On the similar order of convergence of two adjacent sequences related to $\zeta(1/2)$

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Abstract: In this paper we prove the following similar two-sided estimations $\frac{1}{2\sqrt{n+1}} < b_n - l < \frac{1}{2\sqrt{n}}$ and $\frac{1}{2\sqrt{n+1}} < l - a_n < \frac{1}{2\sqrt{n}}$, where $a_n = \left(\sum_{l=1}^{n} \frac{1}{\sqrt{k}}\right) - 2\sqrt{n+1}$, $b_n = \left(\sum_{l=1}^{n} \frac{1}{\sqrt{k}}\right) - 2\sqrt{n}$, and $l = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \zeta(1/2) = -1,46035\ldots$ ($\zeta(1/2)$ being considered in the sense of analytic continuation). These sequences are adjacent. We give a definition of adjacence.

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1 Introduction

Many problems of optimization lead up to various inequalities, asymptotic calculus and numerical calculus.

Generally, the asymptotic analysis is related to functions of real variable, but it is possible also to consider an asymptotic analysis of sequences (see [37]).

Some important theoretical examples were of the domain of the natural variables; there are several old and classical today, e.g.

(a) The asymptotic formula of the harmonic sum $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, $H_n = \ln n + \gamma + \varepsilon_n$, where $\gamma = \lim_{n\to\infty}(H_n - \ln n)$ is the constant of Euler namely $\gamma = 0,577\ldots$ and $\varepsilon_n \to 0$, for $n \to \infty$.

(b) The formula of Stirling, $n! \approx n^n e^{-n}\sqrt{2\pi n}$, which means that the limit of the ratio of the two parts is equal to 1.

(c) The formula of the number of the prime numbers not exceeding n, $\pi(n)$, namely $\pi(n) \approx \frac{n}{\ln n}$, which has a rich and beautiful history, which includes Legendre, Gauss, Tchebycheff, Hadamard, De la Vallée Poussin, and also, Selberg and Erdös.

The first modern form of the Asymptotic Analysis was given by Th. Stieltjes and H. Poincaré in 1886. Later E.Landau introduced the symbols O and o.

Some of the most important texts in this domain are the works [17], [6]-[11], [3], [5], [13], [12], [18], [16], [20], [23].

Several conventions are usual in the asymptotic analysis. If D is a domain in $\mathbb{R}$ or in $\mathbb{C}$, $f, g : D \to \mathbb{C}$, and $x_0 \in D'$, then we denote that (in a neighborhood U of x_0):

a) $f = O(g)$ if there are two constants $M > 0$ and $c > 0$ so that $f(x) < cg(x)$ for any $x \in U$, with $|x| < M$.

b) $f = o(g)$ if $\lim_{\substack{x\to x_0 \\ (x\in U)}} \frac{f(x)}{g(x)} = 0$.

Also:

c) $O(1)$ is a notation for an expression which is bounded for $x \to \infty$;

d) $o(1)$ is a notation for an expression which tends to zero, for $x \to \infty$.

e) The functions f and g are called asymptotic equivalent (in U, for $x \to x_0$) and we write $f \sim g$ if $\lim_{\substack{x\to x_0 \\ (x\in U)}} \frac{f(x)}{g(x)} = 1$.

Now, let $(u_k)_k$ be a sequence of functions defined on D so that $u_{k+1} = o(u_k)$ in U, for $x \to x_0$ and for every $k = 0, 1, 2, \ldots$. If there is a sequence of constants $(a_k)_k$ so that $f(x) \sim$

$\sim a_0u_0(x) + a_1u_1(x) + \ldots + a_ku_k(x)$, for all $k = 0, 1, 2, \ldots, n$ and for all $n \in \mathbb{N}$, we say that the series $\sum_{k=0}^{\infty} a_ku_k(x)$ is an asymptotic expansion of the function f in the point $x \in U$, for $x \to x_0$, respecting the sequence of functions or the asymptotic scale $(u_k)_k$; the coefficients $a_0, a_1, a_2, \ldots, a_n$ are called the coefficients of the expansion, or, still, the iterated limits of the function f, because they are given by the formulas:

$$a_0 = \lim_{x\to x_0} \frac{f(x)}{u_0(x)},$$
$$a_1 = \lim_{x\to x_0} \frac{f(x) - a_0u_0(x)}{u_1(x)},$$
$$a_2 = \lim_{x\to x_0} \frac{f(x) - a_0u_0(x) - a_1u_1(x)}{u_2(x)} \text{ etc.}$$

So we obtain that $f(x)$ is structured respecting the order of the successive functions u_k; the term $a_0u_0(x)$ "extracts" the "principal" part of $f(x)$, the term $a_1u_1(x)$ "extracts" the "principal" part of $f(x_1) - a_0u_0(x)$ etc.

