Synthesis of Generalized Van der Pol Oscillator Systems

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Abstract: In this paper we present the method for synthesis of generalized Van der Pol oscillator systems using Melnikov function. The oscillations in such systems are regarded as limit cycles in perturbed Hamiltonian systems under polynomial perturbations of arbitrary degree. The method of synthesis is based on appropriate computation of perturbation coefficients, so that the prescribed properties are fulfilled.

Key-Words: Van der Pol oscillators, Melnikov function, Limit cycles, Synthesis of oscillator systems

1 Introduction

This paper presents a synthesis of oscillator systems, described by generalized Van der Pol equations and allowing self-sustained oscillations. The results are obtained on the basis of qualitative investigation of differential equations, whereupon the self-sustained oscillations are regarded as limit cycles on the phase plane. Under these conditions the synthesis of such systems consists of finding a differential equation having in advance assigned limit cycles, respectively self-sustained oscillations. Obtaining an electronic circuit from a given differential equation can be made by using the principles given in [6], [7] and these problems are not subject of our investigations.

The synthesis of generalized Van der Pol oscillators equations, allowing limit cycles with preliminarily assigned properties, is done by use of the well known Melnikov function for the perturbed Hamiltonian systems [1],[2],[3].

2 Oscillator equations, limit cycles and Melnikov function

The generalized Van der Pol oscillator systems consist of a nonlinear active element (nonlinear resistor) NE, linear conservative and dissipative elements and a source of direct voltage or current. More common examples of AC equivalent circuits of such oscillators systems are shown in Fig. 1, where NE are nonlinear resistive elements with V-A characteristics, respectively $i = F_1(u)$ for the first circuit and $u = F_2(i)$ for the second one.

The dynamics of the regarded systems is described by the generalized Van der Pol equation

$$\ddot{x} - \varepsilon \frac{dF}{dx}\dot{x} + x = 0\,, \tag{1}$$

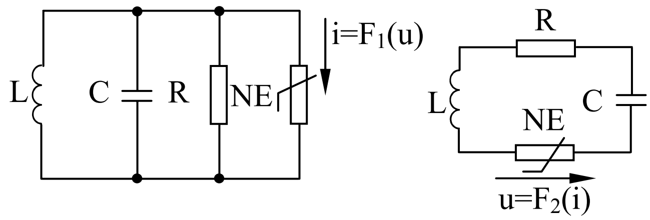

Fig.1. AC equivalent circuits of a generalized van der Pol oscillator system

where x is a physical quantity (scaling), $F(x)$ is a known characteristic (scaling) of the nonlinear active element and ε is a small parameter. Equation (1) can be rewritten in the form of perturbed Hamiltonian system

$$\begin{vmatrix} \dot{x} = y + \varepsilon F(x) \\ \dot{y} = -x \end{vmatrix}. \tag{2}$$

We shall assume that $F(x)$ is represented by a polynomial of arbitrary degree. Moreover, it has been proved in [5], that the arising of limit cycles in system (2) does not depend on the terms with even degrees in $F(x)$. In this case we can write

$$F(x) = a_1 x + a_3 x^3 + ... + a_{2n+1} x^{2n+1} \tag{3}$$

and the system (2) takes the following form

$$\begin{vmatrix} \dot{x} = y + \varepsilon(a_1 x + a_3 x^3 + \cdots + a_{2n+1} x^{2n+1}) \\ \dot{y} = -x \end{vmatrix} \tag{4}$$

Further in our exposition we will give a method for a synthesis of systems of the type (4) or more precisely obtaining the values of the perturbation coefficients a_1,

a_3, …, a_{2n+1}, in such a way, that the system (4) has in advance assigned limit cycles.

The unperturbed system (with $\varepsilon = 0$) has a Hamiltonian

$$H(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

and a single equilibrium point which is "centre", surrounded by a continuous one-parameter family of closed trajectories. The equation of a given closed trajectory is

$$\Gamma_0(h): H(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 = h,$$
$$h \in \Omega \equiv (0,\infty).$$

Following [2], [5] the expression for the Melnikov function of system (4) is

$$M(h) = 4\pi h\left[\frac{a_1}{2^2}\binom{2}{1} + \frac{a_3}{2^3}\binom{4}{2}h + \cdots + \frac{a_{2n+1}}{2^{n+2}}\binom{2n+2}{n+1}h^n\right] \quad (5)$$

