A New Algorithm for the Numerical Solution of Maxwell's Equations in a Class of Generalized Functions

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Abstract

In this paper a new method for the numerical solution of the Cauchy problem for the Maxwell's equations in a class of generalized functions is suggested. For this purpose, the special auxiliary problem having some advantages over the main problem is introduced. On the basis of the auxiliary problem, an effective and economical algorithm for the numerical solution has been developed.

Keywords Computational Electromagnetizm, Numerical Methods in a Class of Generalized Functions

1 Introduction

.It is known that the simplified Maxwell's system of equations given below, (see [1], [2])

$$\frac{\mu}{c_0}\frac{\partial H_y}{\partial t} = \frac{\partial H_z}{\partial x} = 0, \quad \frac{\epsilon}{c_0}\frac{\partial H_y}{\partial t} = -\frac{\partial H_z}{\partial x},$$

$$\frac{\mu}{c_0}\frac{\partial H_z}{\partial t} = -\frac{\partial H_y}{\partial x} = 0, \quad \frac{\epsilon}{c_0}\frac{\partial H_z}{\partial t} = \frac{\partial H_y}{\partial x} = 0$$

have the multiple characteristics roots [3]. However this is not an obstacle to express the system of equations in the canonical form as

$$\frac{\partial(\sqrt{\mu}H_y + \sqrt{\epsilon}E_z)}{\partial t} - \frac{c_0}{\sqrt{\mu\epsilon}}\frac{\partial(\sqrt{\mu}H_y + \sqrt{\epsilon}E_z)}{\partial x} = 0, \qquad (1.1)$$

$$\frac{\partial(\sqrt{\mu}H_z - \sqrt{\epsilon}E_y)}{\partial t} - \frac{c_0}{\sqrt{\mu\epsilon}}\frac{\partial(\sqrt{\mu}H_z - \sqrt{\epsilon}E_y)}{\partial x} = 0, \qquad (1.2)$$

$$\frac{\partial(\sqrt{\mu}H_y - \sqrt{\epsilon}E_z)}{\partial t} + \frac{c_0}{\sqrt{\mu\epsilon}}\frac{\partial(\sqrt{\mu}H_y - \sqrt{\epsilon}E_z)}{\partial x} = 0, \qquad (1.3)$$

$$\frac{\partial(\sqrt{\mu}H_z + \sqrt{\epsilon}E_y)}{\partial t} + \frac{c_0}{\sqrt{\mu\epsilon}}\frac{\partial(\sqrt{\mu}H_z + \sqrt{\epsilon}E_y)}{\partial x} = 0. \qquad (1.4)$$

Here $\vec{E}(x,t)$ and $\vec{H}(x,t)$ are the electrical and magnetic fields respectively. ϵ, μ and c_0 are the parameters of medium known as permittivity, permeability and velocity of light in free space respectively. x is the position usually considered in the form of $x = (x, y, z)$.

In order to obtain the unknown functions $\vec{H}(x,t)$ and $\vec{E}(x,t)$ we add the following initial conditions to the system of equations (1.1)-(1.4)

$$H_y(x,0) = H_y^0(x), \; H_z(x,0) = H_z^0(x), \qquad (1.5)$$

$$E_y(x,0) = E_y^0(x), \; E_z(x,0) = E_z^0(x) \qquad (1.6)$$

The system equations (1.1)-(1.4) together with the following Rienmann invarients form the conservation laws.

$$\sqrt{\mu}H_y + \sqrt{\epsilon}E_z, \quad \sqrt{\mu}H_z - \sqrt{\epsilon}E_y, \quad \sqrt{\mu}H_y - \sqrt{\epsilon}E_z, \quad \sqrt{\mu}H_z + \sqrt{\epsilon}E_y$$

2 The Cauchy Problem

.Let us solve the system equations (1.1)-(1.4) in the frame with the initial conditions of (1.5),(1.6). As it is known that the derivatives with respect to x of the solution of the wave equations posses jumps on the characteristics. Additionally, the equations (1.1)-(1.4) may not have a classical solution at all. Furthermore if the initial values of the problem have singular points then the existence of the classical solution could not be achieved. Let us define the weak solution of the problem as below.

