Intuitionistic fuzzy h-ideals of hemirings

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Abstract: We introduce the notion of intuitionistic fuzzy (left) h-ideals of hemirings and investigate their properties connected with the corresponding level subsets. Methods of constructions of such intuitionistic fuzzy ideals from given sequences of left h-ideals of a hemiring R are presented. Some natural classification of such intuitionistic fuzzy h-ideals is given.

Key–Words: Hemiring, fuzzy set, intuitionistic fuzzy left h-ideal, descending chain.

1 Introduction

Many-valued logic has been proposed to model phenomena in which uncertainty and vagueness are involved. One of the most general classes of the many-valued logic is the BL-logic defined as the logic of continuous t-norms. But in fact, BL-logic is a commutative lattice-ordered semiring. So, some types of logic such as Łukasiewicz logic, Gödel logic and Product logic, as special cases of BL-logic, are semirings. Hemirings, as semirings with zero and commutative addition, appear in a natural manner in some applications to the theory of automata and formal languages (see [1, 11, 12]). It is a well known result that regular languages form so-called star semirings. According to the well known theorem of Kleene, the languages, or sets of words, recognized by finite-state automata are precisely those that are obtained from letters of input alphabets by the application of the operations: sum (union), concatenation (product), and Kleene star (Kleene closure). If a language is represented as a formal series with the coefficients in a Boolean hemiring, then the Kleene theorem can be well described by the Kleen-Schützenberger theorem. Moreover, if the coefficient hemiring is a field, then a series is rational if and only if its syntactic algebra (see [11] and [12] for details) has a finite rank. The so-called min-max-plus computations (and suitable semirings) are used in several areas. Continuous timed Petri nets can be modeled by using generalized polynomial recurrent equations in the $(\min, +)$ semiring (see [5]). Very similar semirings can be used to study fundamental concepts of the automata theory such as nondeterminism (cf. [9]). Many other applications with references can be found in [4].

Ideals of hemirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals. Many results in rings apparently have no analogues in hemirings using only ideals. Henriksen defined in [6] a more restricted class of ideals in semirings, which is called the class of k-ideals, with the property that if the semiring R is a ring then a complex in R is a k-ideal if and only if it is a ring ideal. Another more restricted, but very important, class of ideals, called now h-ideals, has been given and investigated by Izuka [7] and La Torre [10]. Other important results connected with fuzzy ideals in hemirings were obtained in [8].

In this short note, we introduce the notion of intuitionistic fuzzy left h-ideals of hemirings and investigate their properties and connections with chains of left h-ideals of the corresponding hemirings.

2 Preliminaries

By a *semiring* is mean an algebraic system $(R, +, \cdot)$ consisting of a nonempty set R together with two binary operations on R called addition and multiplication (denoted in the usual manner) such that $(R, +)$ and $(R, \cdot)$ are semigroups satisfying for all $x, y, z \in R$ the following distributive laws

$$x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz.$$

By a *zero* we mean an element $0 \in R$ such that $0x = x0 = 0$ and $0 + x = x + 0 = x$ for all $x \in R$. A semiring with zero and a commutative semigroup $(R, +)$ is called a *hemiring*.

A nonempty subset A of R is said to be a *left ideal* if it is closed with respect to the addition and such that

$RA \subseteq A$. A left ideal A is called a *left h-ideal* (cf. [5]) if for any $x, z \in R$ and $a, b \in A$ from $x+a+z = b+z$ it follows $x \in A$.

A fuzzy set μ of a hemiring R is called a *fuzzy left h-ideal* (cf. [6]) if for all $a, b, x, z \in R$ the following three conditions hold:

$$\mu(x+y) \geq \min\{\mu(x), \mu(y)\},$$
$$\mu(xy) \geq \mu(y),$$
$$x + a + z = b + z \longrightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}.$$

As an important generalization of the notion of fuzzy sets, Atanassov introduced in [3] the concept of an *intuitionistic fuzzy set* (IFS for short) defined as objects having the form:

$$A = (\mu_A, \lambda_A) = \{(x, \mu_A(x), \lambda_A(x)) \,|\, x \in R\},$$

where the fuzzy sets μ_A and λ_A denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of non-membership* (namely $\lambda_A(x)$) of each element $x \in R$ to the set A respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in R$.

According to [3], for every two intuitionistic fuzzy sets $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ in R, we define: $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in R$. Obviously $A = B$ means that $A \subseteq B$ and $B \subseteq A$.

3 Intuitionistic fuzzy left h-ideals

Definition 1 An IFS $A = (\mu_A, \lambda_A)$ in a hemiring R is called an *intuitionistic fuzzy left h-ideal* if

(1) $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(2) $\lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$,

(3) $\mu_A(xy) \geq \mu_A(y)$,

(4) $\lambda_A(xy) \leq \lambda_A(y)$,

(5)
$x + a + z = b + z \longrightarrow \mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\}$,

(6)
$x + a + z = b + z \longrightarrow \lambda_A(x) \leq \max\{\lambda_A(a), \lambda_A(b)\}$

hold for all $a, b, x, y, z \in R$.

An IFS $A = (\mu_A, \lambda_A)$ satisfying the first four conditions is called an *intuitionistic fuzzy left ideal*.

The family of all intuitionistic fuzzy left h-ideals of a hemiring R will be denoted by $IFI(R)$.