It is well known that the number, positions and multiplicities of the limit cycles arising in system (4) are determined by the number, positions and multiplicities of the Melnikov function's zeros [1], [2], [3]. Omitting the details we will give a compendium of the general principles concerning bifurcations of limit cycles from centre:

a) if the following conditions hold

$$M(h_0) = M^{(1)}(h_0) = \cdots = M^{(m-1)}(h_0) = 0,$$
$$M^{(m)}(h_0) \neq 0, \quad m \geq 1, \quad (6)$$

then, for a sufficiently small $\varepsilon \neq 0$, there exists h_ε in an $O(\varepsilon)$ neighbourhood of h_0, such that the perturbed Hamiltonian system (4) has a multiple limit cycle $\Gamma_\varepsilon(h_\varepsilon)$, of multiplicity m (when $m = 1$ the limit cycle is simple). The limit cycle $\Gamma_\varepsilon(h_\varepsilon)$ is localized in an $O(\varepsilon)$ neighbourhood of the curve $\Gamma_0(h_0)$ and tend to $\Gamma_0(h_0)$ as $\varepsilon \to 0$;

b) if m is an odd number, $\Gamma_\varepsilon(h_\varepsilon)$ is a multiple limit cycle of odd multiplicity, which is stable when $\varepsilon M^{(m)}(h_0) < 0$, and unstable when $\varepsilon M^{(m)}(h_0) > 0$;

c) if m is an even number, $\Gamma_\varepsilon(h_\varepsilon)$ is a multiple limit cycle of even multiplicity, which is semi-stable. If $\varepsilon M^{(m)}(h_0) < 0$, then $\Gamma_\varepsilon(h_\varepsilon)$ is stable in the region $\{h : h > h_0\}$ and unstable in the region $\{h : h < h_0\}$. If $\varepsilon M^{(m)}(h_0) > 0$ the opposite assertion is valid.

3 Synthesis of the oscillator equations

The limit cycles arising in system (4) are determined by finding the zeros with their multiplicities of the Melnikov function, i.e. by finding the roots of the equation $M(h) = 0$, $h \in (0,\infty)$. It is easy to see that the nonzero roots of this equation coincide with the roots of the following polynomial equation

$$\frac{a_1}{2^2}\binom{2}{1} + \frac{a_3}{2^3}\binom{4}{2}h + \cdots + \frac{a_{2n+1}}{2^{n+2}}\binom{2n+2}{n+1}h^n = 0 \quad (7)$$

Further more without loss of generality we will assume $a_1 = 1$.

The roots of equation (7) are closely related to the perturbation coefficients a_1, $a_3, \ldots, a_{2n+1}$. The opposite assertion is also valid, i.e. the roots of the polynomial equation (7) determine its polynomial coefficients. Thus the main idea for the synthesis of oscillator equations consists of an appropriate choice of the zeros of the Melnikov function corresponding to the necessary limit cycles, and then determining the polynomial coefficients in equation (7) and perturbation coefficients in system (4). Briefly, we have to construct a polynomial under the conditions that its roots are given.

Let h_0 be a zero (of multiplicity m) of the Melnikov function. In this case the system (4) has a limit cycle $\Gamma_\varepsilon(h_\varepsilon)$, which is localized in an $O(\varepsilon)$ neighbourhood of the curve

$$\Gamma_0(h_0): \frac{1}{2}x^2 + \frac{1}{2}y^2 = h_0, \quad h_0 \in \Omega \equiv (0,\infty). \quad (8)$$

In the phase plane the curve $\Gamma_0(h_0)$ is a circle of radius $r = \sqrt{2.h_0}$ and centre $(0,0)$. The limit cycle $\Gamma_\varepsilon(h_\varepsilon)$ coincides in practice with the curve $\Gamma_0(h_0)$. In the time domain the limit cycle corresponds to the following sinusoidal oscillations

$$x(t) = \sqrt{2h_0}\sin t, \; y(t) = \sqrt{2h_0}\cos t.$$

Moreover, the stability of the limit cycle is determined by the sign of $\varepsilon M^{(m)}(h_0)$.

Finally, as an application of the proposed method we shall consider some examples of synthesis of oscillator systems.

Example: *Find a oscillator system of the type (4) having three simple limit cycles, which are circles of radiuses* $r_1 = \sqrt{2}$, $r_2 = 2$, $r_3 = \sqrt{6}$.

