The shape of the line of action optimized by means of DWT

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Abstract: - Gears are one of the most common and important machine components in many advanced machines. This paper shows the importance of determining the shape of the line of action in order to obtain the best kinematical characteristics in terms of quality performance. For this reason a simplified derivation of the mathematical model of parametric tooth profiles was presented. A procedure for the design of spur gear sets using the quadratic parametric tooth profiles has been developed. The improvement of the kinematical quality of gear transmission was tested by means of the Wavelet Transform.

Key-Words: - Spur gear, Discrete Wavelet Transform, line of action, gear transmission.

1 Introduction

Gears are machine elements used to transmit rotary motion between two shafts, normally with a constant ratio. The pinion is the smallest gear and the larger gear is called the gear wheel. A rack is a rectangular prism with gear teeth machined along one side- it is in effect a gear wheel with an infinite pitch circle diameter. In practice the action of gears in transmitting motion is a cam action each pair of mating teeth acting as cams. Gear design has evolved to such a level that throughout the motion of each contacting pair of teeth the velocity ratio of the gears is maintained fixed and the velocity ratio is still fixed as each subsequent pair of teeth come into contact. When the teeth action is such that the driving tooth moving at constant angular velocity produces a proportional constant velocity of the driven tooth the action is termed a conjugate action. The teeth shape universally selected for the gear teeth is the involute profile. The vast majority of gear applications use the standard 20° involute system because of its good combination of bending and surface pressure strength, involute insensitivity to errors in center distance and relative ease of manufacturing. Most gears are produced by hobbing or other generation-type processes, where a straight-tooth rack or equivalent tool produces the involute working gear tooth surface as well as a trochoidal root fillet.

Despite the benefits of this system, it is generally felt that a higher bending strength and hence load carrying capacity should be obtained. This is especially true with small numbers of teeth (less than 14 or 17 depending on the tip radius of the hob), where the standard involute teeth are susceptible to undercutting. This is a situation where the tip of the cutter removes material from the involute profile in a secondary cutting action. The resulting teeth have smaller thicknesses near their roots, where the critical section is usually located, and this severely hampers load carrying capacity.

From an other point of view, the characteristics of motion are studied by the techniques of tooth contact analysis in the fixed coordinate system. Many authors proposed model and solution in order to approach the problem of designing an optimum tooth profile. For example [1,2] proposed a mathematical model of parametric tooth profiles for spur gears using pressure angle as a parametric variable. In [3] was proposed a method of designing high-contact-ratio spur gears using quadratic parametric tooth profiles for the shorter addendum without undercut and [4-6] studied the effects of the linear profile modification on the dynamic tooth load and stress for high-contact ratio gearing. In [7] was proposed an optimum tooth profile of spur gear rotary pumps method to reduce the delivery fluctuation.

The advantage of using both a mathematical model of gear profile derived from the line of action and the Wavelet Transform for processing the regularity of such a line is that the mathematical model comprises only one single variable of the line of action and the multiresolution analysis, performed by means of wavelet transform, provides more information regarding the kinematical quality of the designed gear pair.

2 The Discrete Wavelet Transform

Mother wavelets are special functions, whose first *h* moments are zero [8]. Note that, if ψ is a wavelet whose all moments are zero, also the function ψ_{jk} is a wavelet, where

$$\psi_{jk}(x) = 2^{-j/2}\psi(2^j x - k), k \in Z, j = 1,2,\ldots,N. \quad (1)$$

Wavelets, like sinusoidal functions employed in the Fourier analysis, are used for representing signals. In fact, consider a wavelet ψ (*mother wavelet*) and a function φ (*father wavelet*) such that

$$\left\{\left\{\varphi_{jk}\right\},\left\{\psi_{jk}\right\}, k \in Z, j = 0,1,2,\ldots,N\right\}, \quad (2)$$

is a complete orthonormal system [9][10]. By Parseval theorem, for every signal $s \in L^2(R)$, it follows that

$$s(t) = \sum_k a_{j_0 k}\varphi_{j_0 k}(t) + \sum_{j=j_0}^{j_1}\sum_k d_{jk}\psi_{jk}(t). \quad (3)$$

In particular, the decomposition of signal $s(t)$ performed by means of the Discrete Wavelet Transform (DWT) is represented by the detail function coefficients $d_{jk} = \left\langle s, \psi_{jk}\right\rangle$ and by approximating scaling coefficients $a_{jk} = \left\langle s, \varphi_{jk}\right\rangle$. Observe that d_{jk} can be regarded, for any j, as a function of k. Consequently, it is constant if the signal $s(t)$ is a smooth function, having considered that a wavelet has zero moments.