It is not difficult to see that $\mu_A(x) \leq \mu_A(0)$ and $\lambda_A(0) \leq \lambda_A(x)$ for each $A \in IFI(R)$ and $x \in R$.

Example 2 On a four element hemiring $(R, +, \cdot)$ defined by the following two tables:

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

·	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

consider an IFS $A = (\mu_A, \lambda_A)$, where $\mu_A(0) = 0.4$, $\lambda_A(0) = 0.2$ and $\mu_A(x) = 0.2$, $\lambda_A(x) = 0.7$ for all $x \neq 0$. It is not difficult to verify that $A \in IFI(R)$.

The following three results can be proved by the verification of the corresponding axioms.

Proposition 3 *Let A be a nonempty subset of a hemiring R. Then an IFS (μ_A, λ_A) defined by*

$$\mu_A(x) = \begin{cases} \alpha_2 & \text{for} \quad x \in A, \\ \alpha_1 & \text{for} \quad x \notin A, \end{cases}$$

$$\lambda_A(x) = \begin{cases} \beta_2 & \text{for} \quad x \in A, \\ \beta_1 & \text{for} \quad x \notin A, \end{cases}$$

where $0 \leq \alpha_1 < \alpha_2 \leq 1$, $0 \leq \beta_2 < \beta_1 \leq 1$ and $\alpha_i + \beta_i \leq 1$ for $i = 1, 2$, is an intuitionistic fuzzy left h-ideal of R if and only if A is a left h-ideal of R.

Proposition 4 *An IFS $A = (\mu_A, \overline{\mu_A})$, where $\overline{\mu_A} = 1 - \mu_A$ is an intuitionistic fuzzy left h-ideal of a hemiring R if and only if μ_A is a fuzzy left h-ideal of R.*

Definition 5 Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in a hemiring R and let $\alpha, \beta \in [0, 1]$ be such that $\alpha + \beta \leq 1$. Then the set

$$R_A^{(\alpha,\beta)} = \{x \in R \mid \alpha \leq \mu_A(x), \ \lambda_A(x) \leq \beta\}$$

is called an *(α, β)-level subset* of $A = (\mu_A, \lambda_A)$.

The set of all $(\alpha, \beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\lambda_A)$ such that $\alpha + \beta \leq 1$ is called the *image* of $A = (\mu_A, \lambda_A)$.

Clearly $R_A^{(\alpha,\beta)} = U(\mu_A, \alpha) \cap L(\lambda_A, \beta)$, where $U(\mu_A, \alpha)$ and $L(\lambda_A, \beta)$ are upper and lower level subsets of μ_A and λ_A, respectively.

Theorem 6 *An IFS $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left h-ideal of R if and only if $R_A^{(\alpha,\beta)}$ is a left h-ideal of R for every $(\alpha, \beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\lambda_A)$ such that $\alpha + \beta \leq 1$.*

Corollary 7 *An IFS $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left h-ideal of R if and only if for every $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, all nonempty $U(\mu_A, \alpha)$ and $L(\lambda_A, \beta)$ are left h-ideals of R.*

Lemma 8 *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy left h-ideal of a hemiring R and let $x \in R$. Then $\mu_A(x) = \alpha$, $\lambda_A(x) = \beta$ if and only if $x \in U(\mu_A, \alpha)$, $x \notin U(\mu_A, \gamma)$ and $x \in L(\lambda_A, \beta)$, $x \notin L(\lambda_A, \delta)$ for all $\gamma > \alpha$ and $\delta < \beta$.*

Theorem 9 *Let $\{C_\alpha\}_{\alpha\in\Gamma}$, where $\Gamma \subseteq [0, \frac{1}{2}]$ be a collection of left h-ideals of R such that $R = \bigcup_{\alpha\in\Gamma} C_\alpha$, and for $\alpha, \beta \in \Gamma$, $\alpha < \beta$ if and only if $C_\beta \subset C_\alpha$. Then an IFS (μ, λ) defined by*

$$\mu(x) = \sup\{\alpha \in \Gamma \mid x \in C_\alpha\},$$
$$\lambda(x) = \inf\{\alpha \in \Gamma \mid x \in C_\alpha\}$$

is an intuitionistic fuzzy left h-ideal of R.

Proof: According to Corollary 7, it is sufficient to show that for every $\alpha, \beta \in [0, 1]$, where $\alpha+\beta \leq 1$, the nonempty sets $U(\mu, \alpha)$ and $L(\lambda, \beta)$ are left h-ideals of R. To prove that $U(\mu, \alpha)$ is a left h-ideal, we consider two cases:

$$(i) \qquad \alpha = \sup\{\delta \in \Gamma \mid \delta < \alpha\}$$
$$(ii) \qquad \alpha \neq \sup\{\delta \in \Gamma \mid \delta < \alpha\}.$$

In the first case

$$x \in U(\mu, \alpha) \longleftrightarrow (x \in C_\delta \ \forall\delta < \alpha) \longleftrightarrow x \in \bigcap_{\delta<\alpha} C_\delta.$$