All the precedent considerations are also valid if $\infty \in D'$ and $x_0 \to \infty$.

If we consider the functions of natural variable, all the precedent facts are valid, but for the unique accumulation point of $\mathbb{N}$, namely $x_0 = \infty$. These form the discrete asymptotic analysis. One of the purposes of the discrete asymptotic analysis is to find asymptotic evaluations of first order or asymptotic expansion of the sequences ([2], [16], [18], [19], [25]–[38]).

Let us the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of general term:

$$a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} - 2\sqrt{n+1} \quad (1.1)$$

$$b_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} - 2\sqrt{n}\ ^{1)} \quad (1.2)$$

The convergence of $(b_n)_{n\geq 1}$ is well-known since a long period of time (see the footnote 1)). Namely, from the inequalities (for $k \in \mathbb{N}$, $n \in \mathbb{N}$):

$$2(\sqrt{k+1} - \sqrt{k}) \leq \frac{1}{\sqrt{k}} < 2(\sqrt{k} - \sqrt{k-1}) \quad (1.3)$$

$$2(\sqrt{n+1} - 1) \leq S_n < 2\sqrt{n}, \quad (1.4)$$

where $S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$, it follows that $(b_n)_{n\geq 1}$ is strictly decreasing and lower bounded by the constant -2. According to the theorem of convergence of monotone sequences, $(b_n)_{n\geq 1}$ is convergent. Let l be its limit.

Pass now to $(a_n)_{n\geq 1}$. From the equality:

$$a_n = b_n - 2\left(\sqrt{n+1} - \sqrt{n}\right),$$

it follows that $(a_n)_n$ is also convergent to the same limit l. But, from the equality:

$$a_n - a_{n+1} = 2\left(\sqrt{n+2} - \sqrt{n+1}\right) - \frac{1}{\sqrt{n+1}},$$

according to the left part of (1.3) written for $k = n + 1$, it result that $a_n - a_{n+1} < 0$, therefore the sequence $(a_n)_{n\geq 1}$ is strictly increasing to l. So $(a_n)_{n\geq 1}$ is a natural companion of $(b_n)_{n\geq 1}$, both tending from opposite senses to l.

2 The main result: the velocity of convergence of the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$

It is described by the following

Theorem 1 *Both sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfy the similar two-sided inequalities:*

$$\frac{1}{2\sqrt{n+1}} < l - a_n < \frac{1}{2\sqrt{n}} \quad (2.1)$$

$$\frac{1}{2\sqrt{n+1}} < b_n - l < \frac{1}{2\sqrt{n}} \quad (2.2)$$

Proof: We will prove and use the monotonicity of certain adequate sequences. All the calculations will be elementar. The left part of (2.1) is equivalent to:

$$a_n + \frac{1}{2\sqrt{n+1}} < l. \quad (2.1')$$

Let $u_n = a_n + \frac{1}{2\sqrt{n+1}}$ be; it result that $\lim_{n\to\infty} u_n = l$. After few calculations we obtain:

$$u_n - u_{n+1} = 2\left(\sqrt{n+2} - \sqrt{n+1}\right) - \frac{1}{\sqrt{n+1}} + \frac{1}{2}\left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}\right), \text{ i.e.}$$
$$u_n - u_{n+1} = 2\left(\sqrt{n+2} - \sqrt{n+1}\right) - \frac{1}{2}\left(\frac{1}{\sqrt{n+2}} + \frac{1}{\sqrt{n+1}}\right).$$

[1)] The convergence of $(b_n)_{n\geq 1}$ has represented one of the oldest problems of Gazeta Matematică (vol. 1 (1895), nr. 2, probl. 16, page 39) where it was proposed by the romanian ingenieur and mathematician *Andrei G. Ioachimescu* (1868-1943), a Professor at University of Bucharest and at Polytechnic School of Bucharest; therefore $(b_n)_{n\geq 1}$ is known in the Romanian mathematical literature as the sequence of Ioachimescu. (N.A.)

Now, we find that: $u_n - u_{n+1} =$

$$= \frac{2}{\sqrt{n+2}+\sqrt{n+1}} - \frac{\sqrt{n+2}+\sqrt{n+1}}{2\sqrt{n+2}\sqrt{n+1}} =$$

$$= \frac{4\sqrt{n+2}\sqrt{n+1} - \left(\sqrt{n+2}+\sqrt{n+1}\right)^2}{2\sqrt{n+2}\sqrt{n+1}\left(\sqrt{n+2}\sqrt{n+1}\right)^2} < 0.$$

So, the sequence $(u_n)_n$ is strictly incresing, therefore we obtain $u_n < \lim_{p\to\infty} u_p = l$ and (2.1′) is proved.