Solution: We note that these limit cycles correspond to the following sinusoidal oscillations:

$$x_1(t) = \sqrt{2}.\sin t\,,\; x_2(t) = 2.\sin t\,,\; x_3(t) = \sqrt{6}.\sin t\,.$$

The relations $r_i = \sqrt{2h_i}$, $h_i = r_i^2/2$, $i = 1,2,3$ give the Hamiltonian levels $h_1 = 1$, $h_2 = 2$, $h_3 = 3$. Therefore we have to find the coefficients $a_1, a_3, \cdots$, such that the numbers $h_1 = 1$, $h_2 = 2$, $h_3 = 3$ to be roots of equation (7). Since the cubic equation has three roots, then equation (7) takes the following form

$$\frac{1}{2^2}.\binom{2}{1} + \frac{a_3}{2^3}.\binom{4}{2}.h + \frac{a_5}{2^4}.\binom{6}{3}.h^2 + \frac{a_7}{2^5}.\binom{8}{4}.h^3 = 0\,,$$

or

$$\frac{35}{16}a_7 h^3 + \frac{5}{4}a_5 h^2 + \frac{3}{4}a_3 h + \frac{1}{2} = 0\,.$$

The following relations are valid:

$$h_1 + h_2 + h_3 = -\frac{(5/4)a_5}{(35/16)a_7} = -\frac{4a_5}{7a_7} = 6\,, \quad \text{(9a)}$$

$$h_1h_2 + h_2h_3 + h_3h_1 = \frac{(3/4)a_3}{(35/16)a_7} = \frac{12a_3}{35a_7} = 11\,, \quad \text{(9b)}$$

$$h_1h_2h_3 = -\frac{1/2}{(35/16)a_7} = -\frac{8}{35a_7} = 6\,. \quad \text{(9c)}$$

Equations (9) yields $a_3 = -11/9$, $a_5 = 2/5$ and $a_7 = -4/105$. In this way we obtain the following system

$$\begin{cases} \dot{x} = y + \varepsilon.\left(x - \frac{11}{9}.x^3 + \frac{2}{5}.x^5 - \frac{4}{105}.x^7 \right) \\ \dot{y} = -x \end{cases} \quad (10)$$

The system (10) has three limit cycles - $\Gamma_\varepsilon(h_{1\varepsilon})$, $\Gamma_\varepsilon(h_{2\varepsilon})$ and $\Gamma_\varepsilon(h_{3\varepsilon})$, which are respectively localized in $O(\varepsilon)$ neighbourhoods of the curves $\Gamma_0(h_1)$, $\Gamma_0(h_2)$ and $\Gamma_0(h_3)$.

The function $M(h)$ and its derivative are

$$M(h) = 4\pi h\left[\frac{1}{2} - \frac{11}{12}h + \frac{1}{2}h^2 - \frac{1}{12}h^3\right],$$

$$M'(h) = 4\pi\left[\frac{1}{2} - \frac{11}{6}h + \frac{3}{2}h^2 - \frac{1}{3}h^3\right].$$

Moreover

$$M(h_1) = 0\,,\; M'(h_1) = -4\pi(1/6) < 0\,,$$
$$M(h_2) = 0\,,\; M'(h_2) = 4\pi(1/6) > 0\,,$$
$$M(h_3) = 0\,,\; M'(h_3) = -4\pi(1/2) < 0\,.$$

From the last inequalities it follows that at $\varepsilon > 0$ the limit cycles $\Gamma_\varepsilon(h_{1\varepsilon})$ and $\Gamma_\varepsilon(h_{3\varepsilon})$ are stables and the limit cycle $\Gamma_\varepsilon(h_{2\varepsilon})$ is unstable. At $\varepsilon < 0$ the opposite assertion is valid.

The phase portrait of system (10) at $\varepsilon = 0.35$ obtained by numerical integration is shown in Fig. 2. The numerical computations confirm perfectly the analytical results.

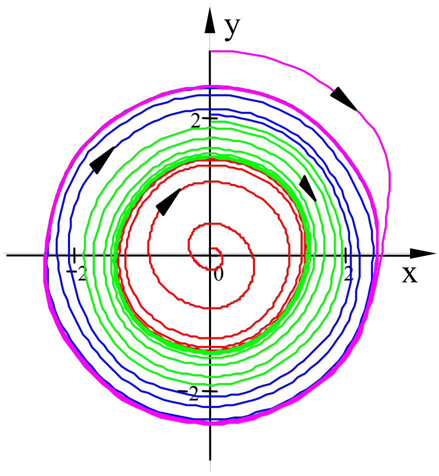

Fig. 2. The phase portrait of system (10) at $\varepsilon = 0.35$

4 Conclusion

A procedure for a synthesis of generalized Van der Pol oscillator systems is proposed. The synthesis of such systems is based on the Melnikov theory and consists in appropriate computation of the perturbation coefficients so that the prescribed properties to be fulfilled. This approach can be used for more complete and precise synthesis of oscillator systems.

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