So, $U(\mu, \alpha) = \bigcap_{\delta<\alpha} C_\delta$, which is a left h-ideal of R. In the second case, we have $U(\mu, \alpha) = \bigcup_{\delta\geq\alpha} C_\delta$. Indeed, if $x \in \bigcup_{\delta\geq\alpha} C_\delta$, then $x \in C_\delta$ for some $\delta \geq \alpha$. Thus $\mu(x) \geq \delta \geq \alpha$, i.e., $x \in U(\mu, \alpha)$. This proves $\bigcup_{\delta\geq\alpha} G_\delta \subset U(\mu, \alpha)$. To prove the converse inclusion consider $x \notin \bigcup_{\delta\geq\alpha} C_\delta$. Then $x \notin C_\delta$ for all $\delta \geq \alpha$. Since $\alpha \neq \sup\{\delta \in \Gamma \mid \delta < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha) \cap \Gamma = \emptyset$. Hence $x \notin C_\delta$ for all $\delta > \alpha-\varepsilon$, which means that if $x \in C_\delta$ then $\delta \leq \alpha-\varepsilon$. Thus $\mu(x) \leq \alpha - \varepsilon < \alpha$, and so $x \notin U(\mu, \alpha)$. Therefore $U(\mu, \alpha) = \bigcup_{\delta\geq\alpha} C_\delta$. Since, as it is not difficult to verify, $\bigcup_{\delta\geq\alpha} C_\delta$ is a left h-ideal of R, we see that $U(\mu, \alpha)$ is a left h-ideal in any case.

For $L(\lambda, \beta)$ the proof is similar. □

In a similar way we can prove

Theorem 10 *If $A = (\mu_A, \lambda_A) \in IFI(R)$, then*

$$\mu_A(x) = \sup\{\alpha \in [0, 1] \mid x \in U(\mu_A, \alpha)\},$$
$$\lambda_A(x) = \inf\{\beta \in [0, 1] \mid x \in L(\lambda_A, \beta)\}$$

for every $x \in R$.

Theorem 11 *For any chain*

$$A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = R$$

of left h-ideals of a hemiring R there exists an intuitionistic fuzzy left h-ideal of R for which upper and lower sets coincide with this chain.

Proof: Let $\{\alpha_k\}_{k=0}^n$ and $\{\beta_k\}_{k=0}^n$ be finite decreasing and increasing sequences in $[0, 1]$ such that $\alpha_i + \beta_i \leq 1$, for $0 \leq i \leq n$. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in R defined by $\mu_A(A_0) = \alpha_0$, $\lambda_A(A_0) = \beta_0$, $\mu_A(A_k \setminus A_{k-1}) = \alpha_k$ and $\lambda_A(A_k \setminus A_{k-1}) = \beta_k$ for $0 < k \leq n$. Let $x, y \in R$. If $x, y \in A_k \setminus A_{k-1}$, then $x + y, xy \in A_k$ and

$$\mu_A(x + y) \geq \alpha_k = \min\{\mu_A(x), \mu_A(y)\},$$
$$\lambda_A(x + y) \leq \beta_k = \max\{\lambda_A(x)), \lambda_A(y)\},$$
$$\mu_A(xy) \geq \alpha_k = \mu_A(y), \quad \lambda_A(xy) \leq \beta_k = \lambda_A(y).$$

For $i > j$, if $x \in A_i \setminus A_{i-1}$ and $y \in A_j \setminus A_{j-1}$, then $\mu_A(x) = \alpha_i = \mu_A(y)$, $\lambda_A(x) = \beta_j = \lambda_A(y)$ and $x + y, xy \in A_i$. Thus

$$\mu_A(x + y) \geq \alpha_i = \min\{\mu_A(x), \mu_A(y)\},$$
$$\lambda_A(x + y) \leq \beta_j = \max\{\lambda_A(x)), \lambda_A(y)\},$$
$$\mu_A(xy) \geq \alpha_i = \mu_A(y), \quad \lambda_A(xy) \leq \beta_j = \lambda_A(y).$$

In the same manner we can verify the conditions (5) and (6). Consequently, $A = (\mu_A, \lambda_A) \in IFI(R)$.

Obviously $\mathrm{Im}(\mu_A) = \{\alpha_k\}_{k=0}^n\}$ and $\mathrm{Im}(\lambda_A) = \{\beta_k\}_{k=0}^n$. It follows that the upper level subsets and the lower level subsets of $A = (\mu_A, \lambda_A)$ are given by the chain of left h-ideals

$$U(\mu_A, \alpha_0) \subset U(\mu_A, \alpha_1) \subset \ldots \subset U(\mu_A, \alpha_n) = R$$

and

$$L(\lambda_A, \beta_0) \subset L(\lambda_A, \beta_1) \subset \ldots \subset L(\lambda_A, \beta_n) = R$$

respectively. Indeed,

$$U(\mu_A, \alpha_0) = \{x \in R \mid \mu_A(x) \geq \alpha_0\} = A_0,$$
$$L(\mu_A, \beta_0) = \{x \in R \mid \lambda_A(x) \leq \beta_0\} = A_0.$$

We now prove that

$$U(\mu_A, \alpha_k) = A_k = L(\lambda_A, \beta_k) \quad \text{for} \quad 0 < k \leq n.$$