Definition 1. The vector functions $\vec{H}(x,t), \vec{E}(x,t)$ satisfing the initial conditions (1.5),(1.6) are the weak solutions of the problem (1.1)-(1.4), and if for the arbitrary test fuctions $f(x,t)$ and $f(x,T)=0$ the following integral expressions hold.

$$\int_{D_T}[(\sqrt{\mu}H_y+\sqrt{\epsilon}E_z)f_t+(\sqrt{\mu}H_y+\sqrt{\epsilon}E_z)f_x]dxdt+\int_R(\sqrt{\mu}H_y^0+\sqrt{\epsilon}E_z^0)dx=0, \tag{2.1}$$

$$\int_{D_T}[(\sqrt{\mu}H_y+\sqrt{\epsilon}E_z)f_t+(\sqrt{\mu}H_y+\sqrt{\epsilon}E_z)f_x]dxdt+\int_R(\sqrt{\mu}H_y^0+\sqrt{\epsilon}E_z^0)dx=0, \tag{2.2}$$

$$\int_{D_T}[(\sqrt{\mu}H_y+\sqrt{\epsilon}E_z)f_t+(\sqrt{\mu}H_y+\sqrt{\epsilon}E_z)f_x]dxdt+\int_R(\sqrt{\mu}H_y^0+\sqrt{\epsilon}E_z^0)dx=0, \tag{2.3}$$

$$\int_{D_T}[(\sqrt{\mu}H_y+\sqrt{\epsilon}E_z)f_t+(\sqrt{\mu}H_y+\sqrt{\epsilon}E_z)f_x]dxdt+\int_R(\sqrt{\mu}H_y^0+\sqrt{\epsilon}E_z^0)dx=0 \tag{2.4}$$

Here, $D_T = x \in R, 0 \le t \le T$. As seen from the integral equalities (2.1)-(2.4), the functions H_y, H_z, E_y, E_z are not necessarily to be continous.

To obtain these H_y, H_z, E_y, E_z functions, according to [4], [5] we introduce the following auxiliary problem.

$$\frac{\partial v_y^z(x,t)}{\partial t}-\frac{c_0}{\sqrt{\mu\epsilon}}(\sqrt{\mu}H_y+\sqrt{\epsilon}E_z)=0, \tag{2.5}$$

$$\frac{\partial v_z^{\bar{y}}(x,t)}{\partial t}-\frac{c_0}{\sqrt{\mu\epsilon}}(\sqrt{\mu}H_z-\sqrt{\epsilon}E_y)=0, \tag{2.6}$$

$$\frac{\partial v_y^{\bar{z}}(x,t)}{\partial t}+\frac{c_0}{\sqrt{\mu\epsilon}}(\sqrt{\mu}H_y-\sqrt{\epsilon}E_z)=0, \tag{2.7}$$

$$\frac{\partial v_z^y(x,t)}{\partial t}+\frac{c_0}{\sqrt{\mu\epsilon}}(\sqrt{\mu}H_z+\sqrt{\epsilon}E_y)=0, \tag{2.8}$$

$$v_y^z(x,0) =^0 v_y^z(x), \quad v_z^{\bar{y}}(x,0) =^0 v_z^{\bar{y}}(x), \tag{2.9}$$

$$v_y^{\bar{z}}(x,0) =^0 v_y^{\bar{z}}(x), \quad v_z^y(x,t) =^0 v_z^y(x) \tag{2.10}$$

Here, the functions ${}^0v_y^z(x)$, ${}^0v_z^{\bar{y}}(x)$, ${}^0v_y^{\bar{z}}(x)$, ${}^0v_y^z(x)$ are any solutions of the equations

$$\frac{d\ {}^0v_y^z(x)}{dx} = H_y^0(x), \quad \frac{d\ {}^0v_z^{\bar{y}}(x)}{dx} = H_z^0(x),$$

$$\frac{d\ {}^0v_y^{\bar{z}}(x)}{dx} = E_y^0(x), \quad \frac{d\ {}^0v_z^y(x)}{dx} = E_z^0(x).$$

Theorem 1. If $v_y^z(x,t)$, $v_z^{\bar{y}}(x,t)$, $v_y^{\bar{z}}(x,t)$ *and* $v_z^y(x,t)$ *are the solutions of the equation system (2.5)-(2.10), then*

$$A_1 \equiv \frac{\partial v_y^z(x,t)}{\partial x} = \sqrt{\mu}H_y + \sqrt{\epsilon}E_z, \tag{2.11}$$

$$A_2 \equiv \frac{\partial v_z^{\bar{y}}(x,t)}{\partial x} = \sqrt{\mu}H_z - \sqrt{\epsilon}E_y, \tag{2.12}$$

$$A_3 \equiv \frac{\partial v_y^{\bar{z}}(x,t)}{\partial x} = \sqrt{\mu}H_y - \sqrt{\epsilon}E_z, \tag{2.13}$$

$$A_4 \equiv \frac{\partial v_z^y(x,t)}{\partial x} = \sqrt{\mu}H_z + \sqrt{\epsilon}E_y \tag{2.14}$$

are the weak solutions of the problem (1.1)-(1.6) in means (2.1)-(2.4).

