On Existence of the Principles of Minimum Dissipated Power for Linear and Nonlinear Electric Circuits

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Abstract: - **The principles of the minimum dissipated power are extremely important in the analysis of electrical circuits, and represent the equilibrium state of all the circuit.. For demonstrate these principles, in the present work, a power functional is built for a circuit in steady-state conditions and its limit is determined. It is shown that the minimum of this functional corresponds to the minimum dissipated power in the circuit. Some applications to electric linear and nonlinear circuits are shown.**

Key-Words: - electric circuits, stationary regime, dissipated power, Hilbert space, minimum of functionals.

1 Introduction

Conservative systems accept the definition of functionals expressed in terms of power or energy. Calculating the limits of these functionals represents an important breakthrough in formulating and solving optimization problems.

The steady states in the mechanical, thermal, electrical conservative systems generally represent limit states from an energy point of view. For example [1], in the classical mechanics, Hamilton's principle of the minimum action states that the development in time of a system, from one steady state to another steady state, happens along a curve $\gamma : [a,b] \rightarrow R^n$, which extremates the functional

$$F_M = \int_a^b (T + U)dt \quad (1)$$

called the integral of action, where T is the kinetic energy (the "life power") of the system and U is the mechanical effect of the power system which works on the system under consideration.

In the theory of the electrical circuits, the results obtained by Millar [2] and Stern [3] related to the co-content function for nonlinear resistive and reciprocal network have a special theoretic importance due to their generality. Desoer and Kuh ([4] pp. 770-772), mentioned the same general properties of minimum dissipated power for linear and resistive networks. Furthermore, Ionescu [5] and Mocanu ([6] pp. 350-353) provided important contributions to the theoretical development of the electrical circuits minimax theorems. All these results are basically consequences of Maxwell's principles of minimum-heat ([7] pp. 407-408).

The electromagnetic field analysis admits a differential formulation and a variational equivalent formulation. The mathematical variational model presumes the establishment of a variational principle capable to supply the equations of the differential mathematical model through the stationarity condition of an adequate functional. The energy functional of the non-stationary electromagnetic field associated to the domain D_c and the volume delimitated by D_c, is:

$$F = \int_{D_c} \{(\int_0^{\overline{E}} \overline{D}d\overline{E} - \int_0^{\overline{B}} \overline{H}d\overline{B}) + (\overline{J}\,\overline{A} - \rho_v V)\}dxdydz \quad (2)$$

where $\overline{D}, \overline{E}, \overline{H}, \overline{B}$ are the vectors associated to the electric and magnetic field, $\overline{A}$ is the magnetic vector potential, V is the electric potential, $\overline{J}$ is the vector of conduction current density, and ρ_v is the volume density of electric charge. The condition of minimum of the functional (2) involve the fundamental equations of the electromagnetic field

$$rot\overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}, div\overline{D} = \rho_v$$

Similarly, the energetic functional for electrokinetic field, where the state quantity is the electric potential V, is defined by:

$$F_1(V) = \int_{D_c} (\int_0^{\overline{J}} \overline{J}d\overline{E})dzxdydz + \int_{\Sigma_n} \overline{J}\overline{n}Vd\Sigma_n \quad (3)$$

This paper presents three original contributions concerning the construction of power functionals for linear electric circuits in the steady-state. The first of them proposes a power functional for the linear circuits derived from the circuit equation properties. The second represents another demonstration which proves that the stationary (equilibrium) electrokinetic state for the direct current (d.c.) networks represents a minimum dissipated power state, and the third one proposes an original and general principle of the minimum dissipated active and reactive power for the a.c. linear circuits.

2 Minimum Dissipated Power in D.C Circuits

Let's consider the case of a reciprocal d.c. electric circuit, with l branches, in an equilibrium state. Let G represent the graph of the circuit, of which A is a tree and C its complementary tree. The branch voltages and branch currents verify the first and, respectively, the second theorem of Kirchhoff. The l-dimensional current and voltage matrix can be partitioned as follows

$$[I] = \begin{bmatrix} [I_c] \\ [I_r] \end{bmatrix} = \begin{bmatrix} [I_c] \\ -[\Lambda_{cr}][I_c] \end{bmatrix} \quad (4)$$

$$[U] = \begin{bmatrix} [U_c] \\ [U_r] \end{bmatrix} = \begin{bmatrix} [\Lambda_{cr}][U_r] \\ [U_r] \end{bmatrix} \quad (5)$$

where an adequate notation has been used to point out the submatrix that refers to the current and voltage links, and the respective branches, whereas $[\Lambda] = -\lfloor C_{B_c r} \rfloor$ is the essential incidence matrix. Then, all the voltage generators become equivalent current generators, the matrix relation among the branch currents becomes:

$$[I] = [G][U] \quad (6)$$

where [G] is the square matrix of the branch conductance, which can be partitioned under the form of:

$$[G] = \begin{bmatrix} [G_{cc}][G_{cr}] \\ [G_{rc}][G_{rr}] \end{bmatrix} \quad (7)$$