Clearly, $A_k \subseteq U(\mu_k, \alpha_k)$ and $A_k \subseteq L(\lambda_A, \beta_k)$. If $x \in U(\mu_A, \alpha_k)$, then $\mu_A(x) \geq \alpha_k$ and so $x \notin A_i$ for $i > k$. Hence $\mu_A(x) \in \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$, which implies $x \in A_i$ for some $i \leq k$. Since $A_i \subseteq A_k$, it follows that $x \in A_k$. Consequently, $U(\mu_A, \alpha_k) = A_k$ for every $0 < k \leq n$. Now if $y \in L(\lambda_A, \beta_k)$, then $\lambda_A(x) \leq \beta_k$ and so $y \notin A_i$ for $j \leq k$. Thus $\lambda_A(x) \in \{\beta_0, \beta_1, \ldots, \beta_k\}$, which implies $x \in A_j$ for some $j \leq k$. Since $A_j \subseteq A_k$, it follows that $y \in A_k$. Consequently, $L(\lambda_A, \beta_k) = A_k$ for $0 < k \leq n$. □

Theorem 12 *If every intuitionistic fuzzy left h-ideal of R has the finite image, then every descending chain of left h-ideal of R terminates at finite step.*

Proof: Suppose that there exists a strictly descending chain $A_0 \supset A_1 \supset A_2 \supset \ldots$ of left k-ideals of R which does not terminate at finite step. We prove that the IFS $A = (\mu_A, \lambda_A)$ defined by

$$\mu_A(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \in A_n \setminus A_{n+1},\ n = 0, 1, \ldots \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} A_n, \end{cases}$$

$$\lambda_A(x) = 1 - \mu_A(x),$$

where $A_0 = R$, is an intuitionistic fuzzy left h-ideal of R with an infinite numbers of values. According to Proposition 4 it is sufficient to prove that μ_A has this property.

Let $x, y \in R$. Assume that $x \in A_n \setminus A_{n+1}$ and $y \in A_k \setminus A_{k+1}$ for some n and k. Without loss of generality, we can assume that $n \leq k$. Then obviously $y, x+y, xy \in A_n$ and

$$\mu_A(x+y) \geq \frac{n}{n+1} = \min\{\mu_A(x), \mu_A(y)\},$$

$$\mu_A(xy) \geq \frac{n}{n+1} = \mu_A(y).$$

If $x, y \in \bigcap_{n=0}^{\infty} A_n$, then $x+y, xy \in \bigcap_{n=0}^{\infty} A_n$. Thus

$$\mu_A(x+y) = 1 = \min\{\mu_A(x), \mu_A(y)\},$$

$$\mu_A(xy) = 1 = \mu_A(y).$$

If $x \notin \bigcap_{n=0}^{\infty} A_n$ and $y \in \bigcap_{n=0}^{\infty} A_n$, then there exists $k \in N$ such that $x \in A_k \setminus A_{k+1}$. So, $x+y, xy \in A_k$ and

$$\mu_A(x+y) \geq \frac{k}{k+1} = \min\{\mu_A(x), \mu_A(y)\},$$

$$\mu_A(xy) \geq \frac{k}{k+1} = \mu_A(y).$$

Finally suppose that $x \in \bigcap_{n=0}^{\infty} A_n$ and $y \notin \bigcap_{n=0}^{\infty} A_n$. Then $y \in A_r \setminus A_{r+1}$ for some $r \in N$. Hence $x+y$, $xy \in A_r$, consequently

$$\mu_A(x+y) \geq \frac{r}{r+1} = \min\{\mu_A(x), \mu_A(y)\},$$

$$\mu_A(xy) \geq \frac{r}{r+1} = \mu_A(y).$$

In a similar way we can verify that μ_A satisfies (5). This, by Proposition 4, proves that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left h-ideal with an infinite number of different values. The obtained contradiction completes our proof. □

Now we prove the converse of Theorem 12.

Theorem 13 *Let R be a hemiring in which very descending chain of left h-ideals terminates at finite step. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left h-ideals of R such that a sequence of elements of $\mathrm{Im}(\mu_A)$ is strictly increasing and a sequence of elements of $\mathrm{Im}(\lambda_A)$ is strictly decreasing, then μ_A and λ_A have finite number of values.*

Proof: Suppose that $\mathrm{Im}(\mu_A)$ is not finite. Let $0 \leq \alpha_1 < \alpha_2 < \ldots \leq 1$ be a strictly increasing sequence of elements of $\mathrm{Im}(\mu_A)$. Then every $U(\mu_A, \alpha_t)$ is a left h-ideal of R. For $x \in U(\mu_A, t)$ we have $\mu_A(x) \geq \alpha_t > \alpha_{t-1}$, which implies $x \in U(\mu_A, \alpha_{t-1})$. Hence $U(\mu_A, \alpha_t) \subseteq U(\mu_A, \alpha_{t-1})$. But for $\alpha_{t-1} \in \mathrm{Im}(\mu_A)$ there exists $x_{t-1} \in R$ such that $\mu_A(x_{t-1}) = \alpha_{t-1}$. This gives $x_{t-1} \in U(\mu_A, \alpha_{t-1})$ and $x_{t-1} \notin U(\mu_A, \alpha_t)$. Thus $U(\mu_A, \alpha_t)$ is a proper subset of $U(\mu_A, \alpha_{t-1})$, and so we obtain a strictly descending chain $U(\mu_A, \alpha_1) \supset U(\mu_A, \alpha_2) \supset U(\mu_A, \alpha_3) \supset \ldots$ of left h-ideals of R which is not terminating. This is a contradiction. So, $\mathrm{Im}(\mu_A)$ must be finite.

