Controlling Chaotic Systems via Nonlinear State Feedback and Linear H_∞ Controller Design

Manuel J. López, Manuel Prian, Francisco M. Verdulla
Universidad de Cádiz
Dpto Ingeniería de Sistemas y Automática
Calle Chile 1, 11003 Cádiz
SPAIN

Abstract: In this paper, a method for controlling chaotic systems, such as Lorenz system, Chen system and Lu system is described; which is composed by a nonlinear state vector feedback and a linear H_∞ controller. With this method the objective is to stabilize chaotic oscillations to a steady state, as well as tracking a desired trajectory. In order to physical implementation a discrete time H_∞ controller is used with adjusted sample time, and analogic implementation is used for nonlinear part of the controller. Computer simulations are given and analyzed for the purpose of illustration and verification.

Key–Words: Chaotic system, chaos control, nonlinear control, state feedback linearization, robust control.

1 Introduction

Chaotic systems have attracted a great deal of attention from various fields of science and engineering. Over last fifteen years, many methods and technologies have been developed for chaos control and chaotification (anti-control chaos) [1], [4], [5], [2], [3], [17], [18].

It is difficult to define what constitutes a chaotic system; in fact, until present time no universally accepted definition has been proposed. Nevertheless, the essential elements of constituting chaotic behavior are the following: A chaotic system is one where trajectories present aperiodic behavior and are critically sensitive with respect to initial conditions. Here aperiodic behavior implies that the trajectories never settle down to fixed points or to periodic orbits. Sensitive dependence with respect to initial conditions means that very small differences in initial conditions can lead to trajectories that deviate exponentially rapidly from each other. It is of great theoretical importance that chaotic behavior cannot exit in autonomous systems of dimension less than three. The justification of this statement comes from the well-known Poincare-Bendixson theorem.

Most of mathematical models found in the literatura on control of chaos can be represented by a system of ordinary differential equations (state equations)

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{p}, \mathbf{u})$$

where $\mathbf{x}$ is the n-dimensional vector of the state variable; $\mathbf{u}$ is the n_u-dimensional vector of inputs control; $\mathbf{p}$ is the n_p-dimensional vector of parameter-control, and $\mathbf{F}$ is a nonlinear multivariable function or vector field; which in some cases is a smooth function such as in Lorenz [9], Chen [6] or Lu [4] attractors; and in some cases is a piecewise-linear function, such as in [7], [8], [14], [15]. The measured system output is denoted by $\mathbf{y}$; which is the n_y-dimensional vector of outputs; which can be defined as a function of the current system state:

$$\mathbf{y} = \mathbf{h}(\mathbf{x})$$

The problems of suppressing the chaotic oscillations by reducing them to the regular oscillations or vanishing them completely can be, in general, classified as three problems [1], [4], [5], [2], [3], [17], [18] which are: 1) The first class problem refers to periodic orbit stabilization or as particular case, stabilization of unstable equilibrium state. 2) The second class of chaotic system control includes the generation of chaotic oscillations, which is also called the chaotification or anti chaos control. This problem arises where chaotic motion is the desired behavior of the system. 3) And the third class of control objective correspond to synchronization of two or more chaotic systems.

The possibilities of using control engineering approaches to chaos control have been analyzed in numerous papers [1], [4], [5], [2], [3], [17], [18], [21]. In this paper, we have focused on the first class problem aforementioned, where a mixed controller composed by a non-linear state feedback and a linear H_∞

controller is proposed and applied to chaotic attractors for which simulation results are given. The rest of the paper is organized as follows: In section two the controller design method is described, in section three simulations results are analyzed and finally conclusions are given in section four.

2 Controller design method

In this paper we have only considered dynamical systems which have a mathematical model such as:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \mathbf{u}$$

Examples of this system class are, for example Lorenz system [9], Chen System [6] or Lu system [13]; when control law is applied as a added term to some of the state space equations (first order ordinary differential equations).
The Lorenz system [9] used here is given by

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1) + u_1\\ \dot{x}_2 &= \rho x_1 - x_2 x_3 - x_2 + u_2\\ \dot{x}_3 &= x_1 x_2 - \beta x_3 + u_3\end{aligned}$$

where simulations are carried out for system parameters $\sigma = 10$, $\beta = 8/3$, $\rho = 28$; and initial condition is given by: $x_1(0) = 10$, $x_2(0) = -10$, $x_3 = 10$.
Chen system [6] is similar to Lorenz attractor,