For reciprocal circuits in stable electrokinetic state, the following power functional can be defined in the Hilbert space R^n

$$F : R^n \rightarrow R \quad (8)$$

$$F \stackrel{\Delta}{=} \frac{1}{2}[U]^T [I] \quad (9)$$

By using the relations (4),..,9) we can calculate

$$\begin{aligned} F([U_r]) &= \frac{1}{2}[U]^T [I] = \\ &= \frac{1}{2}\left[[U_c]^T [U_r]^T\right] \begin{bmatrix} [G_{cc}][G_{cr}] \\ [G_{rc}][G_{rr}] \end{bmatrix} \begin{bmatrix} [U_c] \\ [U_r] \end{bmatrix} = \\ &= \frac{1}{2}\{[U_r]^T [\Lambda_{rc}][G_{cc}][\Lambda_{cr}][U_r] + \\ &+ [U_r]^T [G_{rr}][U_r]\} = \\ &= \frac{1}{2}[U_r]^T \{[\Lambda_{rc}][G_{cc}][\Lambda_{cr}] + [G_{rr}]\}[U_r] \end{aligned} \quad (10)$$

where $[G_{cr}]=[G_{cr}]=0$ for the reciprocal electric circuits.

In the relation (10), we have obtained the expression of the functional under consideration, based only on the matrix of the branch voltages. To determine its extreme, we apply the function of matrix properties. If $F'([U_r]=0$, it results

$$\begin{aligned}&[U_r]^T([\Lambda_{rc}][G_{cc}][\Lambda_{cr}]+[G_{rr}])[U_r]=0\\&[\Lambda_{rc}][G_{cc}][\Lambda_{cr}][U_r]+[G_{rr}][U_r]=0\end{aligned} \qquad (11)$$

that is,

$$[\Lambda_{rc}][I_c]+[I_r]=0 \qquad (12)$$

Consequently, the extreme point of the functional verifies the first theorem of Kirchhoff. To demonstrate that the extreme point of the functional is minimum, we take a different matrix of the branch voltages,

$$[\hat{U}_r]=[U_r]+\delta[U_r] \qquad (13)$$

which introduced in expression (9) leads to:

$$\begin{aligned}&F([\hat{U}_r])=\frac{1}{2}([U_r]+\delta[U_r])^T([\Lambda_{rc}][G_{cc}][\Lambda_{cr}]+\\&+[G_{rr}])([U_r]+\delta[U_r])=F([U_r])+\\&+\delta[U_r]^T\{[\Lambda_{rc}][G_{cc}][\Lambda_{cr}]+[G_{rr}]\}[U_r]+\\&+\frac{1}{2}\delta^2[U_r]^T\{[\Lambda_{rc}][G_{cc}][\Lambda_{cr}]+[G_{rr}]\}[U_r]\end{aligned} \qquad (14)$$

As demonstrated in [8], the matrixes $[\Lambda_{rc}][G_{cc}][\Lambda_{cr}]$ and $[G_{rr}]$ are positive defined, so $[\Lambda_{rc}][G_{cc}][\Lambda_{cr}]+[G_{cc}]>0$, and

$$F([\hat{U}_r])>F([U_r]) \qquad (15)$$

This means that the functional has a minimum. According to definition (9), the functional $F([U_r])$ represents the power dissipated at the terminals of all the branches of a reciprocal circuit in stable electrokinetic state. Therefore, the result obtained in (12) shows that the minimum power dissipated in the branches of the circuit corresponds to the first Kirchhoff's law.

Consequently, we get the following principle (*1st Principle of Minimum dissipated Power – PMP*): *the minimum power dissipated by the branches of a linear and resistive circuit in stationary regime (d.c.) is satisfied by the solutions in the currents and voltages of the circuit, and these are the currents and voltages which verify the 1st and 2nd theorem of Kirchhoff.*

3 Minimum of the Active and Reactive Dissipated Power for Linear A.C. Circuits

We can demonstrate a similar principle for the quasi-stationary regime (a.c.) of a linear electric circuit.