For $\mathrm{Im}(\lambda_A)$ the proof is analogous. □

Theorem 14 *Every ascending chain of left h-ideals of a hemiring R terminates at finite step if and only if for any intuitionistic fuzzy left h-ideal $A = (\mu_A, \lambda_A)$ in R $\mathrm{Im}(\mu_A)$ and $\mathrm{Im}(\lambda_A)$ are well-ordered subsets of* $[0, 1]$.

Proof: Suppose that for an intuitionistic fuzzy left h-ideal $A = (\mu_A, \lambda_A)$ the sets of values of μ_A and λ_A are not well-ordered subsets of $[0, 1]$. Then there exists a strictly decreasing sequence $\{\alpha_n\}$ such that $\alpha_n = \mu_A(x_n)$ for some $x_n \in R$. In this case $B_n = \{x \in R \,|\, \mu_A(x) \geq \alpha_n\}$ form a strictly ascending chain of left h-ideals of R which is not terminating. This is a contradiction. So, $\mathrm{Im}(\mu_A)$ must be well-ordered subset of $[0, 1]$. Similarly $\mathrm{Im}(\lambda_A)$.

Conversely, suppose that there exists a strictly ascending chain $A_1 \subset A_2 \subset A_3 \subset \ldots$ of left h-ideals of R which does not terminate at finite step. Then $A = \bigcup_{k=1}^{\infty} A_k$ is a left h-ideal of R. Define on R an IFS $A = (\mu_A, \lambda_A)$ putting

$$\mu_A(x) = \begin{cases} \frac{1}{k} & \text{for } x \in A_k \backslash A_{k-1}, \\ 0 & \text{for } x \notin A, \end{cases}$$

$$\lambda_A(x) := 1 - \mu(x),$$

where A_0 means the empty set.

Using Proposition 4 we prove that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left h-ideal of R.

At first we consider the case when $x, y \in A$. In this case there are m, n such that $x \in A_n \backslash A_{n-1}$,

$y \in A_m \backslash A_{m-1}$. Obviously $x + y \in A_k \backslash A_{k-1} \subset A_p$, where $k \leq p = \max\{m, n\}$. So, $\mu_A(x) = \frac{1}{n}$, $\mu_A(y) = \frac{1}{m}$ and

$$\mu_A(x+y) = \frac{1}{k} \geq \frac{1}{p} = \min\{\mu_A(x), \mu_A(y)\}.$$

Since any A_k is an ideal, $y \in A_m$, we get $xy \in A_m$. Thus, $y \in A_t \backslash A_{t-1}$ for some $t \leq m$, whence

$$\mu_A(xy) = \frac{1}{t} \geq \frac{1}{m} = \mu_A(y).$$

Now we consider the case $x \notin A$, $y \in A$. In this case $y \in A_m \backslash A_{m-1}$ for some natural m. Hence $\mu_A(x) = 0$, $\mu_A(y) = \frac{1}{m}$, consequently

$$\mu_A(x+y) \geq 0 = \min\{\mu_A(x), \mu_A(y)\}.$$

Because, as in the previous case, $xy \in A_m$ means that $xy \in A_t \backslash A_{t-1}$ for some $t \leq m$, then

$$\mu_A(xy) = \frac{1}{t} \geq \frac{1}{m} = \mu_A(y).$$

The case $x \in A$, $y \notin A$ is analogous. The last case $x, y \notin A$ is obvious.

In this way we have proved that μ_A is a fuzzy left ideal of R.

To verify (5), i.e., that $x + a + z = b + z$ implies $\mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\}$ we must consider the following four cases (a) $a, b \in A$, (b) $a \in A$, $b \notin A$, (c) $a \notin A$, $b \in A$, (d) $a, b \notin A$. The last three cases are obvious because in these cases $\min\{\mu_A(a), \mu_A(b)\} = 0$. We verify (a). If $a, b \in A_k \backslash A_{k-1}$, then, according to the assumption on A_k, we have $x \in A_k$, whence

$$\mu_A(x) \geq \frac{1}{k} = \mu_A(b) = \min\{\mu_A(a), \mu_A(b)\}.$$

If $a \in A_k \backslash A_{k-1}$, $b \in A_p \backslash A_{p-1}$, for some $k \neq p$, then, in the case $k < p$, we get $x \in A_p$. Consequently

$$\mu_A(x) \geq \frac{1}{p} = \mu_A(b) = \min\{\mu_A(a), \mu_A(b)\}.$$

In the case $k > p$, we have $x \in A_k$ and

$$\mu_A(x) \geq \frac{1}{k} = \mu_A(a) = \min\{\mu_A(a), \mu_A(b)\}.$$

This proves that μ_A is a fuzzy left h-ideal. Since the chain $A_1 \subset A_2 \subset A_3 \subset \ldots$ is not terminating, μ_A has a strictly descending sequence of values. This contradicts that the value set of any fuzzy ideal is well-ordered.

Proposition 4 completes the proof. □

4 Characteristic intuitionistic fuzzy left h-ideals of hemirings

For any mapping f from R to S we can define in R a new fuzzy set μ^f putting $\mu^f(x) = \mu(f(x))$ for all $x \in R$. Consequently, $A^f = (\mu^f, \lambda^f)$.

Clearly $\mu^f(x_1) = \mu^f(x_2)$ for $x_1, x_2 \in f^{-1}(x)$.

