On the form of the motion equations of the multibody systems with elastic elements

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Abstract: - In many cases when a study of multi-bodies systems is perform, the hypothesis of rigid elements is considered. In reality the elasticity of the components of the system can be large enough so that the dynamic response can be not only quantitative but also qualitative different. For this reason, in some applications, particularly in the field of robotics and high-speed vehicles, is necessary to consider the elasticity of elements and to use correspondent models. Generally, the multi-bodies systems have a great complexity and the strong non-linearity. To study such system with the classic mechanics theorems is not a practical task because the motion equations have, generally, no analytical solutions. For this reason is necessary to use numerical methods and the finite element methods (FEM) remains one of the most important tools.

Key-Words: - Finite element methods, Motion equations, Multibody systems, Dynamic response, Liaison forces, Elasto-dynamic analysis, Eigenvalues calculus.

1 Introduction

The major difficulty using FEM is the non-linearity of the motion equations. The coefficients that appears in equations are time-position dependent and, in some practical application (mechanisms with a periodical motion) they can be periodical. To solve this problem the motion must be considered "frozen" for a very short interval of time. In this case the obtained equations can be considered linear. Writing the principle of minimum energy is possible to obtain the motion equations for a finite element with a three-dimensional rigid motion. These equations have some important particularities: in the equations exists Coriolis terms (conservative) and the rigidity is modified by the some terms determined by the "rigid motion" of element. They depend on element distribution mass and on the field of velocities and accelerations. More, the force term of equations is modified by the effect of inertia forces and momentum due to the relative motion.

For this reason it exists two difficult and major problems when is used finite element method: one consist in the fact that the equation are more large and with more terms as in the classical procedures and the second is that the equations are only incremental valid, for a very short time interval; after this interval must generate new coefficient for the motions equations and the solutions previously obtained are the initial conditions for the new equations.

In the following we will establish the motion equations for an elastic finite element with a general motion together with an element of the system. The type of the shape function is determined by the type of the finite element. For this reason we will present the motion equations in three different situations: for a three-dimensional finite element with a general three-dimensional motion, for a two-dimensional finite element with a plane motion and for an one-dimensional element with a general three-dimensional motion. We will consider that the small deformations will not affect the general, rigid motion of the system.

2 Motion equations

We consider that, for the all elements of the system, we know the field of the velocities and of the accelerations. We refer the finite element to the local coordinate system Oxyz, mobile, and having a general motion with the part of system considered (fig.1). We note with $\vec{v}_o(\dot{X}_o, \dot{Y}_o, \dot{Z}_o)$ the velocity and with $\vec{a}_o(\ddot{X}_o, \ddot{Y}_o, \ddot{Z}_o)$ the acceleration of the origin of the local coordinate system. The motion of the whole system is refer to the general coordinate system O'XYZ. By [R] is denoted the rotation matrix. The velocity of point M' will be:

$$\{v_{M'}\} = \{\dot{r}_o\} + [\dot{R}]\{r'\} + [\dot{R}]\{f\} + [R]\{\dot{f}\} = \{\dot{r}_o\} + [\dot{R}]\{r'\} + [\dot{R}][N]\{\delta_e\} + [R][N]\{\dot{\delta}_e\}. \quad (1)$$

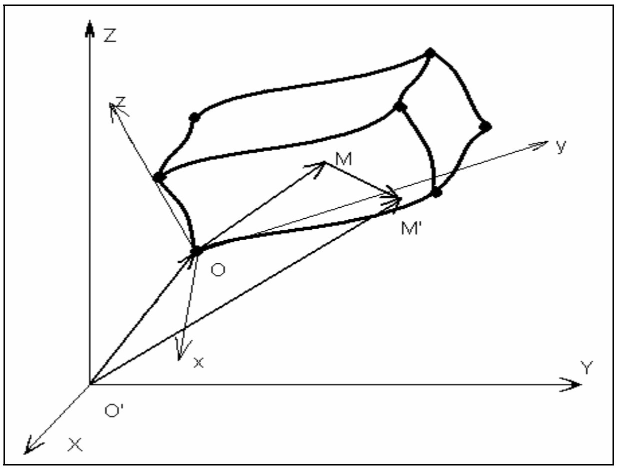

Fig. 1 Finite element in a three-dimensional motion

The kinetic energy of the finite element considered is:

$$E_c = \frac{1}{2}\int_V \rho v^2 dV = \frac{1}{2}\int_V \rho \{v_{M'}\}^T \{v_{M'}\} dV \tag{2}$$

where ρ is the mass density and the deformation energy is:

$$E_p = \frac{1}{2}\int_V \{\delta_e\}^T [k_e]\{\delta_e\} dV, \tag{3}$$

where $[k_e]$ is the rigidity matrix for the e element. If we not with $\{p\} = \{p(x,y,z)\}$ the distributed forces vector, the external work of these is:

$$W = \int_V \{p\}^T \{f\} dV = \left(\int_V \{p\}^T [N] dV\right)\{\delta_e\}, \tag{4}$$

and the nodal forces $\{q_e\}$ produce an external work:

$$W^c = \{q_e\}^T \{\delta_e\}. \tag{5}$$

The Lagrangean for the considered element is obtain with the relation:

$$L = E_c - E_p + W + W^c. \tag{6}$$

If we apply the Lagrange's equations, we obtain:

$$\begin{aligned}&\left(\int_V [N]^T [N]\rho dV\right)\{\ddot{\delta}_e\} + 2\left(\int_V [N]^T [R]^T [\dot{R}][N]\rho dV\right)\{\dot{\delta}_e\} + \\ &+ \left([k_e] + \int_V [N]^T [R]^T [\ddot{R}][N]\rho dV\right)\{\delta_e\} = \\ &= \{q_e\} + \int_V [N]\{p\} dV - \\ &- \left(\int_V [N]^T \rho dV\right)[R]\{\ddot{r}_o\} - \int_V [N]^T [R]^T [\ddot{R}]\{r'\}\rho dV.\end{aligned} \tag{7}$$

If we note by $N_{(1)}, N_{(2)}, N_{(3)}$ the lines of matrix $[N]$ we obtain:

$$\begin{aligned}&\int_V [N]^T [N]\rho dV = \int_V (N_{(1)}^T N_{(1)} + N_{(2)}^T N_{(2)} + N_{(3)}^T N_{(3)})\rho dV = \\ &= [m_{11}] + [m_{22}] + [m_{33}] = [m_e],\end{aligned} \tag{8}$$

$$\begin{aligned}&\int_V [N]^T [R]^T [\dot{R}][N]\rho dV = \int_V [N]^T [\Omega][N]\rho dV = \\ &= \Omega_x([m_{32}] - [m_{23}]) + \Omega_y([m_{13}] - [m_{31}]) + \Omega_z([m_{21}] - [m_{12}]),\end{aligned} \tag{9}$$

$$\begin{aligned}&\int_V [N]^T [R]^T [\ddot{R}][N]\rho dV = \int_V [N]^T [E][N]\rho dV + \int_V [N]^T [\Omega]^T [\Omega][N]\rho dV = \\ &= E_x([m_{32}] - [m_{23}]) + E_y([m_{13}] - [m_{31}]) + E_z([m_{21}] - [m_{12}]) - \\ &- (\Omega_y^2 + \Omega_z^2)[m_{11}] - (\Omega_z^2 + \Omega_x^2)[m_{22}] - (\Omega_x^2 + \Omega_y^2)[m_{33}] + \\ &+ ([m_{12}] + [m_{21}])\Omega_x\Omega_y + ([m_{23}] + [m_{32}])\Omega_y\Omega_z + ([m_{31}] + [m_{13}])\Omega_z\Omega_x,\end{aligned} \tag{10}$$