Lemma 5.4 in [10] implies the recursive relations

$$a_{jk} = \sum_{m \in Z} h_{m-2k} a_{j+1,m} \quad (4)$$

and

$$d_{jk} = \sum_{m \in Z} \lambda_{m-2k} d_{j+1,m}, \quad (5)$$

where $\lambda = (-1)^{k+1} h_{1-k}, \left\{h_k, k \in Z\right\}$ are real-valued coefficients such that only a finite number is not zero and they satisfy the relations

$$\sum_{k \in Z} h_{k+2m}\overline{h_k} = \delta_{0m} \quad (6) \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\sum_{k \in Z} h_k = 1. \quad (7)$$

The sequence of spaces $\left\{V_j, j \in Z\right\}$, generated by φ is called a multiresolution analysis (MRA) of $L^2(R)$ if it satisfies the following main properties

$$V_j \subset V_{j+1}, j \in Z \quad \text{and} \quad \bigcup_{j \geq 0} V_j \text{ is dense in } L^2(R).$$

It follows that if $\left\{V_j, j \in Z\right\}$, is a (MRA) of $L^2(R)$, we say that the function φ generates a (MRA) of $L^2(R)$, and we call φ the father wavelet.

The relation (3) is also called a multiresolution expansion of s. This means that any $s \in L^2(R)$ can be represented as a series (convergent in $L^2(R)$), where a_k and d_{jk} are some coefficients, and $\left\{\psi_{jk}, k \in Z\right\}$, is a basis for W_j, where we define $W_j = V_{j+1} - V_j, j \in Z$. Consequently, we say that $\left\{\psi_{jk}(t)\right\}$ is a general basis for W_j. The space W_j is called resolution level of multiresolution analysis. In the following, by abuse of notation, we frequently write "resolution level *j*" or simply "level *j*". We employ these words mostly to designate not the space W_j itself, but rather the coefficients d_{jk} and the function ψ_{jk} "on the level *j*".

Finally, for evaluating the features of the signal, a parameter (*i.e.*, entropy) was defined as follows.

Given a set $S: = \{x_i,\ i \in \{1,2,\ldots,n\}\}$ and a function $c: x_i \in S \to c(x_i) \in R$, the entropy $H(c)$ of c is defined as

$$H(c) := -\sum_{c(x_i)\neq m} \frac{1}{s}\cdot\frac{c(x_i)-m}{M-m}\cdot\ln\left(\frac{1}{s}\cdot\frac{c(x_i)-m}{M-m}\right) \qquad (8)$$

where

$$s = \sum_{i\in I}\frac{c(x_i)-m}{M-m},$$

and $M: = \max\ \{c(x_i), i \in \{1,2,\ldots,n\}\}$ and $m: = \min\ \{c(x_i), i \in \{1,2,\ldots,n\}\}$.

The entropy measures the best ratio between the maximum dynamic showed by signal and the smallest uniformity of signal. Given $|S| = n$, the entropy, as before defined, riches its maximum value at $\ln(n)$ iff, for any $i \in S$, $c(x_i) = \text{const}$. Finally $H(c) = 0$ iff, for any $i \in \{1, 2,\ldots,n\}$, $c(x_i) = S$ and, for any $j \in \{1, 2,\ldots,n\}-\{i\}$, $c(x_j) = 0$.

3 The mathematical model

With reference to Fig. 1 the position vector of any point P on the line of action referred to the fixed coordinate system S_f is

$$\overline{r}_f = \begin{bmatrix}\lambda\cos\alpha(\lambda)\\ \lambda\sin\alpha(\lambda)\\ 1\end{bmatrix}$$

where λ is the parametric variable of the line of action and α the angle between λ and x-axis. The line of action, with reference to S_1 (rigidly attached to gear 1) and S_2 (rigidly attached to gear 2) coordinate systems respectively, is

$$\overline{r}_1 = \left[M_{1f}\right]\cdot\overline{r}_f = \begin{bmatrix}-r_1\sin\phi_1 + \lambda\cos(\alpha+\phi_1)\\ r_1\cos\phi_1 + \lambda\sin(\alpha+\phi_1)\\ 1\end{bmatrix} \qquad (9)$$

$$\overline{r}_2 = \left[M_{2f}\right]\cdot\overline{r}_f = \begin{bmatrix}-r_1\cdot m_{12}\sin\left(\frac{\phi_1}{m_{12}}\right) + \lambda\cos\left(\alpha+\frac{\phi_1}{m_{12}}\right)\\ -r_1\cdot m_{12}\cos\left(\frac{\phi_1}{m_{12}}\right) + \lambda\sin\left(\alpha-\frac{\phi_1}{m_{12}}\right)\\ 1\end{bmatrix} \qquad (10)$$