Definition 15 A left h-ideal A of a hemiring R is said to be *characteristic* if $f(A) = A$ for all $f \in Aut(R)$, where $Aut(R)$ is the set of all automorphisms of R.

Definition 16 An IFS $A = (\mu_A, \lambda_A)$ of R is called an *intuitionistic fuzzy characteristic* if $A^f = A$, i.e, if $\mu_A^f(x) = \mu_A(x)$ and $\lambda_A^f(x) = \lambda_A(x)$ for all $x \in R$ and $f \in Aut(R)$.

Theorem 17 *$A \in IFI(R)$ is characteristic if and only if each its nonempty level set is a characteristic left h-ideal of R.*

Proof: An IFS $A = (\mu_A, \lambda_A)$ is an IF left h-ideal if and only if all its nonempty level subsets are left h-ideals (Theorem 8). So, we will be prove only that A is characteristic if and only if all its level subsets are characteristic. If $A = (\mu_A, \lambda_A)$ is characteristic, $\alpha \in \mathrm{Im}(\mu_A)$, $f \in Aut(R)$, $x \in U(\mu_A, \alpha)$, then $\mu_A^f(x) = \mu_A(f(x)) = \mu_A(x) \geq \alpha$, which means that $f(x) \in U(\mu_A, \alpha)$. Thus $f(U(\mu_A, \alpha)) \subseteq U(\mu_A, \alpha)$. Since for each $x \in U(\mu_A, \alpha)$ there exists $y \in R$ such that $f(y) = x$ we have $\mu_A(y) = \mu_A^f(y) = \mu_A(f(y)) = \mu_A(x) \geq \alpha$. Therefore $y \in U(\mu_A, \alpha)$. Thus $x = f(y) \in f(U(\mu_A, \alpha))$. Hence $f(U(\mu_A, \alpha)) = U(\mu_A, \alpha)$. Similarly, $f(L(\lambda_A, \beta)) = L(\lambda_A, \beta)$. This proves that $U(\mu_A, \alpha)$ and $L(\lambda_A, \beta)$ are characteristic.

Conversely, if all levels of $A = (\mu_A, \lambda_A)$ are characteristic left h-ideals of R, then for $x \in R$, $f \in Aut(R)$ and $\mu_A(x) = \alpha$, $\lambda_A(x) = \beta$, by Lemma 8, we have $x \in U(\mu_A, \alpha)$, $x \notin U(\mu_A, \gamma)$ and $x \in L(\lambda_A, \beta)$, $x \notin L(\lambda_A, \delta)$ for all $\gamma > \alpha$, $\delta < \beta$. Thus $f(x) \in f(U(\mu_A, \alpha)) = U(\mu_A, \alpha)$ and $f(x) \in f(L(\lambda_A, \beta)) = L(\lambda_A, \beta)$, i.e., $\mu_A(f(x)) \geq \alpha$ and $\lambda_A(f(x)) \leq \beta$. For $\mu_A(f(x)) = \gamma > \alpha$, $\lambda_A(f(x)) = \delta < \beta$ we have $f(x) \in U(\mu_A, \gamma) = f(U(\mu_A, \gamma))$, $f(x) \in L(\lambda_A, \delta) = f(L(\lambda_A, \delta))$, which implies $x \in U(\mu_A, \gamma)$, $x \in L(\mu_A, \delta)$. This is a contradiction. Thus $\mu_A(f(x)) = \mu_A(x)$ and $\lambda_A(f(x)) = \lambda_A(x)$. So, $A = (\mu_A, \lambda_A)$ is characteristic. □

Proposition 18 *Let $f : R \to S$ be a homomorphism of hemirings. If $A = (\mu_A, \lambda_A)$ is an IF left h-ideal of S, then $A^f = (\mu_A^f, \lambda_A^f)$ is an IF left h-ideal of R.*

Proof: Let $x, y \in R$. Then

$$\begin{aligned}\mu_A^f(x+y) &= \mu_A(f(x+y)) = \mu_A(f(x)+f(y)) \\ &\geq \min\{\mu_A(f(x)), \mu_A(f(y))\} \\ &= \min\{\mu_A^f(x), \mu_A^f(y)\},\end{aligned}$$

$$\begin{aligned}\lambda_A^f(x+y) &= \lambda_A(f(x+y)) = \lambda_A(f(x)+f(y)) \\ &\leq \max\{\lambda_A(f(x)), \lambda_A(f(y))\} \\ &= \max\{\lambda_A^f(x), \lambda_A^f(y)\},\end{aligned}$$

$$\begin{aligned}\mu_A^f(xy) &= \mu_A(f(xy)) = \mu_A(f(x)f(y)) \\ &\geq \mu_A(f(y)) = \alpha_A^f(y),\end{aligned}$$

$$\begin{aligned}\lambda_A^f(xy) &= \lambda_A(f(xy)) = \lambda_A(f(x)f(y)) \\ &\leq \lambda_A(f(y)) = \lambda_A^f(y).\end{aligned}$$

If $x+a+z = b+z$, then $f(x)+f(a)+f(z) = f(b)+f(z)$, whence

$$\begin{aligned}\mu_A^f(x) &= \mu_A(f(x)) \geq \min\{\mu_A(f(a))\mu_A(f(b))\} \\ &= \min\{\mu_A^f(a), \mu_A^f(b)\},\end{aligned}$$

$$\begin{aligned}\lambda_A^f(x) &= \lambda_A(f(x)) \leq \max\{\lambda_A(f(a)), \lambda_A(f(b))\} \\ &= \max\{\lambda_A^f(a), \lambda_A^f(b)\}.\end{aligned}$$

