Stabilizability of Linear Periodic Continuous-Time Systems by Double Periodic State Feedback

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Abstract: This report gives a condition on stabilizability by double periodic state feedback for linear periodic continuous-time systems. Firstly stable, unstable and reachable subspaces are extended from linear time invariant case. They are spanned by a smooth double periodic basis and are invariant with respect to the fundamental solution. Then a geometric characterization of stabilizability is given in the sense that a linear periodic continuous-time system is stabilizable by double periodic state feedback iff the unstable subspace is contained in the reachable subspace.

Key–Words: Linear periodic system, linear time-varying system, stabilizability, controllability, geometric approach

1 Introduction

Consider a linear periodic continuous-time system described by

$$\dot{x} = A(t)x + B(t)u \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x \in \mathbb{R}^n =: \mathcal{X}$ is the state, $u \in \mathbb{R}^p =: \mathcal{U}$ is the control input, and $A(t) : \mathcal{X} \to \mathcal{X}$, $B(t) : \mathcal{U} \to \mathcal{X}$ are maps which are continuous and periodic of period T (briefly T-periodic) with respect to time t.

The system (1) is said to be stabilizable if there exists a continuous T-periodic map $F(t) : \mathcal{X} \to \mathcal{U}$ such that the system

$$\dot{x} = (A(t) + B(t)F(t))x \tag{2}$$

is asymptotically stable. Several characterizations of stabilizability had been discussed in [1, 2, 5, 6, 7], until they were shown to be equivalent in [1, 7]. Among them the above definition corresponds to W-stabilizability.

In this report we focus on the notion of K-stabilizability, *i.e. the system (1) is stabilizable iff an uncontrollable part is asymptotically stable*, which makes sense after transforming to the canonical decomposition.

On the contrary, we consider weaker notion of stabilizability and give a condition on stabilizability by continuous $2T$-periodic state feedback (briefly double periodic) in the coordinate free setting.

Theorem 1 *The system (1) is stabilizable by $2T$-periodic state feedback, i.e. there exists a continuous $2T$-periodic map $F(t) : \mathcal{X} \to \mathcal{U}$ such that the system (2) is asymptotically stable, iff*

$$\mathcal{X}_b(\Phi; t) \subset \mathcal{R}(t) \tag{3}$$

iff

$$\mathcal{X}_b(\Phi; 0) \subset \mathcal{R}(0). \tag{4}$$

where Φ denotes a fundamental solution associated to A, $\mathcal{X}_b$ denotes an unstable subspace defined by (5) and $\mathcal{R}$ denotes a reachable subspace defined by (8).

The condition (3) is described by periodically time varying subspaces $\mathcal{X}_b$ and $\mathcal{R}$, and it is reduced to the time independent condition (4). Hence they are natural extension of geometric characterization of stabilizability for linear time invariant systems (Theorem 2.3 in [11]) and are intuitively recognizable in the sense that *"the system is stabilizable by $2T$-periodic state feedback iff unstable modes are controllable"*.

In the subsequent of this section, a stable subspace $\mathcal{X}_g$ and an unstable subspace $\mathcal{X}_b$ are defined in Section 2. A reachable subspace $\mathcal{R}$ is defined in Section 3. Then, using projections onto $\mathcal{R}$ and its orthogonal complement $\mathcal{R}^\perp$, reachable and unreachable subsystems are represented in the coordinate free setting. Since $\mathcal{R}^\perp$ is invariant with respect to Φ, the factor space $\mathcal{X}/\mathcal{R}^\perp$ is well defined. An unreachable subsystem is also represented using the canonical projection onto $\mathcal{X}/\mathcal{R}^\perp$ in the coordinate free setting. It is also shown that controllability is not affected by state feedback as in the time invariant case. Finally the above geometric characterization of stabilizability and the comparison with G-stabilizability are given in Section 4. We make some concluding remarks in Section 5.