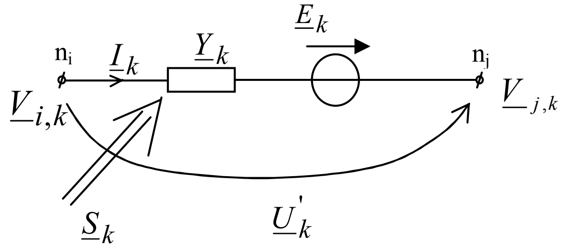

Figure 1. A.c. circuit branch

By using the symbolical method, the voltage at every branch (fig.1) of the circuit is equal to:

$$\underline{U}'_k+\underline{E}_k=\frac{\underline{I}_k}{\underline{Y}_k} \qquad (16)$$

If we denote: $\underline{V}_{i,k}=x_{i,k}+jy_{i,k}$; $\underline{V}_{j,k}=x_{j,k}+jy_{j,k}$; $\underline{Y}_k=G_k-jB_k$; $\underline{E}_k=a_{E,k}+jb_{E,k}$,

where G_k, B_k, $a_{E,k}$ and $b_{E,k}$ are constant, then the complex conjugated current of branch k can be expressed [9]:

$$\begin{aligned}&\underline{I}_k^*=\underline{Y}_k^*(\underline{V}_{i,k}-\underline{V}_{j,k}+\underline{E}_k)^*=\\&[G_k(x_{i,k}-x_{j,k}+a_{E,k})+\\&+B_k(y_{i,k}-y_{j,k}+b_{E,k})]+\\&+j[B_k(x_{i,k}-x_{j,k}+a_{E,k})-\\&-G_k(y_{i,k}-y_{j,k}+b_{E,k})]\end{aligned} \qquad (17)$$

The complex power associated to the admittances of all the L branches of the circuit is:

$$\begin{aligned}&\sum_{k=1}^{L}\underline{S}_k=\sum_{k=1}^{L}\underline{U}_k\underline{I}_k^*=\\&=\sum_{k=1}^{L}(G_k+jB_k)[(x_{i,k}-x_{j,k}+a_{E,k})^2+\\&+(y_{i,k}-y_{j,k}+b_{E,k})^2]\end{aligned} \qquad (18)$$

The real and imaginary parts of the complex power can be defined as functionals in the Hilbert space [10]

$$\begin{aligned}&F_R\equiv\frac{1}{2}\mathrm{Re}[\underline{S}]:R^{2N}\to R\\&F_R(x_1,x_2,...,x_N,y_1,y_2,...,y_N)\equiv\frac{1}{2}\mathrm{Re}[\underline{S}]=\\&\frac{1}{2}\sum_{k=1}^{L}G_k[(x_{i,k}-x_{j,k}+a_{E,k})^2+(y_{i,k}-y_{j,k}+b_{E,k})^2];\\&F_I\equiv\frac{1}{2}\mathrm{Im}[\underline{S}]:R^{2N}\to R\\&F_I(x_1,x_2,...,x_N,y_1,y_2,...,y_N)=\frac{1}{2}\mathrm{Im}[\underline{S}]=\\&\frac{1}{2}\sum_{k=1}^{L}B_k[(x_{i,k}-x_{j,k}+a_{E,k})^2+(y_{i,k}-y_{j,k}+b_{E,k})^2]\end{aligned} \qquad (19)$$

and they are quite obviously a function class C^2 in R^{2N}, and are positive defined, [9], i.e., for all the pairs $(x_i, y_i), i = 1,..., N$, then: $F_R(x_1, x_2,..., x_N, y_1, y_2,..., y_N) > 0$ and $F_I(x_1, x_2,..., x_N, y_1, y_2,..., y_N) > 0$

Consequently, the minimum points of the real and imaginary component of the complex power functionals [10] are the solutions of the system which contains $4N$ equations:

$$\begin{aligned} &\frac{\partial F_R}{\partial x_1} = 0, \frac{\partial F_R}{\partial y_1} = 0,..., \frac{\partial F_R}{\partial x_N} = 0, \frac{\partial F_R}{\partial y_N} = 0, \\ &\frac{\partial F_I}{\partial x_1} = 0, \frac{\partial F_I}{\partial y_1} = 0,..., \frac{\partial F_I}{\partial x_N} = 0, \frac{\partial F_I}{\partial y_N} = 0 \end{aligned} \quad (20)$$

If we calculate the algebraic sum of the solutions, with one of them multiplied with (-1) or ±j , we obtain the expressions:

$$\sum_{l_k \in n_1} \underline{I}_k^* = 0, \quad \sum_{l_k \in n_2} \underline{I}_k^* = 0,..., \quad \sum_{l_k \in n_N} \underline{I}_k^* = 0 \quad (21)$$

which are identical to the Kirchhoff's equations for currents (1st Kirchhoff theorem), expressed in all the N nodes of the circuit.