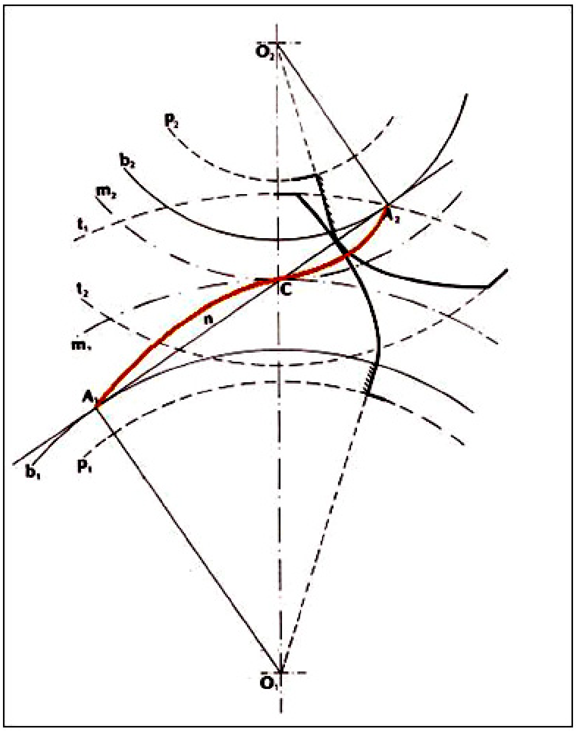

Fig. 1 The geometry of the model

where r_1 and r_2 are the radius of pitch circles.

In particular $r_1 = c/(1+m_{12})$ and $r_2 = cm_{12}/(1+m_{12})$ where c is the centre distance between the centres of gear 1 and 2 and m_{12} is the gear ratio. Observe that $\left[M_{1f}\right]$ and $\left[M_{2f}\right]$ are the coordinate transformation from S_f to S_1 and S_f to S_2 respectively. The determination of the rotation angle of gear 1 is obtained by integrating the equation of meshing gear [3][11]

$$\phi_1 = \frac{1}{r_1}\int_0^{\lambda}\frac{1}{\cos\alpha(\mu)}d\mu,\ \lambda_{\min} < \lambda < \lambda_{\max} \qquad (11)$$

e

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and the maximum value of λ referred to the line of action. The values of $\lambda_{\min}$ and $\lambda_{\max}$, as well known, are limited by the conditions of tooth undercutting and of zero top-land. In order to avoid the undercutting on the tooth profile the condition is as follows

(for the pinion, gear 1)

$$\lambda\left(1+r_1\cos\alpha(\lambda)\right)+r_1\sin\alpha(\lambda)=0 \quad (12)$$

(for the gear 2)

$$\lambda\left(\tau-r_1\cos\alpha(\lambda)\right)-r_1\sin\alpha(\lambda)=0 \quad (13)$$

where τ is the inverse of the transmission ratio.
The surface contact stresses were evaluated by using Hertz's formula [12] as follows

$$\sigma_{c\max}=\frac{2}{\pi}\sqrt{\frac{\pi}{8}}\sqrt{N\frac{\rho}{b}\frac{E}{1-\nu^2}} \quad (14)$$

whit $N=M_1/r_1\cos\alpha(\lambda)$ and $\rho=\left|\rho_1-\rho_2\right|$, where M_1 is the design torque and ρ the relative curvature, with ρ_1 and ρ_2 the curvature of the tooth profile of gear 1 and gear 2 respectively [3]. Note that b is the tooth width, E is the Young's modulus and ν the Poisson's ratio. The specific sliding ratio of meshing profiles is expressed as

$$k_s=\frac{dS_1-dS_2}{dS_1}=\frac{\lambda(1+\tau)}{r_1\sin\alpha(\lambda)+\lambda\left(1+r_1\cos\alpha(\lambda)\frac{d\alpha(\lambda)}{d\lambda}\right)} \quad (15)$$

and

$$k_s'=\frac{dS_2-dS_1}{dS_2}=\frac{-\lambda(1+\tau)}{r_1\sin\alpha(\lambda)-\lambda\left(\tau-r_1\cos\alpha(\lambda)\frac{d\alpha(\lambda)}{d\lambda}\right)} \quad (16)$$

for the pinion and gear respectively.
Finally, the sliding work is

$$L_e(\lambda)=f\frac{M_1(1+\tau)}{r_1^2}\int_0^{\lambda}\frac{\lambda}{\cos^2\alpha(\lambda)}d\lambda,\ \lambda_{\min}<\lambda<\lambda_{\max}. \quad (17)$$

