A Periodic State Feedback Control Law for a Class of Continuous-Time Linear Periodic Systems

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Abstract: In this report, we propose a periodic state feedback control law for a class of continuous-time linear periodic systems. The proposed control law generates a sufficiently high frequency sinusoidal in the coefficient of a closed loop system, then the closed loop system is shown to be exponentially stable based on the averaging method. It is also shown that a closed loop solution is approximated by a solution of the average system. The effectiveness of the proposed method is demonstrated by a numerical example of attitude control of small satellites with magnetic actuator.

Key–Words: Linear periodic system, linear time-varying system, characteristic multiplier, averaging method, satellite, attitude control

1 Introduction

The specification of characteristic multipliers is one of the fundamental problems for linear periodic systems and has been addressed by a number of authors from the following perspectives:

i) When can we arbitrarily assign characteristic multipliers by periodic state feedback ?

ii) How can we practically compute a periodic feedback gain ?

iii) How can we improve the transient response during a period ?

In his fundamental paper, Brunovsky [2] gave the complete solution for i): if the system is controllable, there exists a continuous periodic feedback that allows all characteristic multipliers to be freely assigned. However this approach does not give any solution for ii) and iii). Since the implicit function theorem is used in his constructive proof, it is difficult to compute the periodic feedback gain. Since the periodic gain becomes impulsive, the transient response is stimulated by the impulsive input.

Kabamba [3] gave an alternative solution for i) and ii) in the framework of sampled-data control. He gave an explicit formula for designing a piecewise continuous hold function by reducing the stabilization problem into the pole placement problem for linear time-invariant discrete-time systems. In addition, this approach is extended to multi-rate sampled-data control in order to answer iii) [1, 5]. Then the transient response can be improved at finite points during a period, but it is still vibrating except for those points. So this modification is not sufficient for iii). In addition, a sampled-data control law can not be always implemented by continuous-time feedback. So this approach is not satisfactory for i). Since the state transition matrix and the controllability Gramian is used, it is necessary to numerically integrate them in general. So this approach is not also satisfactory for ii).

Tornambe [7] also gave an alternative solution for i). He gave an explicit procedure to shift characteristic multipliers under the strong assumption that all characteristic multiplies are positive and real. It is necessary to compute left eigenvectors and indefinite integrals, so this approach is not acceptable for ii) in addition to iii).

Therefore we restrict our attention to the following system

$$\dot{x} = Ax + b\, g(t)\, u \tag{1}$$

$$g(t) := \prod_{l=1}^{L} \sin\left(\frac{2\pi}{T}(t - \phi_l T)\right) \tag{2}$$

where $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n$ are constant matrices, L is a positive integer and ϕ_l $(0 \leq \phi_l < 1)$ are rational numbers which are assumed to be different to each other for simplicity.

We note that this class is restrictive but significant, since a class of attitude stabilization problem for small satellites can be described in this framework as shown in Section 4. We also note that it is not possible to apply the input transformation such as $u = \frac{1}{g(t)}v$ for a new input v, since $\frac{1}{g(t)}$ becomes unbounded.

The aim of this report is to give answers for the three questions above for the system (1). The proposed method consists of two steps. In the first step, a constant matrix is selected to specify the desired trajectory. In the second step, a scalar periodic function is selected to generate a sufficiently high frequency trigonometric function in the coefficient of a closed loop system. A linear periodic gain is constructed as the multiplication of the selected constant matrix and the selected scalar periodic function. Based on the averaging method [4], it is shown that a closed loop system is exponentially stable. Furthermore a closed loop solution is approximated by a solution of the average system.

In summary, we give an explicit formula for designing a continuous periodic state feedback control law for the system (1) such that

- i') characteristic multipliers are arbitrarily assigned in the asymptotic sense
- ii') a periodic feedback gain is computed by symbolic computations
- iii') a closed loop solution is arbitrarily approximated by a solution of the averaged linear time invariant system

as shown in Section 3.

Notations: $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n\times m}$ denote the set of real numbers, n-dimensional Euclidean space, and the set of real matrices. $\|X\|$ denotes a Euclidean norm for a vector X, and denotes a matrix induced 2-norm for a matrix X. X' denotes the transpose of a matrix X. In order to express the parameter dependence of functions or solutions of differential equations, we use the following notation $x(t, N)$ which is a function of a time t and depends on a parameter N. Therefore $\dot{x}(t, N)$ denotes the derivative of x by t for a fixed parameter N.