$$\begin{aligned}&\int_V [N]^T [R]^T [\ddot{R}]\{r'\}\rho dV = \int_V [N]^T [E]\{r'\}\rho dV + \int_V [N]^T [\Omega]^T [\Omega]\{r'\}\rho dV = \\ &= E_x([m_{3y}] - [m_{2z}]) + E_y([m_{1z}] - [m_{3x}]) + E_z([m_{2x}] - [m_{1y}]) - \\ &- (\Omega_y^2 + \Omega_z^2)[m_{1x}] - (\Omega_z^2 + \Omega_x^2)[m_{2y}] - (\Omega_x^2 + \Omega_y^2)[m_{3z}] + \\ &+ ([m_{1y}] + [m_{2x}])\Omega_x\Omega_y + ([m_{2z}] + [m_{3y}])\Omega_y\Omega_z + ([m_{3x}] + [m_{1z}])\Omega_z\Omega_x.\end{aligned} \tag{11}$$

In these relations we have denoted:

$$\{m_{ix}\} = \int_V [N_{(i)}]^T x\rho dV; \quad \{m_{iy}\} = \int_V [N_{(i)}]^T y\rho dV; \quad \{m_{iz}\} = \int_V [N_{(i)}]^T z\rho dV, \tag{12}$$

$$[m_{ij}] = \int_V N_{(i)} N_{(j)}^T \rho dV; \quad \{q^*\} = \int_V [N]^T \{p\} dV; \quad [m_{oe}^i] = \int_V [N]^T \rho dV. \tag{13}$$

We obtain the motion equations:

$$\begin{aligned}&([m_{11}] + [m_{22}] + [m_{33}])\{\ddot{\delta}_e\} + \\ &+ 2[\Omega_x([m_{32}] - [m_{23}]) + \Omega_y([m_{13}] - [m_{31}]) + \Omega_z([m_{21}] - [m_{12}])]\{\dot{\delta}_e\} + \\ &+ [[k_e] + E_x([m_{32}] - [m_{23}]) + E_y([m_{13}] - [m_{31}]) + E_z([m_{21}] - [m_{12}]) - \\ &- (\Omega_y^2 + \Omega_z^2)[m_{11}] - (\Omega_z^2 + \Omega_x^2)[m_{22}] - (\Omega_x^2 + \Omega_y^2)[m_{33}] + ([m_{1y}] + [m_{2x}])\Omega_x\Omega_y + \\ &+ ([m_{2z}] + [m_{3y}])\Omega_y\Omega_z + ([m_{3x}] + [m_{1z}])\Omega_z\Omega]\{\delta_e\} = \{q_e\} + \{q_e^*\} - [m_{oe}^i][R]^T\{\ddot{r}_o\} - \\ &- E_x([m_{3y}] - [m_{2z}]) - E_y([m_{1z}] - [m_{3x}]) - E_z([m_{2x}] - [m_{1y}]) - \\ &+ (\Omega_y^2 + \Omega_z^2)[m_{1x}] + (\Omega_z^2 + \Omega_x^2)[m_{2y}] + (\Omega_x^2 + \Omega_y^2)[m_{3z}] + \\ &- ([m_{1y}] + [m_{2x}])\Omega_x\Omega_y - ([m_{2z}] + [m_{3y}])\Omega_y\Omega_z - ([m_{3x}] + [m_{1z}])\Omega_z\Omega_x,\end{aligned} \tag{14}$$

or, if we note:

$$[m_e] = [m_{11}] + [m_{22}] + [m_{33}]; \quad \{q_e^i(\Omega)\} = \int_V [N]^T [\Omega][\Omega]\{r'\}\rho dV, \tag{15}$$

$$[c_e(\Omega)] = \int_V [N]^T [\Omega][N]\rho dV; \quad [k_e(E)] = \int_V [N]^T [E][N]\rho dV, \tag{16}$$

$$[k_e(\Omega^2)] = \int_V [N]^T [\Omega][\Omega][N]\rho dV; \quad \{q_e^i(E)\} = \int_V [N]^T [E]\{r'\}\rho dV, \tag{17}$$