Consequently, the following principle can be issued *(2nd Principle of Minimum Active and Reactive Power – PMARP)*: *the minimum of the active and reactive power associated to the branches of a linear circuit in a quasi-stationary regime (a.c.) is satisfied by the solutions in currents and voltages of the circuit, and these are the currents and voltages that verify the 1st and 2nd theorem of Kirchhoff.*

4. Examples

4.1 Minimum dissipated power for a d.c. linear circuit

We consider the d.c. linear circuit with three branches shown in figure 2, where $R_1 = 1\Omega, R_2 = 2\Omega, R_3 = 3\Omega, E_1 = 4V$.

Given $U_r = V_2 - V_1$, the power functional associated to the branches of the circuit, (9), is calculated depending on the potentials V_1, V_2 :

$$F = \frac{1}{2}\{G_1(V_2 - V_1 + E_1)^2 + G_2(V_1 - V_2)^2 + \\ + G_3(V_1 - V_2)^2\}$$

The minimum of the power functional corresponds to the solutions of the system

$$\frac{\partial F}{\partial V_1} = -G_1(V_2 - V_1 + E_1) + G_2(V_1 - V_2) + \\ + G_3(V_1 - V_2) = -I_1 + I_2 + I_3 = \sum_{l_k \in n_1} I_k = 0$$

$$\frac{\partial F}{\partial V_2} = G_1(V_2 - V_1 + E_1) - G_2(V_1 - V_2) - \\ - G_3(V_1 - V_2) = I_1 - I_2 - I_3 = \sum_{l_k \in n_2} I_k = 0$$

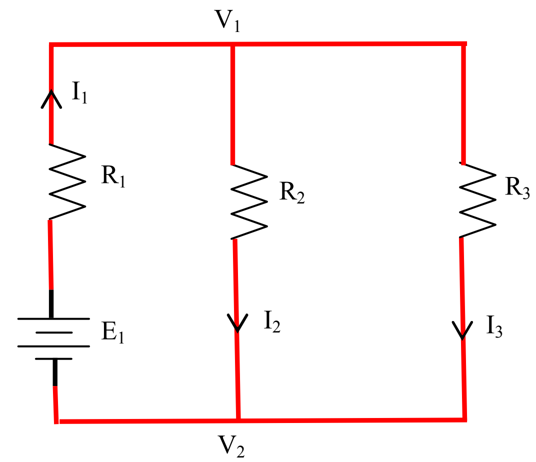

Figure. 2. D.c. circuit with three branches

which represent the 1st theorem of Kirchhoff expressed in node 1 and node 2. We calculated using PSPICE and MATHCAD the variation of the dissipated powers P_1, P_2, P_3 depending by the currents I_1, I_2, I_3, shown respectively in figures 3, 4 and 5. Using the Thèvenin's theorem, we obtain the expressions:

$P_i = (U_{i,0} - R_{e,i} I_i) I_i, i = 2,3,$ where $U_{i,0}, R_{e,i}$ are the open voltage and the equivalent resistance of the circuit at the terminals of branch 2, respectively 3.

It is remarked that the dissipated power in each branch is a minimum value compared with the maximum value of the power, which is obtained as one half of the value of the short-circuit current $(\frac{I_{sc}}{2})$, with

$P_1 \approx 3{,}305W < P_{1\max} = 3{,}33W$; $P_2 \approx 2{,}38W < P_{2\max} = 3W$; $P_3 \approx 1{,}586W < P_{3\max} \approx 2{,}67W$

We observe as well in figures 3, 4, and 5 the existence of two branch current values which correspond to the value of dissipated power for each branch

$I_1^{'} = 1.818A, I_1^{''} = 1.515A; I_2^{'} = 1.09A, I_2^{''} = 2.909A;$

$I_3^{'} = 0.727A, I_3^{''} = 3.272A.$

Only one of these values verifies the Kirchhoff's theorems $(I_1^{'}, I_2^{'}, I_3^{'})$. For the branches with resistances, these current values represent the roots of a second degree equation, that is, $I^{','' } = \frac{U_0}{2R_e}(1 \pm \frac{R_e - R}{R_e + R})$.