Notations:

$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	set of real vectors
$\mathbb{R}^{n\times m}$	set of real matrices
$\mathbb{C}$	set of complex numbers
$\mathbb{C}_b$	$:= \{\lambda \in \mathbb{C} : \lvert\lambda\rvert \geq 1\}$
$\mathbb{C}_g$	$:= \{\lambda \in \mathbb{C} : \lvert\lambda\rvert < 1\}$
$\simeq$	vector space isomorphism
$\oplus$	direct sum
$/$	factor space
$\vert$	map restriction
$\operatorname{Im} M$	image of $M \in \mathbb{R}^{n\times m}$ *i.e.* $\operatorname{Im} M := \{M\xi : \xi \in \mathbb{R}^m\}$
$\operatorname{Ker} M$	image of $M \in \mathbb{R}^{n\times m}$ *i.e.* $\operatorname{Ker} M := \{\xi \in \mathbb{R}^m : M\xi = 0\}$
$\sigma(M)$	spectrum of $M \in \mathbb{R}^{n\times n}$
$1_{\mathcal{X}}$	identity map on $\mathcal{X}$
Φ	fundamental solution associated to A *i.e.* $\frac{\partial}{\partial s}\Phi(s,t) = A(s)\Phi(s,t), \Phi(t,t) = 1_{\mathcal{X}}$
Θ	fundamental solution associated to $A + BF$
$\mathcal{X}$	state space
$\mathcal{X}_b$	unstable subspace defined by (5)
$\mathcal{X}_g$	stable subspace defined by (6)
W_r	reachability Gramian defined by (9)
$\mathcal{R}$	reachable subspace defined by (8)
$\mathcal{R}^\perp$	$:= \{x \in \mathcal{X} : x'y = 0, \forall y \in \mathcal{R}\}$

2 Stable and Unstable Subspaces

In this section, we introduce stable and unstable subspaces, which are natural extension from linear time invariant case (see [11]).

In order to derive a periodic stabilizing feedback consisted of real matrices, it is important to discuss periodic subspace of state space in real vector field. To this end, we regard $A(t)$ as $2T$-periodic. Then there exist a real Floquet factor (see *e.g.* [8]).

Lemma 2 *There exist continuously real differentiable $2T$-periodic isomorphism $Z(t) : \mathcal{X} \to \mathcal{X}$ and real constant map $\Lambda : \mathcal{X} \to \mathcal{X}$ such that*

$$\Phi(s,t) = Z(s)e^{\Lambda(s-t)}Z(t)^{-1}.$$

As is the case with linear time invariant systems, we may wish to distinguish stable modal subspace and unstable modal subspace respectively. For this, factor the characteristic polynomial of $\Phi(T,0)$ in the form

$$\det(\lambda 1 - \Phi(2T,0)) =: \alpha_b(\lambda)\alpha_g(\lambda)$$

where the zeros in $\mathbb{C}$ of α_b (resp. α_g) belongs to $\mathbb{C}_b$ (resp. $\mathbb{C}_g$). Firstly we introduce the preliminary lemma.

Lemma 3

$$\begin{aligned}\Phi(t,0)\operatorname{Ker}\alpha_b(\Phi(2T,0)) &= \operatorname{Ker}\alpha_b(\Phi(t+2T,t))\\ \Phi(t,0)\operatorname{Ker}\alpha_g(\Phi(2T,0)) &= \operatorname{Ker}\alpha_g(\Phi(t+2T,t)).\end{aligned}$$

Now define the *unstable subspace $\mathcal{X}_b$* and the *stable subspace $\mathcal{X}_g$* by

$$\begin{aligned}\mathcal{X}_b(\Phi;t) &:= \Phi(t,0)\operatorname{Ker}\alpha_b(\Phi(2T,0))\\ &= \operatorname{Ker}\alpha_b(\Phi(t+2T,t)) \qquad (5)\\ \mathcal{X}_g(\Phi;t) &:= \Phi(t,0)\operatorname{Ker}\alpha_g(\Phi(2T,0))\\ &= \operatorname{Ker}\alpha_g(\Phi(t+2T,t)). \qquad (6)\end{aligned}$$

Those definitions are extended from linear time invariant systems (see Chap. 2 in [11]) and become periodically time varying. It is worthy to mention that the state $\xi \in \mathcal{X}$ is in $\mathcal{X}_g(\Phi;t)$ at time t iff the state $\Phi(s,t)\xi$ converges to 0 as $s \to \infty$.