it result the motion equations for the finite element analyzed in a compact form:

$$[m_e]\{\ddot{\delta}_e\}+2[c_e]\{\dot{\delta}_e\}+([k_e]+[k_e(E)]+[k_e(\Omega^2)])\{\delta_e\}=$$
$$=\{q_e\}+\{q_e^*\}-\{q_e^i(E)\}-\{q_e^i(\Omega^2)\}-[m_{oe}^i][R]^T\{\ddot{r}_o\}, \quad (18)$$

where $\vec{\Omega}$ represent the angular velocity and $\vec{E}$ the angular acceleration with the components in the local coordinate system.

3 Assembling procedures and Liaison forces eliminating

The unknowns in the elasto-dynamic analysis of a mechanical system with liaisons are the nodal displacements and the liaison forces. By assembling the motion equations written for each finite element we try to eliminate the liaisons forces and the motion equations will contain only nodal displacements as unknowns. The liaison between finite elements are realized by the nodes where the displacements can be equal or can exists other type of functional relations between these. When two finite elements belong to two different elements (bodies) the liaison realized by node can imply relations more complicated between nodal displacement and their derivatives. Generally, the relations between the first order derivative of the nodal displacements can be expressed by the linear formulas:

$$\{\dot{\Delta}\}=[A]\{\dot{q}\}, \quad (19)$$

where by $\{\Delta\}$ we have noted the nodal displacement vector and by $\{q\}$ the nodal independent displacements. By differentiation (14) we obtain:

$$\{\ddot{\Delta}\}=[A]\{\ddot{q}\}+[\dot{A}]\{\dot{q}\}. \quad (20)$$

The transformation relations between the displacements expressed in the global fix coordinate system $\{\Delta_e\}$ and the displacements expressed in the local mobile coordinate system $\{\delta_e\}$ are:

$$\{\Delta_e\}=[R_e]\{\delta_e\}, \quad (21)$$

where index *e* denote the e-th element.

For a single finite element that belong to an elastic component of the system that has a general three-dimensional rigid motion with the angular velocity $\vec{\omega}$ and the angular acceleration $\vec{\varepsilon}$ (or $\vec{\Omega}$ and $\vec{E}$ in the mobile co-ordinate system) we consider the motion equations obtained by the relation (21). For the other cases the procedures are the same.

The equations are expressed in the local mobile reference system. If we write these equations in the global fix coordinate system, they keep there form:

$$[M_e]\{\ddot{\Delta}_e\}+2[C_e]\{\dot{\Delta}_e\}+([K_e]+[K_e(\varepsilon)]+[K_e(\omega^2)])\{\Delta_e\}=$$
$$=\{Q_e\}+\{Q^*_e\}-\{Q^i_e(E)\}-\{Q^i_e(\Omega^2)\}-[M^i_{oe}][R]^T\{\ddot{r}_o\}. \quad (22)$$

We will note in the following:

$$\{Q_e\}^{inertia}=-\{Q^i_e(E)\}-\{Q^i_e(\Omega^2)\}-[M^i_{oe}][R]^T\{\ddot{r}_o\}, \quad (23)$$

and we can obtain finally the motion equations for the whole structure, referred to the global coordinate system, under the form:

$$[M]\{\ddot{\Delta}\}+2[C]\{\dot{\Delta}\}+([K]+[K(\varepsilon)]+[K(\omega^2)])\{\Delta\}=$$
$$=\{Q\}^{ext}+\{Q^*\}^{ext}+\{Q\}^{leg}+\{Q\}^{inertie}. \quad (24)$$

If we take into account the relations (23) and (24) we can write:

$$[M]\;([A]\{\ddot{q}\}+[\dot{A}]\{\dot{q}\})+2[C][A]\{\dot{q}\}+([K]+[K(\varepsilon)]+[K(\omega^2)])[A]\{q\}=$$
$$=\{Q\}^{ext}+\{Q^*\}^{ext}+\{Q\}^{leg}+\{Q\}^{inertia}. \quad (25)$$

It can be shown that the work of the liaison forces for system can be written:

$$dL=\{\dot{\Delta}\}^T\{Q\}^{leg}dt=\{\dot{q}\}^T[A]^T\{Q\}^{leg}dt. \quad (26)$$

But the work due to the liaison forces is null for an ideal system and the independence of the nodal coordinates q offer the relation:

$$[A]^T\{Q\}^{leg}=0, \quad (27)$$

that is the basic relation in the following.