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1) + u_1\\ \dot{x}_2 &= (\rho - \sigma)x_1 - x_2 x_3 - \rho x_2 + u_2\\ \dot{x}_3 &= x_1 x_2 - \beta x_3 + u_3\end{aligned}$$

In this case, simulations are made for system parameters $\sigma = 35$, $\beta = 3$, $\rho = 28$; and initial condition given by: $x_1(0) = 10$, $x_2(0) = -10$, $x_3 = 10$. Although similar to Lorenz attractor, nevertheless, Chen system is not topologically equivalent in the sense defined in [4]. The Lorenz system satisfies the condition $a_{12}a_{21} > 0$, while Chen system satisfies $a_{12}a_{21} < 0$; where a_{12}, a_{21} are the corresponding elements in the constant matrix $A = (a_{ij})_{3\times 3}$ for the linear part of the system. Lu system [13]is given by,

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1) + u_1\\ \dot{x}_2 &= -x_1 x_3 + \rho x_2 + u_2\\ \dot{x}_3 &= x_1 x_2 - \beta x_3 + u_3\end{aligned}$$

for which in simulations we have used for system parameters $\sigma = 10$, $\beta = 8/3$, $\rho = 28$; and initial condition given by: $x_1(0) = 10$, $x_2(0) = -10$, $x_3 = 10$. Lu system satisfies the condition $a_{12}a_{21} = 0$, thereby bridging the gap between the Lorenz and Chen attractors [13].

In our method, the control law is composed with two components: linear part (u_L) and non-linear part (u_{NL}),

$$\mathbf{u} = \mathbf{u}_{NL}(\mathbf{x}) + \mathbf{u}_L$$

where the non-linear part is chosen in such away that the resulting feedback system is linear and time invariant (LTI), and it is given by

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{u}_L$$

where this system has only one equilibria point at origin $\mathbf{x} = \mathbf{0}$; and the control system design problem is reduced to design a linear control which stabilizes the closed loop system at origin. For that problem, linear H_∞ control theory is used.

The linear part of the controller is characterized by

$$\begin{aligned}\mathbf{e} &= \mathbf{r} - \mathbf{y}\\ \mathbf{u}_L &= C_c\mathbf{x}_c + D_c\mathbf{e}\\ \dot{\mathbf{x}}_c &= A_c\mathbf{x}_c + B_c\mathbf{e}\end{aligned}$$

where $\mathbf{e} = \mathbf{r} - \mathbf{y}$ is the error vector, $\mathbf{r}$ is the reference vector or set-point, and $\mathbf{y} = \mathbf{h}(\mathbf{x})$ is the output measurement vector (in this work $\mathbf{y} = \mathbf{x}$ has been considered). Where controller matrices are computed using the respective linearized system, which is obtained by means of the corresponding nonlinear state feedback. The nonlinear state feedback structure depends on the attractor. In case of Lorenz, Chen and Lu systems, the following nonlinear control law is used:

$$\mathbf{u}_{NL} = \begin{pmatrix} 0 \\ x_1 x_3 \\ -x_1 x_2 \end{pmatrix}$$

With this non-linear state feedback the resulting linearized system for Lorenz attractor has the following A matrix

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

meanwhile, matrices B, C and D are as follows:

$$B = (b_{ij}), \quad C = (c_{ji}), \quad D = \mathbf{0}$$

where $i = 1, 2, 3$ and $j = 1, 2, 3$ for multivariable systems (MIMO) with three inputs and three outputs. For scalar or simple-input-simple-output (SISO) systems, one input and one output: $i = 1, 2, 3$, $j = 1$, and so on. In this work different combinations of inputs and outputs for MIMO and SISO systems have been analyzed, but only the results obtained with

MIMO system (3×3, where $B = \text{diag}\{1, 1, 1\}$, $C = \text{diag}\{1, 1, 1\}$), will be presented.

In case of Chen system, A matrix is as follows:

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - \sigma & -\rho & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

For Lu system, the respective A matrix is given by,

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ 0 & \rho & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

Due to equations structure of Lorenz, Chen and Lu systems, the same nonlinear state feedback is used. Nevertheless, for different system equations different nonlinear state feedback is implemented, but the procedure of the proposed method does not change.