Then it follows from Lemma 2 and Lemma 3 that $\mathcal{X}_b(\Phi;t)$ and $\mathcal{X}_g(\Phi;t)$ satisfy the following properties:

Proposition 1 1) *Dimensions of $\mathcal{X}_b$ and $\mathcal{X}_g$ are independent of time.*

$$\mathcal{X}_b(\Phi;t) \simeq \mathcal{X}_b(\Phi;0),\ \mathcal{X}_g(\Phi;t) \simeq \mathcal{X}_b(\Phi;0).$$

2) *$\mathcal{X}_b(\Phi;t)$ and $\mathcal{X}_g(\Phi;t)$ are spanned by a continuously differentiable $2T$-periodic basis.*

3) *$\mathcal{X}_b$ and $\mathcal{X}_g$ are Φ-invariant both forward and backward in time.*

$$\begin{aligned}\Phi(s,t)\mathcal{X}_b(\Phi;t) &= \mathcal{X}_b(\Phi;s)\\ \Phi(s,t)\mathcal{X}_g(\Phi;t) &= \mathcal{X}_g(\Phi;s).\end{aligned}$$

4) *$\mathcal{X}_b$ and $\mathcal{X}_g$ decompose $\mathcal{X}$ at each time.*

$$\mathcal{X} = \mathcal{X}_b(\Phi;t) \oplus \mathcal{X}_g(\Phi;t)$$

Proposition 1 has importance from the aspect of geometric control. In case of linear time invariant systems, the stable subspace $\mathcal{X}_g$ and the unstable subspace $\mathcal{X}_b$ are independent of time t. And their invariance properties are described by $A\mathcal{X}_b \subset \mathcal{X}_b$ and $A\mathcal{X}_g \subset \mathcal{X}_g$ using A. In case of linear periodic systems, $\mathcal{X}_u$ and $\mathcal{X}_g$ become periodically time varying and spanned by a smooth basis. Furthermore their invariance properties are described using the fundamental solution Φ, which corresponds to $e^{A(s-t)}$ for linear time invariant systems.

3 Reachable Subspace

In this section, we introduce a reachable subspace, which is also natural extension from linear time invariant case (see [11]).

A linear continuous-time system of the form (1), which is not necessarily to be periodic, is said to be completely controllable if there exits an $s \geq t$ such that the integral equation

$$\Phi(s,t)x_0 + \int_t^s \Phi(s,\tau)B(\tau)u(\tau)d\tau = x_1 \qquad (7)$$

has a solution u in the admissible class of control functions for arbitrarily given $x_0, x_1 \in \mathcal{X}$ and $t \in \mathbb{R}$ [4]. For general linear continuous-time systems, (1) is completely controllable iff there exists a finite $s \geq t$ such that (7) has an admissible solution, while s depends on t.

For linear periodic continuous-time systems, Brunovsky showed that (1) is completely controllable iff (7) has an admissible solution for $s = t+nT$ where n is the dimension of the state space $\mathcal{X}$ [3].

In the study of controllability, a state x reachable from 0 is of particular importance. Now define the *reachable subspace* $\mathcal{R}$ by

$$\mathcal{R}(t) := \{L(t, t-nT)u \; : u \text{ is admissible on } [t-nT, t)\} \qquad (8)$$

where

$$L(s,t)u := \int_t^s \Phi(s,\tau)B(\tau)u(\tau)d\tau$$

is a linear map from the space of piecewise continuous control functions into $\mathcal{X}$. $\mathcal{R}(t)$ denotes the reachable set by the admissible control $\{u(\tau) : t-nT < \tau < t\}$ starting from $x(t-nT) = 0$.

In analogy with general linear continuous-time systems, $\mathcal{R}$ can be represented by the image of the reachability Gramian W_r defined by

$$W_r(s,t) := \int_t^s \Phi(s,\tau)B(\tau)B(\tau)'\Phi(s,\tau)'d\tau. \qquad (9)$$

Lemma 4

$$\mathcal{R}(t) = \operatorname{Im} W_r(t, t-nT) = \Phi(t,0)\operatorname{Im} W_r(0, -nT)$$

It follows from Lemma 2, Lemma 4 and Theorem 3 in [9] that $\mathcal{R}$ satisfy the following properties:

Proposition 2 1) *A dimension of $\mathcal{R}$ is independent of time.*

$$\mathcal{R}(t) \simeq \mathcal{R}(0).$$

2) *$\mathcal{R}(t)$ is spanned by a continuously differentiable $2T$-periodic basis.*

3) *$\mathcal{R}$ is Φ-invariant both forward and backward in time.*

$$\Phi(s,t)\mathcal{R}(t) = \mathcal{R}(s).$$

It is important to note that the statement of Proposition 2 2) is not always satisfied for a T-periodic basis, since it is not always possible to factor a T-periodic $W_r(t, t-nT)$ by a continuously differentiable real T-periodic invertible matrix $V(t)$ and a continuously differentiable real T-periodic positive definite matrix $E(t)$ (see Section 5 in [9]).

We also note that the statement of Proposition 2 3) is not always satisfied for general time-varying systems. In this case, it can be only shown that $\mathcal{R}(s) \supset \Phi(s,t)\mathcal{R}(t)$, *i.e.* $\mathcal{R}$ is Φ-invariant forward in time (see Theorem 1 and Theorem 2 in [10]).

3.1 Reachable and Unreachable Subsystems

Based on the invariance property of Φ, a reachable subsystem (resp. an unreachable subsystem) is represented by the restriction of Φ to $\mathcal{R}$ (resp. to the orthogonal complement of $\mathcal{R}$) in the coordinate free setting.

Let $\mathcal{R}^\perp(t)$ denotes the orthogonal complement of $\mathcal{R}(t)$ in $\mathcal{X}$

$$\mathcal{R}^\perp(t) := \{x \in \mathcal{X} : x'y = 0, \forall y \in \mathcal{R}\},$$

$P_{\mathcal{R}}(t) : \mathcal{X} \to \mathcal{R}(t)$ (resp. $P_{\mathcal{R}^\perp}(t) : \mathcal{X} \to \mathcal{R}^\perp(t)$) denotes the natural projection on $\mathcal{R}(t)$ along $\mathcal{R}^\perp(t)$ (on $\mathcal{R}^\perp(t)$ along $\mathcal{R}(t)$), $P^\dagger_{\mathcal{R}}(t) : \mathcal{R}(t) \to \mathcal{X}$ (resp. $P^\dagger_{\mathcal{R}^\perp}(t) : \mathcal{R}^\perp(t) \to \mathcal{X}$) denotes the insertion map of $\mathcal{R}(t)$ in $\mathcal{X}$ (of $\mathcal{R}^\perp(t)$ in $\mathcal{X}$). Then we have

$$P_{\mathcal{R}}(t)P^\dagger_{\mathcal{R}}(t) = 1_{\mathcal{R}} \qquad (10)$$

$$P_{\mathcal{R}^\perp}(t)P^\dagger_{\mathcal{R}^\perp}(t) = 1_{\mathcal{R}^\perp} \qquad (11)$$

$$P^\dagger_{\mathcal{R}}(t)P_{\mathcal{R}}(t) + P^\dagger_{\mathcal{R}^\perp}(t)P_{\mathcal{R}^\perp}(t) = 1_{\mathcal{X}}. \qquad (12)$$

By the proof of Proposition 2, $P_{\mathcal{R}}(t)$, $P_{\mathcal{R}^\perp}(t)$ are continuously differentiable $2T$-periodic epic. $P^\dagger_{\mathcal{R}}(t)$, $P^\dagger_{\mathcal{R}^\perp}(t)$ are continuously differentiable $2T$-periodic monic.

Let $\Phi_{\mathcal{R}}(s,t) : \mathcal{R}(t) \to \mathcal{R}(s)$ (resp. $\Phi_{\mathcal{R}^\perp} : \mathcal{R}^\perp(t) \to \mathcal{R}^\perp(s)$) denotes the restriction of Φ in $\mathcal{R}$ (resp. $\mathcal{R}^\perp$).

$$\begin{aligned}\Phi_{\mathcal{R}}(s,t) &:= \mathcal{R}(s)|\Phi(s,t)|\mathcal{R}(t) \\ &= P_{\mathcal{R}}(s)\Phi(s,t)P^\dagger_{\mathcal{R}}(t) \qquad (13)\end{aligned}$$

$$\begin{aligned}\Phi_{\mathcal{R}^\perp}(s,t) &:= \mathcal{R}^\perp(s)|\Phi(s,t)|\mathcal{R}^\perp(t) \\ &= P_{\mathcal{R}^\perp}(s)\Phi(s,t)P^\dagger_{\mathcal{R}^\perp}(t) \qquad (14)\end{aligned}$$