This proves that $A^f = (\mu_A^f, \lambda_A^f)$ is an IF left h-ideal of R. □

Proposition 19 *Let $f : R \to S$ be an epimorphism of hemirings. If $A^f = (\mu_A^f, \lambda_A^f)$ is an IF left h-ideal of R, then $A = (\mu_A, \lambda_A)$ is an IF left h-ideal of S.*

Proof: Since f is a surjective mapping, for $x, y \in S$ there are $x_1, y_1 \in R$ such that $x = f(x_1)$, $y = f(y_1)$. Thus

$$\begin{aligned}\mu_A(x+y) &= \mu_A(f(x_1)+f(y_1)) = \mu_A(f(x_1+y_1)) \\ &= \mu_A^f(x_1+y_1) \geq \min\{\mu_A^f(x_1), \mu_A^f(y_1)\} \\ &= \min\{\mu_A(x), \mu_A(y)\},\end{aligned}$$

proves that μ_A satisfies the first condition on Definition 1. In a similar way we can verify others conditions. □

As a consequence of the above two propositions we obtain the following theorem.

Theorem 20 *Let $f : R \to S$ be an epimorphism of hemirings. Then $A^f = (\mu_A^f, \lambda_A^f)$ is an IF left h-ideal of R if and only if $A = (\mu_A, \lambda_A)$ is an IF left h-ideal of S.*

5 Equivalence relations

For any $\alpha \in [0,1]$ define on $IFI(R)$ two binary relations $\mathcal{U}^\alpha$ and $\mathcal{L}^\alpha$ as follows:

$$(A, B) \in \mathcal{U}^\alpha \longleftrightarrow U(\mu_A, \alpha) = U(\mu_B, \alpha)$$

and

$$(A, B) \in \mathcal{L}^\alpha \longleftrightarrow L(\lambda_A, \alpha) = L(\lambda_B, \alpha),$$

respectively, where $A = (\mu_A, \lambda_A)$, $B = (\mu_B, \lambda_B)$. Then clearly $\mathcal{U}^\alpha$ and $\mathcal{L}^\alpha$ are equivalence relations on $IFI(R)$.

For any $A = (\mu_A, \lambda_A) \in IFI(R)$, let $[A]_{\mathcal{U}^\alpha}$ (resp. $[A]_{\mathcal{L}^\alpha}$) denote the equivalence class of a modulo $\mathcal{U}^\alpha$ (resp. $\mathcal{L}^\alpha$), and denote by $IFI(R)/\mathcal{U}^\alpha$ (resp. $IFI(R)/\mathcal{L}^\alpha$) the system of all equivalence classes modulo $\mathcal{U}^\alpha$ (resp. $\mathcal{L}^\alpha$).

Now let $I(R)$ denote the family of all left h-ideals of R and let $\alpha \in [0,1]$. Define two maps f_α and g_α from $IFI(R)$ to $I(R) \cup \{\emptyset\}$ by

$$f_\alpha(A) = U(\mu_A, \alpha), \quad g_\alpha(A) = L(\lambda_A, \alpha)$$

for all $A = (\mu_A, \lambda_A) \in IFI(R)$. Then f_α and g_α are clearly well-defined.

Theorem 21 *For any $\alpha \in (0,1)$ the maps f_α and g_α are surjective from $IFI(R)$ to $I(R) \cup \{\emptyset\}$.*

Proof: Let $\alpha \in (0,1)$. Note that $0_\sim = (0,1)$ is in $IFI(R)$, where 0 and 1 are fuzzy sets in R defined by $0(x) = 0$ and $1(x) = 1$ for all $x \in R$. Obviously $f_\alpha(0_\sim) = U(0, \alpha) = \emptyset = L(1, \alpha) = g_\alpha(0_\sim)$. Let $\emptyset \neq B \in I(R)$. For $B_\sim = (\chi_B, \overline{\chi_B}) \in IFI(R)$, we have $f_\alpha(B_\sim) = U(\chi_B, \alpha) = B$ and $g_\alpha(B_\sim) = L(\overline{\chi_B}, \alpha) = B$. Hence f_α and g_α are surjective. □

Theorem 22 *The quotient sets $IFI(R)/\mathcal{U}^\alpha$ and $IFI(R)/\mathcal{L}^\alpha$ are equipotent to $I(R) \cup \{\emptyset\}$ for every $\alpha \in (0,1)$.*

Proof: For $\alpha \in (0,1)$ let f_α^* (resp. g_α^*) be a map from $IFI(R)/\mathcal{U}^\alpha$ (resp. $IFI(R)/\mathcal{L}^\alpha$) to $I(R) \cup \{\emptyset\}$ defined by $f_\alpha^*([A]_{\mathcal{U}^\alpha}) = f_\alpha(A)$ (resp. $g_\alpha^*([A]_{\mathcal{L}^\alpha}) = g_\alpha(A)$) for all $A = (\mu_A, \lambda_A) \in IFI(R)$. If $U(\mu_A, \alpha) = U(\mu_B, \alpha)$ and $L(\lambda_A, \alpha) = L(\beta_B, \alpha)$ for $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \beta_B)$ in $IFI(R)$, then $(A, B) \in \mathcal{U}^\alpha$ and $(A, B) \in \mathcal{L}^\alpha$. Thus $[A]_{\mathcal{U}^\alpha} = [B]_{\mathcal{U}^\alpha}$ and $[A]_{\mathcal{L}^\alpha} = [B]_{\mathcal{L}^\alpha}$. This proves that the maps f_α^* and g_α^* are injective.