For linear H_∞ controller design, the linear time invariant system G is employed, which has a state space representation composed with the following set of matrices:

$$\{\ A,\ \ B,\ \ C,\ \ D\ \}$$

where A depends on the particular system used for designing (see matrices given previously), and in case of using multivariable controller (MIMO) with three inputs and three outputs, B, C and D are given as before (unit diagonal matrices 3×3). The relation between state space matrices and its matrix transfer function is given by,

$$G(s) = C(sI - A)^{-1}B + D$$

where "s" is the Laplace transform variable and I is unit matrix with adequate dimensions.

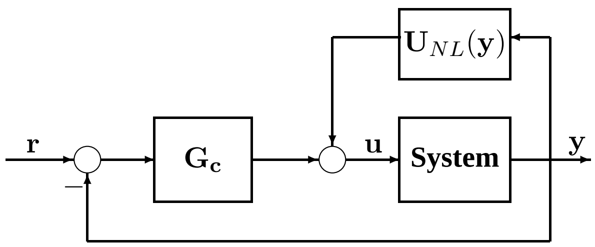

Figure 1: Control system structure

3 H_∞ Controller Design

For linear controller component design, an equivalent conventional closed loop is considered, which consists of system to control or plant (G), controller G_c, set-point or reference signal ($\mathbf{r}$), measurement noise ($\mathbf{n}$), disturbances acting at the plant input ($\mathbf{d}_i$) and at the plant output ($\mathbf{d}_o$). The signals to evaluate the performance of the system are the control signal ($\mathbf{u}_L$), the output signal ($\mathbf{y}$), and the error signal ($\mathbf{e} = \mathbf{r} - \mathbf{y}$); where signals are multivariable and nominal mathematical models for G and G_c are considered LTI (Linear Time Invariant). In this scheme, it is considered a vector $\mathbf{z}$, which is used to include signals required to characterize the behavior of the closed-loop system, and a vector w which contains external inputs (setpoints, disturbances and noise). So that, the behavior of the closed-loop system is given by $\mathbf{z} = T_{zw}\mathbf{w}$, where input-output transfer function T_{zw} depends on the weighting transfer functions employed in the design problem [16], [19], [23].

The suboptimal H_∞ control problem is to find a controller G_c which *stabilizes internally* the system G, such that $\|T_{zw}\|_\infty < \gamma$, where γ is a positive real number; and the H_∞ norm for a stable transfer matrix, T_{zw}, is defined in frequency domain ($s = j\omega$) as

$$\|T_{zw}(j\omega)\|_\infty = \sup_\omega \overline{\sigma}(T_{zw}(j\omega))$$

where $\overline{\sigma}$ represents the upper singular value.

Internal stability of the closed loop system is an additional important requirement. In order to define internal stability, previously we remember that for continuous-time systems, a rational transfer-function matrix (or its corresponding state space representation) is *exponentially stable* if and only if it is proper and has no poles (or eigenvalues of its state space representation matrix, A) in the closed right half-plane. When considering discrete-time systems we need to replace "closed right half-plane" by "boundary and exterior of the origin-centered unit circle". Now *internal stability* is defined: A linear time invariant control system is internally stable if each element of the transfer-function matrix, Q between input vector $[\ \mathbf{r}\ \ \ \mathbf{d}_i\ ]^T$ and output vector $[\ \mathbf{y}\ \ \ \mathbf{u}_L\ ]^T$ is exponentially stable; where Q is given by,

$$Q = \begin{pmatrix} GG_c(I + GG_c)^{-1} & (I + GG_c)^{-1}G \\ G_c(I + GG_c)^{-1} & -G_c(I + GG_c)^{-1}G \end{pmatrix}$$

In this way, the concept of internal stability is more complete and restrictive than the usual exponential stability concept used in control systems.

In our approach, H_∞ controller is designed for robustness and performance specifications expressed in the frequency domain; but indicators based on time domain response such as overshoot, rise time and settle time, are considered too. In practice, it is difficult to obtain specified time responses, for what we employ a tuning method [10], [11] which gives parameters for controller pre-tuning, controller evaluation indicators and a method for controller fine tuning. In the first step, pre-tuning values for performance and robustness are established by means of the suboptimal H_∞ problem, $\|T_{zw}\|_\infty < \gamma$ (where γ is a positive real number) is considered, with

$$T_{zw} = [W_S S \quad W_R R \quad W_T T]^T$$

where $S = (I + GG_c)^{-1}$ is the sensitivity function, $T = (I + GG_c)^{-1} GG_c$ is the complementary sensitivity function, and $R = G_c S$ is the control sensitivity function; where the weighting transfer functions, $W_m(s) = \text{diag}\{W_{mj}(s)\}$, $m = S, R, T$, $j = 1, 2, 3$ are obtained from the linearized model of the system [10], [11].

$$W_m(s) = \text{ diag } \left\{ \frac{N_{mj}(s)}{D_{mj}(s)} \right\}$$

where $m = S, R, T$; $j = 1, 2, 3$; and the polynomials coefficients are obtained from the linearized model of the system (which has been previously carried out via nonlinear state feedback) following the pre-tuning formulae given in [10], [11], which are valid for scalar systems (SISO) and multivariable (MIMO) processes.