Then we have the following propositions:

Proposition 3 1) *Φ and $\Phi_{\mathcal{R}}$ commute with $P^{\dagger}_{\mathcal{R}}$.*

$$\Phi(s,t)P^{\dagger}_{\mathcal{R}}(t) = P^{\dagger}_{\mathcal{R}}(s)\Phi_{\mathcal{R}}(s,t)$$

2) *$\Phi_{\mathcal{R}}$ represents a fundamental solution of a controllable part of (1).*

$$\frac{\partial \Phi_{\mathcal{R}}(s,t)}{\partial s} = A_{\mathcal{R}}(s)\Phi_{\mathcal{R}}(s,t)$$

where

$$A_{\mathcal{R}}(t) := (\dot{P}_{\mathcal{R}}(t) + P_{\mathcal{R}}(t)A(t))P^{\dagger}_{\mathcal{R}}(t).$$

3) *$(A_{\mathcal{R}}, B_{\mathcal{R}})$ is controllable, where*

$$B_{\mathcal{R}}(t) := \mathcal{R}(t)|B(t) = P_{\mathcal{R}}(t)B(t).$$

Moreover it will be shown that characteristic multipliers of the reachable subsystem are arbitrarily assigned by state feedback (see Proposition 7).

Proposition 4 1) *Φ and $\Phi_{\mathcal{R}^\perp}$ commute with $P^{\perp}_{\mathcal{R}}$.*

$$P_{\mathcal{R}^\perp}(s)\Phi(s,t) = \Phi_{\mathcal{R}^\perp}(s,t)P_{\mathcal{R}^\perp}(t)$$

2) *$\Phi_{\mathcal{R}^\perp}$ represents a fundamental solution of an uncontrollable part of (1).*

$$\frac{\partial \Phi_{\mathcal{R}^\perp}(s,t)}{\partial s} = A_{\mathcal{R}^\perp}(s)\Phi_{\mathcal{R}^\perp}(s,t)$$

where

$$A_{\mathcal{R}^\perp}(t) := (\dot{P}_{\mathcal{R}^\perp}(t) + P_{\mathcal{R}^\perp}(t)A(t))P^{\dagger}_{\mathcal{R}^\perp}(t).$$

3) *The uncontrollable part of (1) is not affected by any input.*

$$B_{\mathcal{R}^\perp}(t) := \mathcal{R}^{\perp}(t)|B(t) = P_{\mathcal{R}^\perp}(t)B(t) = 0.$$

Proposition 5 *The spectrum of Φ is divided by those of a controllable part $\Phi_{\mathcal{R}}$ and an uncontrollable part $\Phi_{\mathcal{R}^\perp}$.*

$$\sigma(\Phi(s,t)) = \sigma(\Phi_{\mathcal{R}}(s,t)) \cup \sigma(\Phi_{\mathcal{R}^\perp}(s,t))$$

where $\cup$ denotes the union with any common elements repeated.

3.2 Factor Space and Canonical Projection

In the previous subsection, we showed that $\Phi_{\mathcal{R}^\perp}$ represents an uncontrollable part of Φ. But it is worth mentioning that an unreachable set from 0 in $\mathcal{X}$ is $\mathcal{X} \setminus \mathcal{R}(t)$, and it would be more pertinent to introduce the set of equivalent class mod $\mathcal{R}(t)$. So we also characterize an unreachable subsystem on the factor space.