Now let $\emptyset \neq D \in I(R)$ and $D_\sim = (\chi_D, \overline{\chi_D})$. Then obviously $D_\sim \in IFI(R)$,

$$f_\alpha^*([D_\sim]_{\mathcal{U}^\alpha}) = f_\alpha(D_\sim) = U(\chi_D, \alpha) = D$$

and

$$g_\alpha^*([D_\sim]_{\mathcal{L}^\alpha}) = g_\alpha(D_\sim) = L(\overline{\chi_D}, \alpha) = D.$$

Finally, for $0_\sim$ we get

$$f^*_\alpha([0_\sim]_{\mathcal{U}^\alpha}) = f_\alpha(0_\sim) = U(0,\alpha) = \emptyset$$

and

$$g^*_\alpha([0_\sim]_{\mathcal{L}^\alpha}) = g_\alpha(0_\sim) = L(1,\alpha) = \emptyset.$$

This shows that f^*_α and g^*_α are surjective. This completes the proof. □

For any $\alpha \in [0,1]$, we define another relation $\mathcal{R}^\alpha$ on $IFI(R)$ as follows:

$$(A,B) \in \mathcal{R}^\alpha \longleftrightarrow R_A^{(\alpha,\alpha)} = R_B^{(\alpha,\alpha)}.$$

Then the relation $\mathcal{R}^\alpha$ also is an equivalence relation on $IFI(R)$.

Theorem 23 *For any $\alpha \in (0,1)$ the map*
$$\varphi_\alpha : IFI(R) \to I(R) \cup \{\emptyset\}$$
defined by $\varphi_\alpha(A) = R_A^{(\alpha,\alpha)}$ is surjective.

Proof: Let $\alpha \in (0,1)$. Then $\varphi_\alpha(0_\sim) = R_A^{(\alpha,\alpha)} = U(0,\alpha) \cap L(1,\alpha) = \emptyset$. For any $H \in IFI(R)$, there exists $H_\sim = (\chi_H, \overline{\chi_H}) \in IFI(R)$ such that $\varphi_\alpha(H_\sim) = R_A^{(\alpha,\alpha)} = U(\chi_H,\alpha) \cap L(\overline{\chi_H},\alpha) = H$. So, φ_α is surjective. □

Theorem 24 *For any $\alpha \in (0,1)$, the quotient set $IFI(R)/\mathcal{R}^\alpha$ is equipotent to $I(R) \cup \{\emptyset\}$.*

Proof: Let $\alpha \in (0,1)$ and let
$$\varphi^*_\alpha : IFI(R)/\mathcal{R}^\alpha \to I(R) \cup \{\emptyset\}$$
be a map defined by $\varphi^*_\alpha([A]_{\mathcal{R}^\alpha}) = \varphi_\alpha(A)$ for all $[A]_{\mathcal{R}^\alpha} \in IFI(R)/\mathcal{R}^\alpha$.

If $\varphi^*_\alpha([A]_{\mathcal{R}^\alpha}) = \varphi^*_\alpha([B]_{\mathcal{R}^\alpha})$ for any $[A]_{\mathcal{R}^\alpha}$, $[B]_{R^\alpha} \in IFI(R)/\mathcal{R}^\alpha$, then $R_A^{(\alpha,\alpha)} = R_B^{(\alpha,\alpha)}$, i.e., $(A,B) \in \mathcal{R}^\alpha$. It follows that $[A]_{\mathcal{R}^\alpha} = [B]_{\mathcal{R}^\alpha}$ so that φ^*_α is injective.

Moreover, $\varphi^*_\alpha([0_\sim]_{\mathcal{R}^\alpha}) = \varphi_\alpha(0_\sim) = R_{0_\sim}^{(\alpha,\alpha)} = \emptyset$. For any $H \in I(R)$ we consider $H_\sim = (\chi_H, \overline{\chi_H})$. Then $H_\sim \in IFI(R)$ and

$$\varphi^*_\alpha([H_\sim]_{\mathcal{R}^\alpha}) = \varphi_\alpha(H_\sim) = R_{H_\sim}^{(\alpha,\alpha)}$$
$$= U(\chi_H,\alpha) \cap L(\overline{\chi_H},\alpha) = H.$$

This proves that φ^*_α is surjective. □

6 Conclusion

In the present paper we present the basic results on intuitionistic fuzzy left h-ideals of hemirings. It is clear that the most of these result can be simply extended to intuitionistic (S,T)-fuzzy left h-ideals, where S and T are given imaginable triangular norms. In our opinion the future study of (intuitionistic) fuzzy ideals in hemirings and semirings can be connected with (1) investigating semiprime and prime (intuitionistic) fuzzy h-ideals; (2) finding intuitionistic and/or interval valued fuzzy sets and triangular norms. The obtained results can be used to solve some social networks problems and decide whether the corresponding graph is balanced or clusterable.

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