In order to find the optimum shape of the line of action and, consequently, the tooth profile generating the line, it was studied the problem of minimizing the functions (15), (16) and (17). The function $\alpha(\lambda)$ which minimize the above functions for all λ values, will provide the optimum shape of the line of action. By deriving the (15) and (16) we obtain

$$\lambda^2\frac{d^2\alpha(\lambda)}{d\lambda^2}-\lambda^2\tan\alpha(\lambda)\left(\frac{d\alpha(\lambda)}{d\lambda}\right)^2+ \quad (18)$$
$$+\lambda\frac{d\alpha(\lambda)}{d\lambda}-\tan\alpha(\lambda)=0$$

where $\cos\alpha(\lambda)\neq 0\rightarrow\alpha(\lambda)\neq\mp\pi/2$.

4 Wavelet applications

In the Fig. 2 and 3, reported below, it is shown the ability of wavelet transform in order to detect and to localize, starting from the line of action (18), the area where the energy and the entropy, defined in (8), assume consistent value indicating the potential sliding and stress surface of profile.

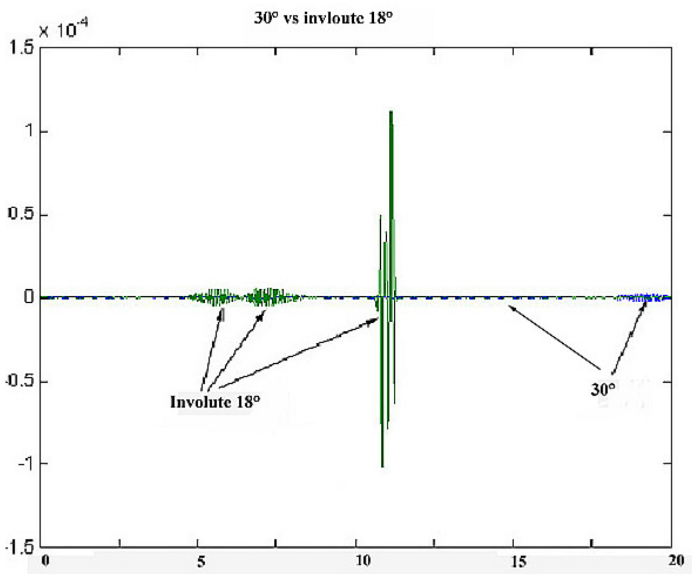

Fig. 2 Wavelet resolution response at 2^{-6} level (comparison of generalized profile 30° *vs* involute 18°)

In the Fig. 4 is shown the comparison of the sliding work for different pressure angle. It is evident the best result obtained by employing the generalized profile named as 30° instead of the involute profiles having the pressure angle α of 18° or 20°.
In the Fig. 5 is represented the comparison of surface contact stress, calculated by (14), for each pressure angle. It is quit evident the concentration of the energy in the same area individuated through the application of wavelet to the line of action.

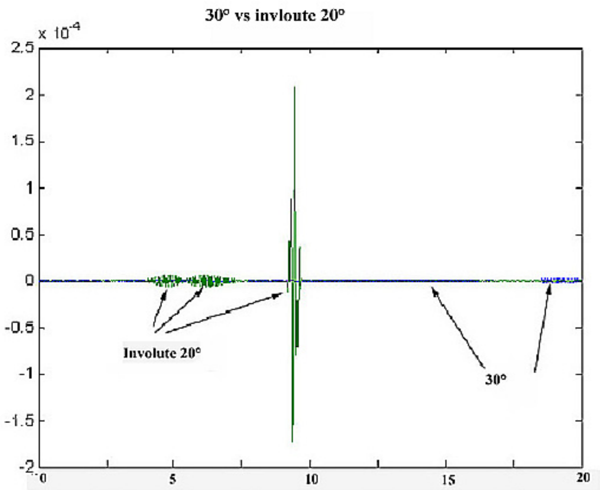

Fig. 3 Wavelet resolution response at 2^{-6} level (comparison of generalized profile 30° *vs* involute 20°)