Due to the invariance property of Φ, the factor space $\mathcal{X}/\mathcal{R}(t)$ and the canonical projection $P_{\mathcal{X}/\mathcal{R}}(t) : \mathcal{X} \to \mathcal{X}/\mathcal{R}(t)$ are well-defined and are shown that the following properties:

Proposition 6 1) *$P_{\mathcal{X}/\mathcal{R}}(t)$ is a continuously differentiable $2T$-periodic epimorphism.*

2) *There exists a unique differentiable isomorphism $\Phi_{\mathcal{X}/\mathcal{R}}(s,t) : \mathcal{X}/\mathcal{R}(t) \to \mathcal{X}/\mathcal{R}(s)$ in $\mathcal{X}/\mathcal{R}$ by Φ*

$$P_{\mathcal{X}/\mathcal{R}}(s)\Phi(s,t) = \Phi_{\mathcal{X}/\mathcal{R}}(s,t)P_{\mathcal{X}/\mathcal{R}}(t).$$

3) *An uncontrollable part of (1) is not affected by any input.*

$$P_{\mathcal{X}/\mathcal{R}}(t)B(t) = 0.$$

3.3 Controllability and Feedback

In analogy with linear time invariant systems, it can be shown that controllability is not affected by state feedback.

Let $F(t) : \mathcal{X} \to \mathcal{U}(t)$ denotes a continuous $2T$-periodic map, $\Theta(s,t) : \mathcal{X} \to \mathcal{X}$ denotes the fundamental solution of the closed loop system (2), $\Theta_{\mathcal{R}} : \mathcal{R}(t) \to \mathcal{R}(s)$ denotes the restriction of Θ in $\mathcal{R}$

$$\begin{aligned}\Theta_{\mathcal{R}}(s,t) &:= \mathcal{R}(s)|\Theta(s,t)|\mathcal{R}(t) \\ &= P_{\mathcal{R}}(s)\Theta(s,t)P^{\dagger}_{\mathcal{R}}(t), \qquad (15)\end{aligned}$$

$F_{\mathcal{R}}$ denotes the restriction of F in $\mathcal{R}$

$$F_{\mathcal{R}}(t) := F(t)|\mathcal{R}(t) = F(t)P^{\dagger}_{\mathcal{R}}(t).$$

Then the following properties are satisfied:

Proposition 7 1) *$\mathcal{R}$ is Θ-invariant both backward and forward in time.*

$$\Theta(s,t)\mathcal{R}(t) = \mathcal{R}(s)$$

2) *$\Theta_{\mathcal{R}}$ denotes a fundamental solution of a controlled part of (2).*

$$\frac{\partial \Theta_{\mathcal{R}}(s,t)}{\partial s} = (A_{\mathcal{R}}(s) + B_{\mathcal{R}}(s)F_{\mathcal{R}}(s))\Theta_{\mathcal{R}}(s,t)$$

3) *Θ and $\Phi_{\mathcal{X}/\mathcal{R}}$ commute with $P_{\mathcal{X}/\mathcal{R}}$.*

$$P_{\mathcal{X}/\mathcal{R}}(s)\Theta(s,t) = \Phi_{\mathcal{X}/\mathcal{R}}(s,t)P_{\mathcal{X}/\mathcal{R}}(t)$$

4) *The spectrum of Θ is divided by those of a controlled part of (2), i.e. $\Theta_{\mathcal{R}}$, and an uncontrollable part of (2), i.e. $\Phi_{\mathcal{R}^\perp}$ or $\Phi_{\mathcal{X}/\mathcal{R}}$.*

$$\begin{aligned}\sigma(\Theta(s,t)) &= \sigma(\Theta_{\mathcal{R}}(s,t)) \cup \sigma(\Phi_{\mathcal{R}^\perp}(s,t)) \\ &= \sigma(\Theta_{\mathcal{R}}(s,t)) \cup \sigma(\Phi_{\mathcal{X}/\mathcal{R}}(s,t))\end{aligned}$$

where $\cup$ denotes the union with any common elements repeated.

This proposition implies that controllability is not affected by state feedback. Since $(A_{\mathcal{R}}, B_{\mathcal{R}})$ is controllable by Proposition 3 3), $\Theta_{\mathcal{R}}(s,t)$ denotes a controllable part of the closed loop system Θ and characteristic multipliers of $\Theta_{\mathcal{R}}(s,t)$ are arbitrarily assigned. On the contrary, from Proposition 4 3), characteristic multipliers of $\Phi_{\mathcal{X}/\mathcal{R}}(s,t)$ are invariant by state feedback. Therefore 4) implies that a controllable part of characteristic multipliers $\sigma(\Phi_{\mathcal{R}}(2T,0))$ is arbitrarily shifted to desired places, while an uncontrollable part of characteristic multipliers $\sigma(\Phi_{\mathcal{X}/\mathcal{R}}(2T,0))$ cannot be shifted by any state feedback.