2 Basic Idea

In this section we introduce our basic idea with an unstable scalar AR model with the sinusoidal coefficient

$$\dot{x} = ax + \sin\left(\frac{2\pi}{T}t\right)u, \quad a > 0 \tag{3}$$

where $x \in \mathbb{R}$ is the state variable, $u \in \mathbb{R}$ is the input. The control objective is to find a stabilizing state feedback control as well as to improve the transient response during a period.

2.1 Proportional Control

Let us start with the proportional control

$$u = k\,x.$$

A closed loop solution is given by

$$x(t) = \exp\left(at + \frac{Tk}{2\pi}\left(1 - \cos\left(\frac{2\pi}{T}t\right)\right)\right)x(0),$$

whose absolute value is bounded from below

$$|x(t)| \geq \exp\left(at - \frac{T|k|}{\pi}\right)|x(0)|.$$

It then follows that

$$\lim_{t\to\infty} |x(t)| = \infty$$

for any k and $x(0) \neq 0$, therefore there exists no stabilizing proportional feedback for (3).

2.2 Periodic Control

The main difficulty arises from the sign indefiniteness of the input coefficient, which disables us to utilize the concept of negative feedback control. In order to avoid this difficulty, let us continue with the periodic control

$$u = 2k\sin\left(\frac{2\pi}{T}t\right)x.$$

A closed loop solution is given by

$$x(t) = \exp\left((a+k)t - \frac{Tk}{4\pi}\sin\left(\frac{4\pi}{T}t\right)\right)x(0), \tag{4}$$

whose absolute value is bounded from above

$$|x(t)| \leq \exp\left((a+k)t + \frac{T|k|}{4\pi}\right)|x(0)|.$$

It then follows that

$$\lim_{t\to\infty} |x(t)| = 0$$

for any $k < -a$ and $x(0)$. We note that the pole of the average system

$$\dot{x} = (a+k)x \tag{5}$$

is shifted by the use of the identity

$$1 - \cos(2p) = 2\sin^2(p),$$

and this is the key idea to stabilize the closed loop system.

Now we have the freedom of the choice of k. Let us compare two extreme cases to choose an appropriate k.

Case 1: For sufficiently small k, *i.e.* $k \to -\infty$, the closed loop solution (4) converges to 0 sufficiently fast. However, since $|\frac{Tk}{4\pi}| \to \infty$, the closed loop solution (4) is extremely corrupted by $\frac{Tk}{4\pi}\sin(\frac{4\pi}{T}t)$.

Case 2: If we could choose k satisfying $|Tk| \simeq 0$, the closed loop solution (4) converges to the solution of the average system (5). However, for fixed a and T, the closed loop system becomes unstable for $k \simeq 0$. Hence it is not possible to choose such k.

Therefore the choice of k involves a certain trade-off between the stabilizability and the improvement of the closed loop solution.

2.3 Periodic Control by Raising Frequency

In order to achieve the additional freedom to shape the closed loop solution, let us continue with the periodic control

$$u = 2k \sum_{l=1}^{L} \sin\left((2l-1)\frac{2\pi}{T}t\right) x \tag{6}$$

where L is the positive integer. A closed loop solution is given by

$$x(t) = \exp\left((a+k)t - \frac{Tk}{4\pi L}\sin\left(\frac{4\pi L}{T}t\right)\right) x(0) \tag{7}$$

by the use of the following identity

$$2(\sin p)\sum_{l=1}^{L}\sin((2l-1)p) = 1 - \cos(2Lp).$$

The closed loop system is stabilized by choosing k satisfying $k < -a$. In addition, the closed loop solution (7) converges to the solution of its average system (5) by choosing sufficiently large L.

3 Main Results

In this section, we extend the idea of feedback control by raising frequency in (6) to a multidimensional case. Since the closed loop solution cannot be analytically computed for this case, the averaging method is applied to prove the closed loop stability as well as to show the asymptotic approximation by its average system.

Firstly we introduce the key lemma:

Lemma 1 *Given a positive integer N and a periodic function $g(t)$ defined in (2). Factor rational numbers ϕ_l contained in $g(t)$ as the ratio of coprime integers ν_l and δ_l $(\geq \nu_l)$ for each $l = 1, \cdots, L$, i.e.*

$$\phi_l =: \frac{\nu_l}{\delta_l}.$$

Then there exists a function $f(t, N)$ which is T-periodic and continuous for t, and satisfies

$$g(t)f(t,N) = 1 - \cos\frac{4\pi NG}{T}t, \tag{8}$$

where G is the least common multiple of δ_l.