Time domain specifications

In conventional approach of H_∞ control, controller is designed for robustness and performance specifications expressed in the frequency domain; but usual indicators based on time domain response such as overshoot, rise time and settle time, are not considered a priori. In practice, it is difficult to obtain the specified time responses using this approach, for what we use a method to satisfy time response specifications as well as robustness properties [10],[12]. For that, H_∞ control theory and genetic algorithms (H_∞-GA) are employed in our design. We have implemented the following procedure for controller design:

1) Time domain specifications are established for rise time (t_r), overshoot (M_p), settle time (t_s) and stationary error (e_{ss}) for a step change in set-point.

2) Suboptimal H_∞ problem, $\|T_{zw}\|_\infty < \gamma$ ($\gamma > 0$) is considered.

3) Weighting transfer functions are given by,

$$W_m(s) = \text{ diag } \left\{ \frac{c_j s + d_j}{a_j s + b_j} \right\}$$

where $m = S, R, T$, and $j = 1, 2, 3$. With this election, and if $\gamma = 1$, performance and robustness lower bounds are guaranteed. If it is necessary, higher order transfer functions may be used [10]. Selection is made in order to guarantee an lower bound of relative uncertainty (80%) for which robust stability is achieved.

5) Similar weighting transfer functions are used if more or different components of the set (see Fig. 2) $\{W_H\}$ are employed, where

$$\{W_H\} = \{W_r, W_n, W_{do}, W_{di}, W_T, W_R, W_S\}$$

4) Controller is solved through standards algorithms for H_∞ problem. If more adjusted restriction on time response is necessary, genetic algorithms are used, employing the following objective function [12]:

$$f(k) = w_1|M_p(k) - M_{po}| + w_2|t_r(k) - t_{ro}| + w_3|e_{ss}|$$

where M_{po}, t_{ro}, e_{ss} are respectively the specified (objective) overshoot, rising time and steady-state error; and w_i ($i = 1, 2, 3$) are weighting factors. Here, GA are used for obtaining weighting transfer functions parameters $\{a_j, b_j, c_j, d_j\}$, such that closed loop system satisfaces design specifications. Once these parameters are fixed, conventional algorithms based on algebraic Ricatti equations (ARE) or on linear matrix inequalities (LMI) can be used for solving $\|T_{zw}\|_\infty < \gamma$. This method has been implemented combining TuhiCo Toolbox and GARCO (Genetic Algorithms for Robust Control) Toolbox [11],[12].

