{D}(\Omega) = C_0^\infty(\Omega)$ be given and, for a fixed N, let us consider

$$v^\varepsilon = v^+ - w_N^\varepsilon v^-.$$

Obviously, $v^\varepsilon \in K^\varepsilon$ and $v^\varepsilon \to v$ strongly in $L^2(\Omega)$. Using v^ε in (1), we get

$$J^\varepsilon(u^\varepsilon) \leq J^\varepsilon(v^\varepsilon). \tag{7}$$

Computing $J^\varepsilon(v^\varepsilon)$, we get

$$J^\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} A^\varepsilon Dv^+Dv^+dx +$$

$$\int_{\Omega^\varepsilon} A^\varepsilon (w_N^\varepsilon)^2 Dv^- Dv^- dx + \int_{\Omega^\varepsilon} A^\varepsilon Dw_N^\varepsilon Dw_N^\varepsilon (v^-)^2 dx$$

$$-2\int_{\Omega^\varepsilon} A^\varepsilon w_N^\varepsilon Dv^+ Dv^- dx + 2\int_{\Omega^\varepsilon} G(v^+ - w_N^\varepsilon v^-) dx -$$

$$-2\int_{\Omega^\varepsilon} A^\varepsilon v^- Dv^+ Dw_N^\varepsilon dx +$$

$$2\int_{\Omega^\varepsilon} A^\varepsilon w_N^\varepsilon v^- Dv^- Dw_N^\varepsilon dx - 2\int_{\Omega^\varepsilon} f(v^+ - w_N^\varepsilon v^-) dx.$$

Due to our hypotheses and using (3) written for G, it is easy to pass to the limit in all the terms of the above equation. Hence, taking the supremum in N, we have

$$\lim_{\varepsilon\to 0} J^\varepsilon(v^\varepsilon) = J^0(v).$$

So, from (7) we get

$$\limsup_{\varepsilon\to 0} J^\varepsilon(u^\varepsilon) \le J^0(v),\ \forall v \in \mathcal{D}(\Omega). \tag{8}$$

Third step. Let us decompose $\widehat{u^\varepsilon}$ as $(\widehat{u^\varepsilon})^+ - (\widehat{u^\varepsilon})^-$. Obviously, since $(\widehat{u^\varepsilon})^+$ is bounded in $H_0^1(\Omega)$, it converges weakly in $H_0^1(\Omega)$ to u^+.

The classical lower semicontinuity of the energy implies

$$\liminf_{\varepsilon\to 0} \int_\Omega A^\varepsilon D(\widehat{u^\varepsilon})^+ D(\widehat{u^\varepsilon})^+ dx \ge$$

$$\int_\Omega \overline{A} Du^+ Du^+ dx. \tag{9}$$

On the other hand, $(\widehat{u^\varepsilon})^-$ is also bounded in $H_0^1(\Omega)$ and it converges weakly in $H_0^1(\Omega)$ to u^-.

Let $\varphi \in \mathcal{D}(\Omega)$. Consider, for fixed N, the integral $X^\varepsilon = \int_{\Omega^\varepsilon} A^\varepsilon (Du_\varepsilon^- - \varphi Dw_N^\varepsilon - w_N^\varepsilon D\varphi)(Du_\varepsilon^- - \varphi Dw_N^\varepsilon - w_N^\varepsilon D\varphi) dx$. By construction $Dw_N^\varepsilon = w_N^\varepsilon = 0$ on the holes and also, due to the fact that $\widehat{u^\varepsilon} \ge 0$ on the holes, $D(\widehat{u^\varepsilon})^- = (\widehat{u^\varepsilon})^- = 0$ on holes. So, we get $(\widehat{u^\varepsilon})^- \in H_0^1(\Omega^\varepsilon)$ and $X^\varepsilon = \int_\Omega A^\varepsilon (D(\widehat{u^\varepsilon})^- - \varphi Dw_N^\varepsilon - w_N^\varepsilon D\varphi)(D(\widehat{u^\varepsilon})^- - \varphi Dw_N^\varepsilon - w_N^\varepsilon D\varphi) dx$. Expanding X^ε, which is obviously nonnegative, we get

$$\int_\Omega A^\varepsilon D(\widehat{u^\varepsilon})^- D(\widehat{u^\varepsilon})^- dx + \int_\Omega A^\varepsilon \varphi^2 Dw_N^\varepsilon Dw_N^\varepsilon dx +$$

$$\int_\Omega A^\varepsilon |w_N^\varepsilon|^2 D\varphi D\varphi dx - 2\int_\Omega A^\varepsilon w_N^\varepsilon D(\widehat{u^\varepsilon})^- D\varphi dx$$

$$-2\int_\Omega A^\varepsilon Dw_N^\varepsilon D(\widehat{u^\varepsilon})^- \varphi dx +$$

$$+2\int_\Omega A^\varepsilon w_N^\varepsilon Dw_N^\varepsilon \varphi D\varphi dx \ge 0. \tag{10}$$

Let us extract a subsequence, still denoted by ε, such that

$$\lim_{\varepsilon\to 0} \int_\Omega A^\varepsilon D(\widehat{u^\varepsilon})^- D(\widehat{u^\varepsilon})^- dx =$$

$$\liminf_{\varepsilon\to 0} \int_\Omega A^\varepsilon D(\widehat{u^\varepsilon})^- D(\widehat{u^\varepsilon})^- dx.$$