Proof: ϕ_l is represented by the following fractional form

$$\phi_l = \frac{\psi_l}{2NG}, \quad \psi_l := \frac{2\nu_l NG}{\delta_l}$$

for each l. We note that ψ_l are nonnegative integers by their construction. Then there exist nonnegative integers φ_m $(m = 1, \cdots, 2NG - L)$ such that the set $\{\varphi_m\}$ is a complementary of $\{\psi_l\}$ in $\{0, \cdots, 2NG - 1\}$. Define a function

$$f(t,N) := \frac{2^{2NG-1}}{(-1)^{NG}} \prod_{m=1}^{2NG-L} \sin\left(\frac{2\pi}{T}\left(t - \frac{\varphi_m T}{2NG}\right)\right),$$

then it is clear that $f(t, N)$ is T-periodic and continuous for t. Since the identity

$$\prod_{r=1}^{s}\sin\left(z + \frac{2(r-1)\pi}{s}\right) = \frac{(-1)^{\frac{s}{2}}}{2^{s-1}}(1 - \cos sz)$$

is satisfied for even $s > 0$, it can be shown that $f(t, N)$ satisfies the identity (8). ■

Remark : For even G, it can be shown that there exists a function $f(t, N)$ which is T-periodic and continuous for t, and satisfies the identity

$$g(t)f(t,N) = 1 - \cos\frac{2\pi NG}{T}t. \tag{9}$$

Next the periodic feedback control law is derived for the linear periodic system (1). The candidate of the periodic feedback control law is given by

$$u = f(t, N)\, k\, x, \tag{10}$$

where $k \in \mathbb{R}^{1\times n}$ is the constant matrix and f is the scalar continuous function satisfying (8) for a given positive integer N. The choice of k and N will be clear in the subsequent of this section.

A closed loop system consisted of (1) and (10) is given by

$$\dot{x} = \left(A + bk - bk\cos\frac{4\pi NG}{T}t \right) x. \quad (11)$$

Define a parameter

$$\varepsilon := \frac{T}{4\pi NG}$$

and transform the time scale as follows

$$\tau = \frac{t}{\varepsilon}.$$

Then the closed loop system is transformed to be

$$\frac{dx}{d\tau} = \varepsilon(A + bk - bk\cos\tau)x. \quad (12)$$

Now we choose k and N so that the linear periodic system (12) becomes exponentially stable. In the first step, we choose k such that $A + bk$ is stable, *i.e.* the real parts of all eigenvalues of $A+bk$ are negative. Then the average system

$$\frac{dx}{d\tau} = \varepsilon(A + bk)x \quad (13)$$

becomes exponentially stable. In the second step, we choose sufficiently large N such that the closed loop system (11) becomes exponentially stable. Since we have chosen k such that the average system (13) is exponentially stable, it follows from the averaging method that the linear periodic system (12) is exponentially stable for sufficiently small ε [4].

Transforming the time scale into the original time scale t, the closed loop system (11) becomes exponentially stable for sufficiently large N. In order to prove that properties i')–iii') are satisfied for certain N, we need to quantitatively evaluate the statement of the averaging method. But we skip this analysis to save space.

Our main theorem is given as follows:

Theorem 2 *Suppose that (A, b) is stabilizable. Let $k \in \mathbb{R}^{1\times n}$ denotes a constant matrix such that $A + bk$ is stable. Let $P = P' > 0$ and $Q = Q' > 0$ denote the solutions of the Lyapnov equation*

$$(A + bk)'P + P(A + bk) = -Q.$$

Choose a constant $M \in \mathbb{R}$ satisfying

$$M > \frac{T}{4\pi Gc} \quad (14)$$
$$c := \|bk\|(2(2\|A + bk\| + \|bk\|)P_{\min}Q_{\min}^{-1} + 1,$$

where $P_{\max}$, $P_{\min}$, $Q_{\min}$ are the maximum, minimum eigenvalues of P and the minimum eigenvalue of Q respectively. Then the closed loop system (11) consisted of the system (1) and the control input (10) is exponentially stable for all integer $N \geq M$. Furthermore if the initial condition satisfy

$$\|x(0, N) - x_{ave}(0)\| \leq \frac{\rho_1}{N}$$
$$\|x(0, N)\| \leq \rho_2,$$

where x is the solution of (11) and x_{ave} is the solution of its average system

$$\dot{x}_{ave} = (A + bk)x_{ave}, \quad (15)$$

the initial response of (11) is approximated by

$$\|x(t, N) - x_{ave}(t)\| \leq \frac{\kappa(\rho_1, \rho_2)}{N} \quad (16)$$
$$\kappa(\rho_1, \rho_2) := \frac{P_{\max}^{\frac{1}{2}}}{P_{\min}^{\frac{1}{2}}}(\rho_1 + \rho_3\rho_2)$$
$$\rho_3 := \|bk\| + \frac{cP_{\max}}{P_{\min}}$$

for all $t \geq 0$ and for all integer $N \geq M$.