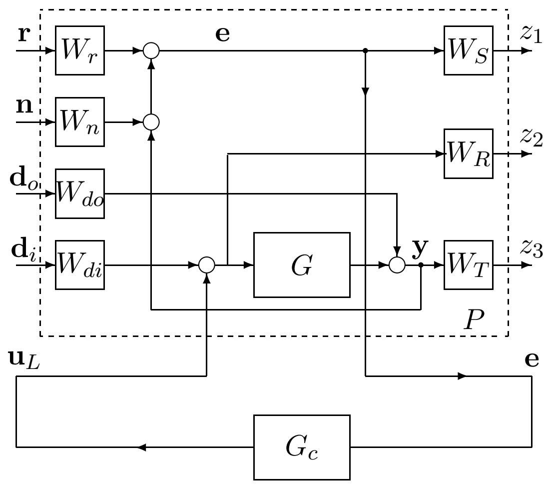

Figure 2: Generalized plant for H_∞ and H_2 control

4 Simulation results

In this section we present simulations results obtained with Lorenz system using multivariable control with three inputs and three outputs. If continuos time controller is designed, its physical implementation must be with analogic electronics or by means of digital processor which requires high sample frequency, for what DSP (digital signal processor) is needed. In order to avoid high sample frequency for H_∞ controller implementation, we have used bilinear transformation for a fixed sample time, since the infinity norm of a given discrete problem is preserved under such a transformation in the continuous domain [23], [19]. For that, the following procedure is used in our design: 1) $G(s)$ is transformed in its discrete time equivalence, $G(z)$, using the zero order hold equivalence for a sample time T_m (where z is the Z-transform variable). 2) w-plane transform, special case of inverse bilinear transform, is used for obtaining $G(w)$. 3) With $G(w)$ is solved the H_∞ problem, obtaining the controller $G_c(w)$. 5) The discrete time controller used for real time implementation, $G_c(z)$, is obtained using the bilinear transform of $G_c(w)$ for the sample time T_m.

Simulations are carried out with the following parameters: $\sigma = 10$, $\beta = 8/3$, $\rho = 28$; and initial condition is given by: $x_1(0) = 10$, $x_2(0) = -10$, $x_3 = 10$. Simulation results for a reference state vector (state stabilization problem) are given in Fig. 3; and Fig. 4 shows the time response of the system for tracking problem, where reference trajectory is $r(t) = \sin(t)$. In all simulations the controller is activated at time $t = 20$ and the sample time is 0.001 seconds, which can be implemented in real time using conventional data acquisition cards. For $T_m = 0.01$ seconds, satisfactory results are obtained too, but time response is slower than for $T_m = 0.001$. Similar results are obtained for Chen system and Lu system, but due to space limitations these results are not described here.

Noise and uncertainty effects are not considered in this paper in explicit way, except for $H\infty$ controller design where robust stability is guaranteed for relative unstructured uncertainty of 80% at the output of the system model. This is made by means of weighting transfer function selection of $W_T(s)$, and if it is satisfied the inequality $\|W_T T\|_\infty < 1$. Significative noise in measurements and parametric uncertainty will be considered in next works.

5 Conclusions

In this paper a method for controlling chaotic systems has been proposed, which is composed with a nonlinear state feedback and a linear $H\infty$ controller. State feedback linearizes system dynamics and the resulting system is used for linear $H\infty$ controller design; for which internal stability is achieved and robust stability is obtained for relative unstructured uncertainty of 80% at the output of the system model. This procedure has been applied to Lorenz system, Chen system and Lu system; for which satisfactory performances have been obtained. In next works, sensitivity of the control system to significative noise in measurements, and parametric uncertainty in the system dynamics will be considered.

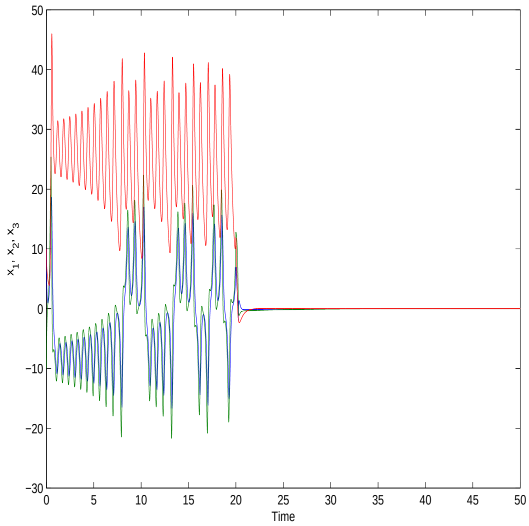

Figure 3: Time responses of the states, x_1, x_2 and x_3 for the controlled Lorenz system. Regulation problem: $\mathbf{r}(t) = [0\ \ 0\ \ 0]^T$

References:

[1] E.H. Abed, H.O. Wang, A. Tesi, *Control of Bifurcations and Chaos*. The Control Handbook, CRC Press and IEEE Press, 1995.

[2] B.R. Andrievskii and A.L. Fradkov, *Control of Chaos: Methods and Applications. I. Methods.*. Automation and Remote Control, vol. 64, no. 5, pp 673-713, 2003.

[3] S. Boccaletti, C. Grebogi, Y.C. Liai, H. Mancini, D. Maza, *The Control of Chaos: Theory and Applications*. Physics Reports 329, pp 103-197, 2000.

[4] S. Celikovsky, G. Chen, *On a Generalized Lorenz Canonical Form of Chaotic Systems*. International Journal of Bifurcation and Chaos, vol. 12, no. 8, pp 1789-1812, 2002.

[5] G. Chen, *Controlling Chaos and Bifurcations in Engineering Systems*. CRC Press, 1999.