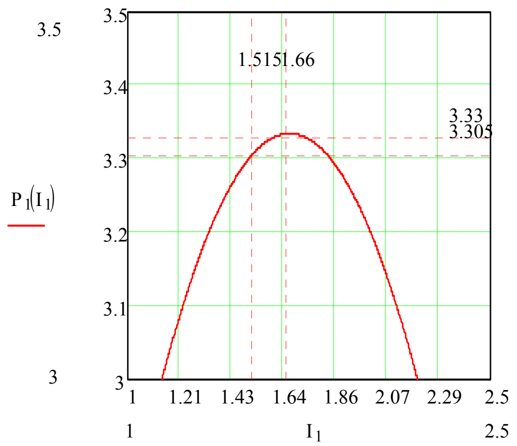

Figure 3. The variation of P_1

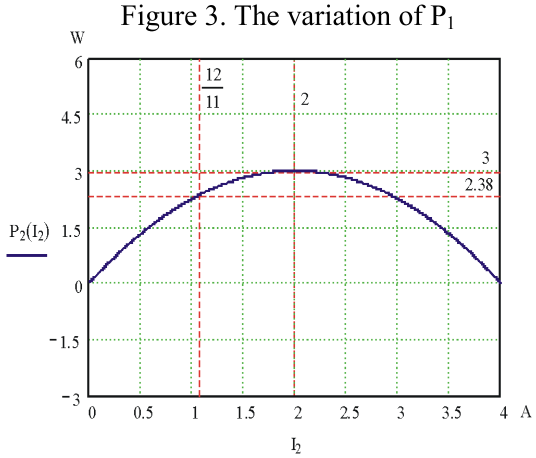

Figure 4. The variation of P_2

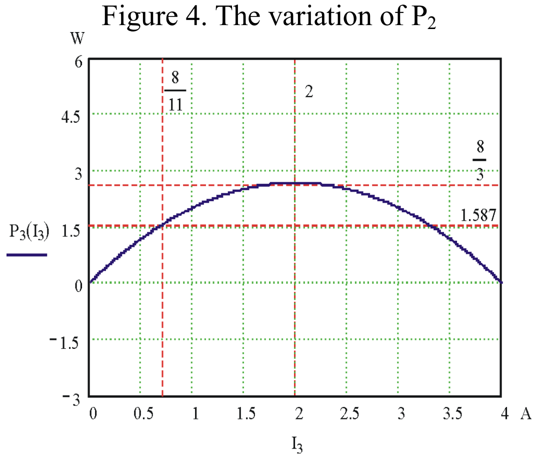

Figure 5. The variation of P_3

The graphic positions of these two values of current compared with the value $\frac{I_{sc}}{2}$ are depending on the resistance value: if $R < R_e$ then $I' > \frac{I_{sc}}{2}$, and if $R > R_e$ then $I'' < \frac{I_{sc}}{2}$, as presented in figure 4 and figure 5, respectively . In conclusion, the currents in a circuit in d.c. regime are distributed to verify the 1st Kirchhoff theorem and the power dissipated by the circuit is minimum.

4.2 Minimum dissipated power for a d.c. nonlinear circuit

We consider the d.c. nonlinear circuit shown in figure 6. The linear elements have the values $R_1 = 1K\Omega, R_2 = 4K\Omega, R_3 = 450\Omega,, E_1 = 4V, I_5 = 4mA$, and the nonlinear conductance G_4 is characterized by the dependence $i_4 = 0{,}9 - 1{,}2u_4 + 0{,}4u_4^2$, shown with red in figure 7.