In this method, two independent designing parameters k and N are available. Firstly k is used to stabilize the average system. Then N is used to shape the closed loop solution by generating the sufficiently high frequency trigonometric function in the coefficient of the closed loop system. Both procedures are easily carried out via simple symbolic computations and the requirement ii') is satisfied. And it is clear from (16) that the requirement iii') is satisfied by choosing sufficiently large N.

Lastly we show that the requirement i') is satisfied. Let $\Phi(t, N)$ denotes the fundamental matrix of the closed loop system (11), *i.e.* Φ is the solution of the following initial value problem:

$$\dot{\Phi} = \left(A + bk - bk\cos\frac{4\pi NG}{T}t \right)\Phi, \ \Phi(0, N) = I.$$

Consider the same initial condition $x(0, N) = x_{ave}(0) =: x_0$ for (11) and (15). Substitute $t = T$ into (16), we have

$$\|(\Phi(T, N) - e^{(A+bk)T})x_0\| \leq \frac{\rho_3 P_{\max}^{\frac{1}{2}}}{NP_{\min}^{\frac{1}{2}}}\|x_0\|,$$

then it follows that $\Phi(T, N)$ uniformly converges to $e^{(A+bk)T}$, *i.e.*

$$\lim_{N\to\infty} \|\Phi(T, N) - e^{(a+bk)T}\| = 0.$$

Therefore characteristic multipliers of (1) can be assigned in the following sense:

Corollary 3 *Suppose that* (A, b) *is stabilizable. Let* $k \in \mathbb{R}^{1\times n}$ *a constant matrix such that* $A+bk$ *is stable. Then, as the integer* $N \to \infty$, *characteristic multipliers of the closed loop system (11) consisted of (1) and (10) converge to eigenvalues of* $e^{(A+bk)T}$.

Moreover, if (A, b) is controllable, poles of the average system (15) can be arbitrarily assigned. Then characteristic multipliers of the closed loop system (11) are arbitrarily assigned to corresponding characteristic multipliers in the asymptotic sense ($N \to \infty$). Hence the requirement i') is satisfied.

4 Illustrative Example

In this section we study the problem of attitude stabilization for small satellites with magnetic actuators. Observing the periodic nature of the geomagnetic field, the Euler's equation linearlized around the yaw axis is given by

$$\dot{x} = Ax + bg(t)u$$
$$A = \begin{bmatrix} 0 & 1 \\ 28.8 & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ g(t) = \cos(2\pi t)$$

where $x := [\ \gamma\ \dot{\gamma}\ ]'$ consists of the yaw angle γ and the yaw rate $\dot{\gamma}$. A period T is normalized to be $T = 1$. The control objective is to design a periodic state feedback control law such that the satellite is stabilized within a half period. So the requirement iii') is of crucial importance in this application. In addition, the magnitude of the control input is restricted to be less than 10 [A-m^2] from the practical reason [6].

The 1-periodic function g is represented by the form (2)

$$g(t) = \sin\left(2\pi\left(t - \frac{3\pi}{4}\right)\right),$$

the least common divisor G is chosen to be 4.

Since (A, b) is controllable, a linear periodic state feedback control law is designed based on Corollary 3. In the first step, we design the state feedback gain to be

$$k = [\ 8.52 \quad 1\ ]$$

such that the average system almost converges to 0 at $t = 0.5$ and the magnitude of the input for the average system to be less than 10 [A-m^2]. In the second step, we compute the periodic coefficient $f(t, N)$ based on (9), *e.g.*

$$f(t, 1) = 2(\cos 2\pi t - \cos 6\pi t)$$
$$f(t, 4) = 2(\cos 2\pi t - \cos 6\pi t + \cos 10\pi t - \cos 14\pi t + \cos 18\pi t - \cos 22\pi t + \cos 26\pi t - \cos 30\pi t).$$

A set of simulation is carried out to evaluate the effectiveness of the proposed method. The initial response is computed for $N = 1$ and $N = 4$ with the initial value $\gamma(0) = 2\pi/360$ [rad], $\dot{\gamma}(0) = 0$ [rad/period]. The initial responses of $\gamma, \dot{\gamma}$ (blue solid line) converge to those of the average system (red dot line) by raising frequency (see Fig. 1–4). The magnitude of the input u is bounded by 10 [A-m^2], therefore the control objective is satisfied (see Fig. 5–6).