The right part of (2.1) is equivalent to:

$$l < a_n + \frac{1}{2\sqrt{n}}, \qquad (2.1'').$$

Let $v_n = a_n + \frac{1}{2\sqrt{n}}$ be; it results that $\lim_{n\to\infty} v_n = l$. We have:

$$v_n - v_{n+1} = 2\left(\sqrt{n+2}-\sqrt{n+1}\right) - \\ -\frac{1}{\sqrt{n+1}} + \frac{1}{2}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right),$$

i.e.

$$v_n - v_{n+1} = 2\left(\sqrt{n+2}-\sqrt{n+1}\right) + \\ +\frac{1}{2}\left(\frac{1}{\sqrt{n}} - \frac{3}{\sqrt{n+1}}\right).$$

We will prove that the right part of the last equality is strictly positive. Also the following calculations also will be elementary, but less elegant and more complicated. We will prove that

$$2\left(\sqrt{n+2}-\sqrt{n+1}\right) > \frac{1}{2}\left(\frac{3}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right).$$

This is succesively equivalent with:

$$\frac{4\sqrt{n(n+1)}}{\sqrt{n+2}+\sqrt{n+1}} > 3\sqrt{n} - \sqrt{n+1},$$

or:

$$4\sqrt{n(n+1)} + (n+1) > \\ > \left(\sqrt{n+2}+\sqrt{n+1}\right)(3\sqrt{n}-\sqrt{n+1}.$$

After the multiplications in the right part and a reduction of similar terms, this is equivalent to:

$$\sqrt{n(n+1)} + (n+1) > 3\sqrt{n(n+2)} - \sqrt{(n+1)(n+2)}.$$

Taking the squares and making some calculations, this is equivalent to:

$$(8n+14)\sqrt{n(n+1)} > 8n^2 + 18n + 1.$$

Taking again the squares, we obtain:

$$64n^4 + 288n^3 + 420n^2 + 196n > \\ > 64n^4 + 288n^3 + 340n^2 + 36n + 1,$$

which is true. So (2.1″) is proved.

The whole two-sided estimation (2.1) is proved.

The left part of (2.2) is equivalent to:

$$l < b_n - \frac{1}{2\sqrt{n+1}}. \qquad (2.2')$$

Let $u_n = b_n - \frac{1}{2\sqrt{n+1}}$ be. Of course $u_n \to l$.

We have $u_n - u_{n+1} = 2\left(\sqrt{n+1}-\sqrt{n}\right) -$

$$-\frac{1}{\sqrt{n+1}} - \frac{1}{2}\left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}\right) =$$

$$= 2\left(\sqrt{n+1}-\sqrt{n}\right) - \frac{1}{2}\left(\frac{3}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}\right).$$

We will prove that the right part of the last equality is strictly positive. We will prove that:

$$2\left(\sqrt{n+1}-\sqrt{n}\right) > \frac{3\sqrt{n+2}-\sqrt{n+1}}{2\sqrt{n+1}\sqrt{n+2}}.$$

This is equivalent to:

$$\frac{2}{\sqrt{n+1}+\sqrt{n}} > \frac{3\sqrt{n+2}-\sqrt{n+1}}{2\sqrt{n+1}\sqrt{n+2}},$$

$$4\sqrt{(n+1)(n+2)} > \\ > \left(\sqrt{n+1}+\sqrt{n}\right)\left(3\sqrt{n+2}-\sqrt{n+1}\right).$$

Opening the brakets in the right part and making a few calculations, the last inequality is equivalent to:

$$\sqrt{n(n+1)} + (n+1) + \sqrt{(n+1)(n+2)} > \\ > 3\sqrt{n(n+2)}$$

or

$$\sqrt{n+1}\cdot\frac{\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}}{3} > \\ > \sqrt{n(n+2)}. \qquad (*)$$

Because of the concavity of the function $t \to \sqrt{t}$, $(t > 0)$, we have:

$$\left(\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right)/3 > \sqrt{n+1},$$

therefore the left part of the inequality $(*)$ is greater than $n+1$, which proves it.