4 Geometric Characterization of Controllability and Stabilizability

Now we characterize controllability and stabilizability of the linear periodic continuous-time system (1) in the geometric framework.

Proposition 8 1) *The system (1) is controllable iff*

$$\mathcal{X} = \mathcal{R}(t).$$

2) *The system (1) is stabilizable by continuous $2T$-periodic state feedback iff*

$$\mathcal{X}_b(\Phi;t) \subset \mathcal{R}(t).$$

Proof: 1) is clear. So we prove 2).
Sufficiency: By the Φ-invariance of $\mathcal{R}$, we have

$$\Phi(s,t)P_{\mathcal{R}}^\dagger(t) = P_{\mathcal{R}}^\dagger(s)\Phi_{\mathcal{R}}(s,t)$$

Let $X_b(t) : \mathcal{X}_b(\Phi;t) \to \mathcal{X}$ denotes the insertion of $\mathcal{X}_b(\Phi;t)$ in $\mathcal{X}$, we also have

$$\Phi(s,t)X_b(t) = X_b(s)\Phi_{\mathcal{X}_b}(s,t).$$

where $\Phi_{\mathcal{X}_b}(s,t) : \mathcal{X}_b(\Phi;t) \to \mathcal{X}_b(\Phi;s)$ denotes the restriction of Φ in $\mathcal{X}_b(\Phi;\cdot)$

$$\Phi_{\mathcal{X}_b}(s,t) := \mathcal{X}_b(\Phi;s)|\Phi(s,t)|\mathcal{X}_b(\Phi;t).$$

By assumption, there exists an $2T$-periodic insertion map $Q(t) : \mathcal{X}_b(\Phi;t) \to \mathcal{R}(t)$ such that

$$X_b(t) = P_{\mathcal{R}}^\dagger(t)Q(t).$$

Then we have

$$\begin{aligned}P_{\mathcal{R}}^\dagger(s)\Phi_{\mathcal{R}}(s,t)Q(t) &= \Phi(s,t)P_{\mathcal{R}}^\dagger(t)Q(t) \\ &= \Phi(s,t)X_b(t) \\ &= X_b(s)\Phi_{\mathcal{X}_b}(s,t) \\ &= P_{\mathcal{R}}^\dagger(s)Q(s)\Phi_{\mathcal{X}_b}(s,t).\end{aligned}$$

Since $P_{\mathcal{R}}^\dagger$ is monic, we have

$$\Phi_{\mathcal{R}}(s,t)Q(t) = Q(s)\Phi_{\mathcal{X}_b}(s,t).$$

Since Q is $2T$-periodic, we have

$$\begin{aligned}\Phi_{\mathcal{R}}(t+2T,t)Q(t) &= Q(t+2T)\Phi_{\mathcal{X}_b}(t+2T,t) \\ &= Q(t)\Phi_{\mathcal{X}_b}(t+2T,t),\end{aligned}$$

by which

$$\sigma(\Phi_{\mathcal{R}}(t+2T,t)) \supset \sigma(\Phi_{\mathcal{X}_b}(t+2T,t)).$$

We note that

$$\begin{aligned}\sigma(\Phi_{\mathcal{R}}(t+2T,t)) &\cup \sigma(\Phi_{\mathcal{X}/\mathcal{R}}(t+2T,t)) \\ &= \sigma(\Phi(t+2T,t))\end{aligned}$$

holds from Proposition 5 and

$$\begin{aligned}\sigma(\Phi_{\mathcal{X}_b}(t+2T,t)) &\cup \sigma(\Phi_{\mathcal{X}_g}(t+2T,t)) \\ &= \sigma(\Phi(t+2T,t))\end{aligned}$$

by the definition of $\mathcal{X}_b$ and $\mathcal{X}_g$, where

$$\Phi_{\mathcal{X}_g}(s,t) := \mathcal{X}_g(s)|\Phi(s,t)|\mathcal{X}_g(t).$$

Hence

$$\sigma(\Phi_{\mathcal{X}/\mathcal{R}}(t+2T,t)) \subset \sigma(\Phi_{\mathcal{X}_g}(t+2T,t)) \subset \mathbb{C}_g.$$