If we write down the unknowns as H_y, H_z, E_y, E_z, then the algebrical system equations (2.11)-(2.14) can be expressed as

$$\begin{cases} \sqrt{\mu}H_y + 0.H_z + 0.E_y + \sqrt{\epsilon}E_z = A_1, \\ 0.H_y + \sqrt{\mu}.H_z - \sqrt{\epsilon}E_y + 0.E_z = A_2, \\ \sqrt{\mu}H_y + 0.H_z - 0.E_y + \sqrt{\epsilon}E_z = A_3, \\ \sqrt{\mu}.0 + \sqrt{\mu}H_z + \sqrt{\epsilon}E_y + 0.E_z = A_4. \end{cases} \tag{2.15}$$

The necessary and satisfactory condition for existence and uniqueness of the solution of the algebric equation system above is to have non-zero valued determinant of the coeficient matrix. The calculation of this determinant yields $4\epsilon\mu$.

2.1 The Numerical Algorithm of the Cauchy Problem

In order to organize a numerical algorithm for the problem (1.1)-(1.6), at first, we cover the region D_T by the grid with the steps h_x, h_y, h_z and h_t.

At any point (x_i, y_j, z_l, t_k) of the grid $\omega_{h_x,h_y,h_z,\tau}$, we approximate the auxiliary problem (2.1)- (2.6) by the finite difference scheme as follows

$$^{k+1}V_y^z =^k V_y^z + \frac{c_0}{\sqrt{\mu\epsilon}}\Big(\sqrt{\mu}^k H_y + \sqrt{\epsilon}^k E_z\Big), \tag{2.16}$$

$$^{k+1}V_z^{\bar{y}} =^k V_z^{\bar{y}} + \frac{c_0}{\sqrt{\mu\epsilon}}\Big(\sqrt{\mu}^k H_z - \sqrt{\epsilon}^k E_y\Big), \tag{2.17}$$

$$^{k+1}V_y^{\bar{z}} =^k V_y\bar{z} - \frac{c_0}{\sqrt{\mu\epsilon}}\Big(\sqrt{\mu}^k H_y - \sqrt{\epsilon}^k E_z\Big), \tag{2.18}$$

$$^{k+1}V_z^y =^k V_z^y - \frac{c_0}{\sqrt{\mu\epsilon}}\Big(\sqrt{\mu}^k H_z + \sqrt{\epsilon}^k E_y\Big), \tag{2.19}$$

$$^0V_y^z =^0 v_y^z(x_i), \quad ^0V_z^{\bar{y}} =^0 v_z^{\bar{y}}(x_i), \tag{2.20}$$

$$^0V_y\bar{z} =^0 v_y^{\bar{z}}(x_i), \quad ^0V_z^y =^0 v_y^z(x_i). \tag{2.21}$$

Here, the grid functions $^kV_y^z$, $^kV_z^{\bar{y}}$, $^kV_y\bar{z}$, $^kV_z^y$, ^kH_y, ^kH_z, ^kE_y and ^kE_z represent approximate values of the functions $v_y^z(x,t)$, $v_z^{\bar{y}}(x,t)$, $v_y^{\bar{z}}(x,t)$, $v_z^y(x,t), H_y(x,t)$, $H_z(x,t)$, $E_y(x,t)$ and $E_z(x,t)$ at any point of the grid respectively.

Theorem 2. If the grid functions $^{k+1}V_y^z$, $^{k+1}V_z^{\bar{y}}$, $^{k+1}V_y^{\bar{z}}$, *and* $^{k+1}V_z^y$ *are the numerical problem (2.16)-(2.21), then*

$$^{k+1}H_y = \Big(^{k+1}V_y^z\Big)_{\bar{x}}, \quad ^{k+1}H_z = \Big(^{k+1}V_z^{\bar{y}}\Big)_{\bar{x}},$$

$$^{k+1}E_y = \Big(^{k+1}V_y^{\bar{z}}\Big)_{\bar{x}}, \quad ^{k+1}E_z = \Big(^{k+1}V_z^y\Big)_{\bar{x}}$$

are the numerical solutions of the main problem (1.1)-(1.6).

3 Conclusion

In this study, on the basis of a newly introduced auxiliary problem, an original method for finding the numerical solution of the Cauchy problem for Maxwell's equations in a generalized functions is suggested. Using this auxiliary problen an economical and effective finite differences scheme has been developed.

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