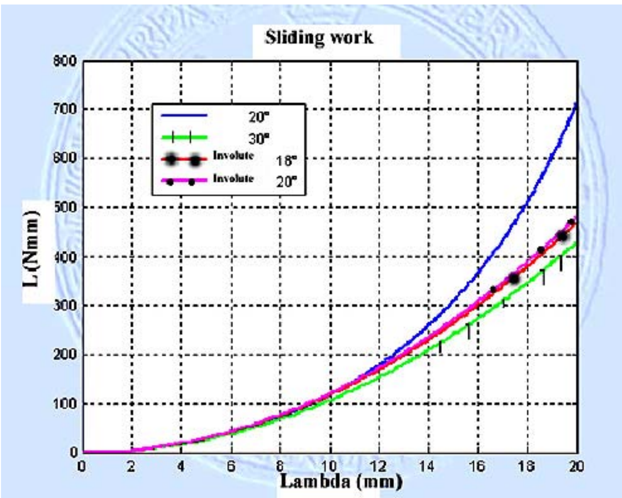

Fig. 4 Comparison of sliding work

The minimum sliding work in conjunction with the minimum surface contact stress are the main factors for determining the best tooth profile. In fact we have also to consider the conditions of tooth undercutting and of zero top-land. These two conditions will be analyzed deeper in the future.

4 Conclusions

Gears are one of the most common and important machine components in many advanced machines. An improved understanding of sliding work and surface contact stress is required both for the early detection of incipient gear failure and to achieve high reliability.

A brief theory of wavelet transforms and their effective computation method with an emphasis on the considerations of the choice of pressure angle are presented in this paper.
This is followed by the numerical results with related graphs.

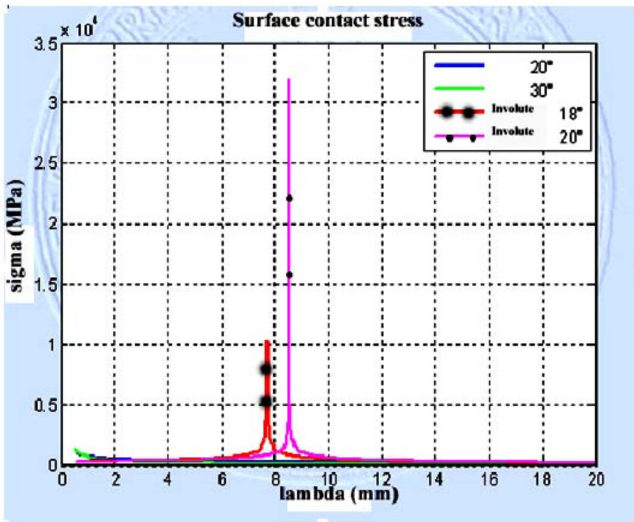

Fig. 5 Comparison of surface contact stress

It is shown the ability of wavelet transform in order to detect and to localize, starting from the line of action, the area where the energy and the entropy assume consistent value indicating the potential sliding and stress surface of profile.
In order to avoid such a problem, probably, we have to modify the pressure angle and corresponding arches of circumference, constituting the profile, until the first derivative. The results are interesting and show that if we design a line of action that fulfils the kinematical behaviour requirements of the gear pair it will be much easier, by applying the wavelets, to ensure the kinematical quality of the designed gear pair.
Then the proposed methodology will be investigated in order to investigate deeper on the regularity of the line of action.
The objective is to calculate the shape of the required modified hobbing tools and to show that the generation process will be no more complicated than that used currently for the production of standard gears.
The work suggests some directions for future investigations:

- reduction of undercutting and interference problem
- reduction of slipping speeds

- increase loading capacity
- increase rigidity of toothing
- reduction of noise and radial forces.

Finally, in order to improve the reliability of the proposed method, investigation will be conducted further on a wider casewise as well as to verify this design, a series of numerical simulations was carried out using the boundary element method (BEM), and the results will be confirmed with subsequent laboratory testing.
In this case the application of Wavelet Transform is also able for the identification and quantification of damaged tooth based on the numerically generated vibration signal.
In fact the main objective of this work is the use of vibration signature analysis procedures for health monitoring and diagnostics of a gear transmission system [13].
The procedure used in this paper include the numerical simulation of the dynamics of a gear transmission system.
Important advancements in preventive maintenance of gear transmission systems are currently being sought for the development of an accurate machine health diagnostic system. Such a diagnostic system would use vibration or acoustic signals from the gear transmission system for

- rapid on-line evaluation of gear wear or damage status
- prediction of remaining gear life.

Such health diagnostic capabilities would be essential for effective machine event/life management and advance warning before critical component failures.

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