The right part of the inequality is equivalent to

$$b_n - \frac{1}{2\sqrt{n}} < l. \qquad (2.2'')$$

But we observe that:

$$b_n - \frac{1}{2\sqrt{n}} = \left(1 + \ldots + \frac{1}{\sqrt{n}} - 2\sqrt{n}\right) - \frac{1}{2\sqrt{n}} =$$

$$= \left(1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n-1}} - 2\sqrt{n}\right) + \frac{1}{2\sqrt{n}} =$$

$$= a_{n-1} + \frac{1}{2\sqrt{n}} = u_{n-1}.$$

Because of the increasing character of the sequence $(u_n)_n$ we have $u_{n-1} < l$, i.e. $(2.2'')$ holds. The theorem is completely proved. ■

Consequence: We have:

$$\lim_{n\to\infty} \sqrt{n}(l - a_n) = \lim_{n\to\infty} \sqrt{n}(b_n - l) = \frac{1}{2}. \qquad (2.3)$$

This shows that $l - a_n = O\left(\frac{1}{2\sqrt{n}}\right)$ and $b_n - l = O\left(\frac{1}{2\sqrt{n}}\right)$.

3 Going to a definition of the adjacent pair of sequences

The pair of convergent sequences previously mentioned satisfy, related to a given limit l, the conditions of Cantor Dedekind type in a strict form:

$$a_1 < \ldots < a_n < \ldots < b_n < \ldots < b_1 \qquad (3.1)$$

(which express simultaneously two monotonicities and two boundednesses) and also the condition:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l. \qquad (3.2)$$

[Of course, if we have the hypothsis (2.1) satisfyied, the second condition of Cantor-Dedekind:

$$\lim_{n\to\infty} (b_n - a_n) = 0 \qquad (3.3)$$

implies that it exists an unique real number l, such that we have the equality (3.2).]

But these conditions are not sufficient to assure that is suitable to call that the sequences $(a_n)_n$ and $(b_n)_n$ constitute a pair of adjacent sequences. We see a necessity to impose a certain condition of analytic relationship between the two sequences. Moreover, we must also put a condition of equal "velocity" of tending to its common limit of the two sequnces.

So we formulate the following

Definition 2 *Two sequences $(a_n)_n$ and $(b_n)_n$ are called to be adjacent related to a given limit l if its satisfy the conditions (2.1) and (2.2) and, moreover, it exists a nondegenerate interval $I \subset \mathbb{R}$ and a function $f : I \times \mathbb{N} \to \mathbb{R}$, such that:*

(i) For any $t \in I$, we have $\lim_{n\to\infty} f(t,n) = l$;

(ii) It exists $\alpha, \beta \in I$, $\alpha < \beta$, such that $f(\alpha,n) = a_n$ and $f(\beta,n) = b_n$, for any $n \in \mathbb{N}$;

(iii) We have $\lim_{n\to\infty} \frac{b_n - l}{l - a_n} = 1$.

So, for our pair, we have $f(t,n) = 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} - 2\sqrt{n + (1-t)}$, $\alpha = 0$, $\beta = 1$.

Also a certain convergent and monotonic sequence admits not only an unique adjacent pair; if $(a_n)_n$ and $(b_n)_n$ are as in the definition, then, for any $\beta_1 > \beta$ ($\beta_1 \in I$), the sequence $n \mapsto f(\beta_1, n)$ is a pair of $(a_n)_n$; also, for any $\alpha_1 < \alpha$ ($\alpha_1 \in I$), the sequence $n \mapsto f(\alpha_1, n)$ is a pair of $(b_n)_n$. To illustrate the necessity of the condition (iii) of our definition, let us the sequences of general term:

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{3n} - \ln n;$$

$$y_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \ln n.$$

Both these sequences converge to the Euler's constant γ; these are obtained modifying not the logarithm, but the last term of the harmonic sum H_n. The sequence $(x_n)_n$ is strictly increasing and the sequence $(y_n)_n$ is strictly decreasing. Also, we can choose the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$, $f(t,n) = = H_{n-1} + \frac{t}{n} - \ln n$ and the values $\alpha = 1/3$ and $\beta = 1/2$. But the sequences $(x_n)_n$ and $(y_n)_n$ are not adjacent because the condition (iii) from definition is not satisfyied; we have: $\lim_{n\to\infty} \frac{y_n - \gamma}{\gamma - x_n} = 0$. [More precisely, we obtain that $\lim_{n\to\infty} n^2(\gamma - y_n) = = 1/(12)$ and $\lim_{n\to\infty} n(\gamma - x_n) = 1/6$. All the three last results can be obtained using the so called lemma of Stolz-Cesàro for the case 0/0 (see [15], p. 54). In one of our older papers the two-estimate $\frac{1}{12(n+1)^2} < \gamma - y_n < \frac{1}{12n^2}$ is proved. The explanation consists in the asymptotic expansion of

H_n, namely:

$$H - n = \ln n + \gamma + \frac{1}{2n} + \frac{1}{2n} + \ldots]$$

4 Conclusion

The main result is original. Also the definition of adjacent sequences.

The importance of the processes of modelling and optimization is well-known. So the help given by the asymptotic analysis can be important.

A more detalied developement of asymptotic Analysis is to consider in our future works.

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