Using our hypotheses and the techniques developed in [4] and [5], we can easily pass to the limit in each term of the above inequality. Hence, passing to the limit and taking the supremum in N, we obtain

$$\liminf_{\varepsilon\to 0} \int_{\Omega^\varepsilon} A^\varepsilon D(\widehat{u^\varepsilon})^- D(\widehat{u^\varepsilon})^- dx \ge$$

$$2\int_\Omega \overline{A} Du^- D\varphi dx -$$

$$-\int_\Omega \overline{A} D\varphi D\varphi dx - \langle \mu_0, \varphi^2 \rangle + 2\langle \mu_0, \varphi u^- \rangle. \tag{11}$$

The above inequality holds true for all $\varphi \in \mathcal{D}(\Omega)$. By density, we get:

$$\liminf_{\varepsilon\to 0} \int_\Omega A^\varepsilon D(\widehat{u^\varepsilon})^- D(\widehat{u^\varepsilon})^- dx \ge$$

$$\int_\Omega \overline{A} Du^- Du^- dx + \langle \mu_0, (u^-)^2 \rangle. \tag{12}$$

Finally, from (9) and (12) we get

$$\liminf_{\varepsilon\to 0} \int_\Omega A^\varepsilon D\widehat{u^\varepsilon} D\widehat{u^\varepsilon} dx \ge$$

$$\int_\Omega \overline{A} Du Du dx + \langle \mu_0, (u^-)^2 \rangle. \tag{13}$$

On the other hand, since $\widehat{u^\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$,

$$\lim_{\varepsilon\to 0} \int_{\Omega^\varepsilon} f u^\varepsilon dx = \int_\Omega f u dx \tag{14}$$

and

$$\lim_{\varepsilon\to 0} \int_{\Omega^\varepsilon} G(u^\varepsilon) dx = \int_\Omega G(u) dx. \tag{15}$$

Hence

$$\liminf_{\varepsilon\to 0} J^\varepsilon(u^\varepsilon) \geq J^0(u). \tag{16}$$

Recalling (8), we have just proved that $u \in H_0^1(\Omega)$ satisfies

$$\int_\Omega \overline{A} Du Du dx + 2\int_\Omega G(u)dx +$$

$$\langle \mu_0, (u^-)^2 \rangle - 2\int_\Omega f u dx \leq$$

$$\leq \int_\Omega \overline{A} Dv Dv dx + 2\int_\Omega G(v)dx +$$

$$\langle \mu_0, (v^-)^2 \rangle - 2\int_\Omega f v dx,$$

for any $v \in \mathcal{D}(\Omega)$ and hence, by density, for any $v \in H_0^1(\Omega)$. So, the function u is the unique solution of (4) and also of the minimization problem

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ J^0(u) = \inf\limits_{v\in H_0^1(\Omega)} J^0(v). \end{cases}$$

As u is uniquely determined, the whole sequence $\widehat{u^\varepsilon}$ converges to u and the proof is finished. Notice that u also satisfies (2).

4 Conclusions

The general question which made the object of this paper was the homogenization of some nonlinear Signorini's type-like problems in periodically perforated domains. Such problems involve the existence of two sources of oscillations, one coming from the geometrical structure of the domain and the other one from its heterogeneity. It is shown how these sources interact to produce the limit behavior of the system. In the case of a critical size of the perforations, the limit problem is a Dirichlet one, associated to a new operator which is the sum of a standard homogenized operator and two extra terms, coming from the special geometry and the nonlinearity of the problem.

References:

[1] H. Brézis, Problèmes unilatéraux, *J. Math. Pures et Appl.*, 51, 1972, pp. 1-168.

[2] D. Cioranescu and P. Donato, *An Introduction To Homogenization*, Oxford University Press, 1999.

[3] D. Cioranescu and F. Murat, *A Strange Term Coming from Nowhere*, in Topics in the Mathematical Modelling of Composite Materials, A. Cherkaev et. al. eds., Progress in Nonlinear Differential Equations and Their Applications, **31**, Birkhäuser, Boston, pp. 45–93, 1997.

[4] C. Conca, J.I. Díaz and C. Timofte, Effective chemical processes in porous media, *Math. Models Methods Appl. Sci.*, Vol. 13, No. 10, 2003, pp. 1437-1462.

[5] C. Conca, F. Murat and C. Timofte, A Generalized Strange Term in Signorini's Type Problems, *Math. Model. Numer. Anal. (M2AN)*, 37(5), 2003, pp 773-806.

[6] J.L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.*, **20**, 1967, pp. 493–519.

[7] F. Murat and L. Tartar, *H-convergence*, in Topics in the Mathematical Modelling of Composite Materials, A. Cherkaev et. al. eds., Progress in Nonlinear Differential Equations and Their Applications **31**, Birkhäuser, Boston, pp. 21–43, 1997.

[8] L. Tartar, *Quelques remarques sur l'homogénéisation*, in Functional Analysis and Numerical Analysis, Proceedings of the Japan-France Seminar 1976, H. Fujita ed., Japan Society for the Promotion of Science, pp. 469–482, Tokyo, 1978.