5 Conclusion

A periodically time-varying control law was proposed for a class of continuous-time linear periodic systems. The designing procedure consists of two steps. Firstly a constant matrix is selected to specify the desired closed loop solution. Then a scalar periodic function is selected to stabilize the closed loop systems as well as to make the closed loop solution to asymptotically converges to the solution of the average system. Both procedures are computed via symbolic computations, therefore the proposed controller can be easily computed.

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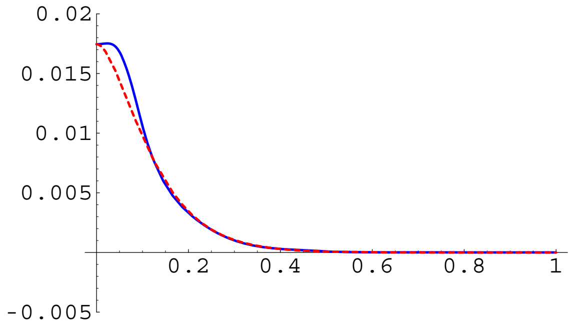

Figure 1: Yaw Angle γ $(N = 1)$

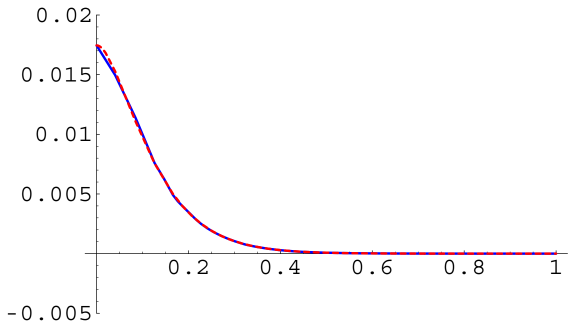

Figure 2: Yaw Angle γ $(N = 4)$

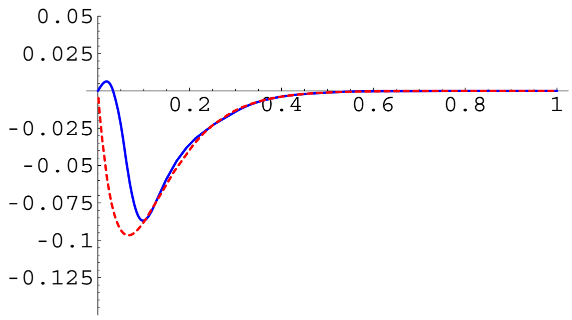

Figure 3: Yaw Rate $\dot{\gamma}$ $(N = 1)$

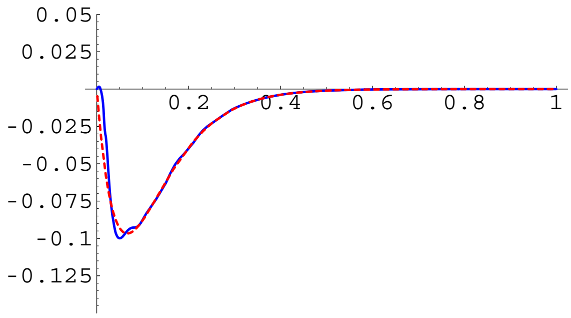

Figure 4: Yaw Rate $\dot{\gamma}$ $(N = 4)$

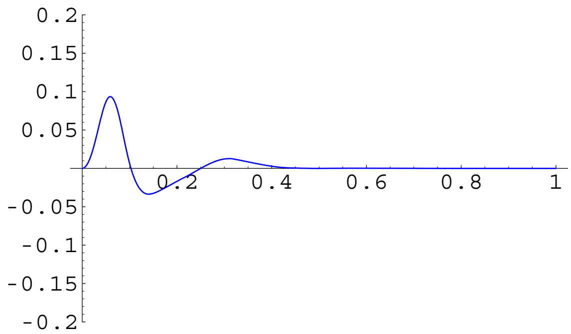

Figure 5: Input u $(N = 1)$

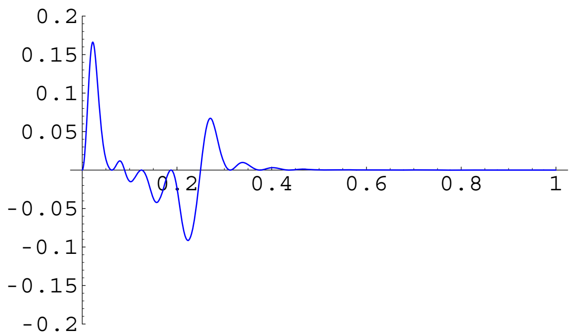

Figure 6: Input u $(N = 4)$