4 Motion equations assembling

We consider relation (24) and we pre-multiply this with $[A]^T$. We obtain:

$$[A]^T[M][A]\{\ddot{q}\}+([A]^T[M][\dot{A}]+2[C][A])\{\dot{q}\}+$$
$$[A]^T([K]+[K(\varepsilon)]+[K(\omega^2)])[A]\{q\}=$$
$$=[A]^T\{Q\}^{ext}+[A]^T\{Q^*\}^{ext}+[A]^T\{Q\}^{leg}+[A]^T(\{Q^*\}^{leg}+\{Q\}^{inertie}). \quad (28)$$

If we take into account the relation (27) the Liaison forces (the nodal forces) vanish and it result a system of equations without liaison forces and the unknown are only the nodal displacements. This result justify the assembling methods used in the case of the mechanical systems with Liaisons analyzed via finite element method.

$$[A]^T[M][A]\{\ddot{q}\} + ([A]^T[M][\dot{A}] + 2[C][A])\{\dot{q}\} + [A]^T([K] + [K(\varepsilon)] + [K(\omega^2)])[A]\{q\} = = [A]^T\{Q\}^{ext} + [A]^T\{Q^*\}^{ext} + [A]^T\{Q\}^{inertia} \quad (29)$$

The system of differential equations obtained is nonlinear, the matrix of the left term depending on the configuration of the multi-body system. These equations contain the "rigid motion" of the system and for these they have one or more singularities. To solve the equations the rigid motion must be eliminated.

5 The influence of the Coriolis terms

The matrix $[c]$ is skew-symmetric. If we want to obtain the energy balance by integration, we obtain that the variation of energy due to the term skew-symmetric is null. Consequently, the Coriolis term only transfer the energy between the independent coordinates of the system and had no role in the dissipation of the energy. If we consider now a motion mode on the form:

$$\{q\} = \{A\}sin(\omega t + \varphi), \quad (30)$$

and we introduce in the motion equations, where the forces are considered null, we obtain:

$$-\omega^2[m]\{A\}sin(\omega t+\varphi)+\omega[c]\{A\}cos(\omega t+\varphi)+([k]+ +[k_\varepsilon]+[k_{\omega^2}])\{A\}sin(\omega t+\varphi)=\{0\}. \quad (31)$$

If we pre-multiply with $\{A\}^T$ and we consider the relations:

$$\omega\{A\}^T[c]\{A\} = 0, \text{ and } \{A\}^T[k_\varepsilon]\{A\} = 0, \quad (32)$$

([c] and $[k_\varepsilon]$ are skew-symmetric), it results:

$$\omega^2 = \frac{\{A\}^T[k]\{A\} + \{A\}^T[k_{\omega^2}]\{A\}}{\{A\}^T[m]\{A\}}. \quad (33)$$

This relation can not express, in a direct way, the influence of the matrix [c] in the eigenvalues calculus, but this influence is present by the eigenvectors $\{A\}$. The terms $[c]$ has an influence on the values of the eigenvalues. Some of the eigenvalues increase and the other decrease. This variation is presented, extended, in the paper. Between these values there exist some interesting relations.

6 Conclusions

The problems involved by finite element analysis of an elastic system are the followings:
- the strong geometric non-linearity of the motions equations and the additional term that appear in these;
- the motion equations are valid only for the "frozen" system, for a very short interval of time.

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