[6] G. Chen, T. Ueta *Yet Another Chaotic Attractor*, International Journal of Bifurcation and Chaos vol. 9, no. 7, pp 1465-166. CRC Press, 1999.

[7] L. O. Chua, C. W. Wu, A. Huang, G. Zhong *A Universal Circuit for Studying and Generating Chaos. Part I: Routes to Chaos*. IEEE Transactions on Circuits and Systems, vol. 40, no. 10, pp 732-744, 1993.

[8] L. O. Chua, C. W. Wu, A. Huang, G. Zhong *A Universal Circuit for Studying and Generating*

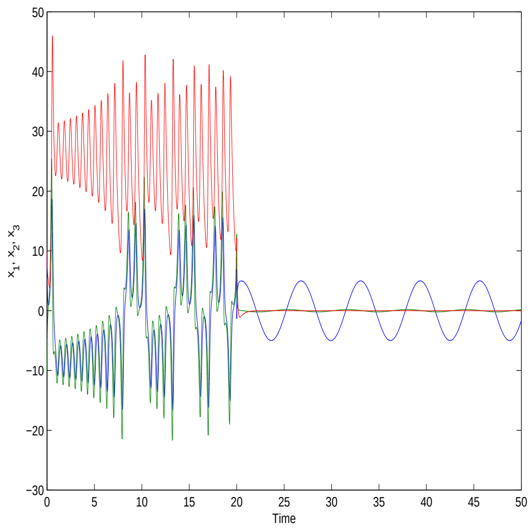

Figure 4: Time responses of the states, x_1, x_2 and x_3 for the controlled Lorenz system. Tracking problem: $\mathbf{r}(t) = [5\sin(t) \;\; 0 \;\; 0]^T$

Chaos. Part II: Strange Attractors. IEEE Transactions on Circuits and Systems, vol. 40, no. 10, pp 745-761, 1993.

[9] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 1993.

[10] M.J. L.ópez, *Contributions to Advanced Process Control. Methods for H_∞ Controller Tuning*. Cadiz University. 1999.

[11] M.J. L.ópez, *System identification and H_∞ controller tuning. TuHiCo Toolbox.* . Cadiz University. 1999.

[12] M.J. Lopez, L. Garcia *Multivariable H_2 and H_∞ Controller Synthesis with Fault Detection Method Based on Genetic Algorithms*. WSEAS Transactions on Systems, Issue 5, Vol. 5, pp 939-946, 2006.

[13] J. Lu, G. Chen, D. Cheng, S. Celikovsky *Bridge the Gap Between the Lorenz System and the Chen System*. International Journal of Bifurcation and Chaos, vol 12, no. 12 pp 2917-2926, 2002.

[14] R. Gonzalez, M. Prian, M.A. Fernandez, J.L. Rojas, E. Romero *A Symmetric Piecewise-linear Chaotic system witn a Single Equilibrium Point*. International Journal of Bifurcation and Chaos, vol 15, no. 4 pp 1411-1415, 2005.

[15] T. Kousaka, T. Ueta, Y. Ma, H. Kawakami *Control of chaos in a piecewise smooth nonlinear system*. Chaos, Solitons and Fractals, vol. 27, pp 1019-1025, 2006.

[16] Maciejowski, J.M., *Multivariable Feedback Design*, . Adison Wesley, 1989.

[17] M.J. Ogorzalek, *Taming Chaos: Part I: Synchonization*, . IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, vol. 40, No. 10, pp 693-699, 1993.

[18] M.J. Ogorzalek, *Taming Chaos: Part II: Control*, . IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, vol. 40, No. 10, pp 700-707, 1993.

[19] S. Skogestad, I. Postlethwaite, *Multivariable Feedback Control*. Wiley, 2003.

[20] T. Ueta, G. Chen, *Bifurcation analysis of Chen's Equation*. International Journal of Bifucation and Chaos, vol. 10, no. 8, pp 1917-1931, 2000.

[21] M.T. Yassen, *Feedback and adaptive synchronization of chaotic Lu system*. Chaos, Solitons and Fractals, vol. 25, pp 379-386, 2005.

[22] M.T. Yassen, *Chaos control of chaotic dynamical systems using backstepping design*. Chaos, Solitons and Fractals, vol. 27, pp 537-548, 2006.

[23] K. Zhou, J. C. Doyle and K. Glover, *Robust and Optimal Control*. Prentice Hall, 1996.