On the other hand, since $(A_{\mathcal{R}}, B_{\mathcal{R}})$ is controllable, there exists an $2T$-periodic map $F_{\mathcal{R}}(t) : \mathcal{R}(t) \to \mathcal{U}$ such that $A_{\mathcal{R}} + B_{\mathcal{R}}F_{\mathcal{R}}$ is asymptotically stable. Choose the state feedback gain $F(t) := F_{\mathcal{R}}(t)P_{\mathcal{R}}^\dagger(t)$, then it follows from Proposition 7 2) that

$$\sigma(\Theta_{\mathcal{R}}(t+2T,t)) \subset \mathbb{C}_g.$$

Hence

$$\sigma(\Theta(t+2T,t)) \\ = \sigma(\Theta_{\mathcal{R}}(t+2T,t)) \cup \sigma(\Phi_{\mathcal{X}/\mathcal{R}}(t+2T,t)) \subset \mathbb{C}_g.$$

Necessity: Since the induced map in $\mathcal{X}/\mathcal{R}$ by Φ is invariant by state feedback, if there exists a stabilizing feedback gain, we have

$$\sigma(\Phi_{\mathcal{X}/\mathcal{R}}(t+2T,t)) \subset \mathbb{C}_g.$$

So if there exits a $\lambda \in \sigma(\Phi(t+2T,t)) \cap \mathbb{C}_b$, then $\lambda \in \sigma(\Phi_{\mathcal{R}}(t+2T,t))$. □

Theorem 1 is a direct consequence of the above proposition and Φ-invariance of $\mathcal{X}_b(\Phi;t)$ and $\mathcal{R}(t)$. We note that the time independent condition (4) is represented as follows:

Corollary 5 *Let Λ_j denotes real Jordan blocks consisted of unstable characteristic multipliers of $\Phi(2T,0)$ and Ξ_j denotes real matrices whose column vectors consist of real generalized eigenvectors, i.e.*

$$\Phi(2T,0)\Xi_j = \Xi_j\Lambda_j, \ \|\Lambda_j^{-1}\| \leq 1.$$

for all $j = 1, \cdots, l$, where l denotes a number of real Jordan blocks Λ_j consisted of unstable characteristic multipliers of $\Phi(2T,0)$, and $\|M\|$ denotes a matrix induced 2-norm of $M \in \mathbb{R}^{n\times m}$. Then the system (1) is stabilizable by $2T$-periodic state feedback iff

$$\operatorname{Im}\Xi_j \subset \operatorname{Im}W_r(0,-nT) \tag{16}$$

for all $j = 1, \cdots, l$.

Let us compare (16) with the G-stabilizability in [1]: let ξ_j denotes left eigenvectors of $\Phi(T,0)$, i.e.

$$\Phi(T,0)'\xi_j = \lambda_j\xi_j, |\lambda_j| \geq 1$$

then the system (1) is stabilizable by T-periodic state feedback iff

$$\xi_j \in \operatorname{Ker}W_c(nT,0) \ \Rightarrow \ \xi_j = 0 \tag{17}$$

for all j where

$$W_c(s,t) := \int_t^s \Phi(t,\tau)B(\tau)B(\tau)'\Phi(t,\tau)'d\tau.$$

The condition (16) consists of *real right* generalized eigenvectors of the Monodromy matrix, while (17) consists of *complex left* eigenvectors of the Monodromy matrix. Although we have discussed weaker notion of stabilizability, we have discussed in real vector field, therefore the condition (16) allows an explicit geometric interpretation in the sense that the system (1) is stabilizable by $2T$-periodic state feedback iff unstable modes of (1) are controllable at time $t = 0$ as shown in (4).

5 Conclusion

This report gives a geometric viewpoint for linear periodic continuous-time systems. Stable, unstable and reachable subspaces are defined as in the time invariant case. Then controllable and unreachable subsystems are represented in the coordinate free setting. It is also shown that controllability is not affected by state feedback as in the time invariant case. Finally a new geometric characterization of stabilizability is given in the sense that a linear periodic continuous-time system is stabilizable by double periodic state feedback iff the unstable subspace is contained in the reachable subspace.

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