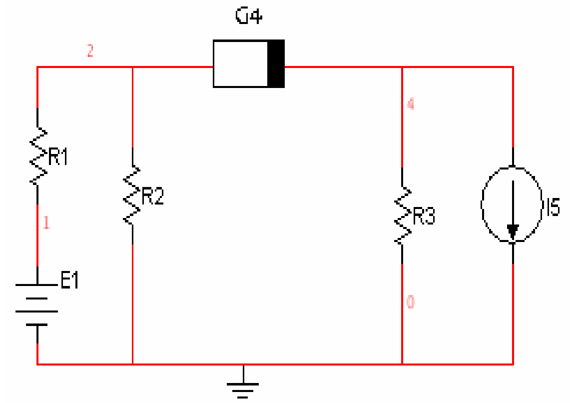

Figure 6. D.C. nonlinear circuit

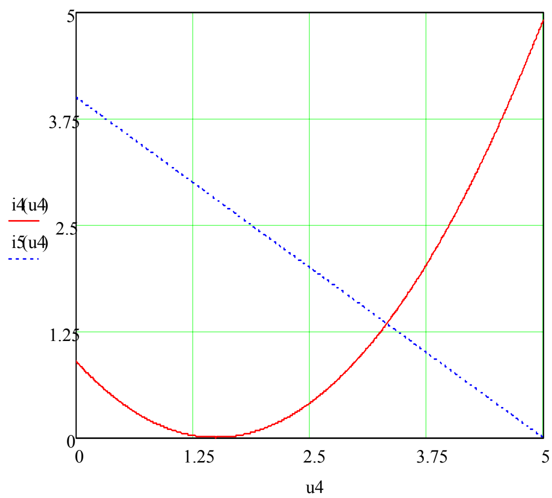

Figure 7. The nonlinear characteristic i_4 (u_4) - red - and the load characteristic of circuit of the nodes 2-4 - blue.

We can calculate the currents of circuit branches using a PSPICE algorithm. For the nonlinear conductance the graphical solution, shown in figure 7, is given by the intersection point – functioning point A (3,32V; 1,34mA), and results $G_4 = 0{,}393mS$.. The dissipated powers in the linear resistances and in the nonlinear conductance are $P_1 = 3.316W, P_2 = 1.0857W, P_3 = 3.2805W, P_4 = 0.00429W$. Thus, the total dissipated power is $P_{dis} = 7{,}68649W$.

For comparing these dissipated powers with the maximum powers absorbed by each resistance, we can use the Thevenin' theorem for each element. We obtain the maximum dissipated power for each resistance, as if the regime of circuit is adapted for all the consumers. The maximum values are $P_{1,\max} = 20.636W, P_{2,\max} = 0.240W$, $P_{3,\max} = 7.718W, P_{4,\max} = 3.2W$, and the total maximum power is: $P_{\max} = 31,794W$. It is remarked that the total dissipated power and the dissipated power for each resistance are a minimum values compared with the total maximum power and the maximum values of power for the same resistance. Thus, for the currents which verifies 1st Kirchhoff theorem, the powers dissipated by the considered nonlinear circuit are minimum.

4.3.Determination of the minimum power functional for a.c. circuits

We consider the a.c. circuit shown in figure 8. The expressions of the complex potentials, of the complex source and of the complex conjugated currents (15), are:

$$\underline{V}_1 = x_1 + jy_1; \underline{V}_2 = x_2 + jy_2; \underline{E}_1 = a + jb;$$

$$\underline{I}_1^* = (G_1 + jB_1)[(x_2 - x_1 + a) - j(y_2 - y_1 + b)];$$

$$\underline{I}_2^* = (G_2 + jB_2)[(x_1 - x_2) - j(y_1 - y_2)];$$

$$\underline{I}_3^* = (G_3 + jB_3)[(x_1 - x_2) - j(y_1 - y_2)].$$

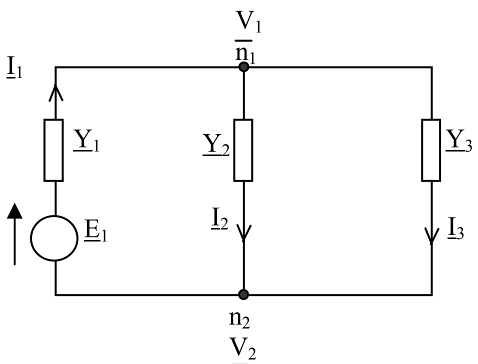

Figure 8. A.c. circuit with three branches

The total complex power associated to the admittances of the circuit is (16):

$$\underline{S} = \sum_{k=1,2,3} \underline{S}_k = (G_1 + jB_1)[(x_2 - x_1 + a)^2 + (y_2 - y_1 + b)^2] +$$

$$+ (G_2 + jB_2)[(x_1 - x_2)^2 + (y_1 - y_2)^2] +$$

$$+ (G_3 + jB_3)[(x_1 - x_2)^2 + (y_1 - y_2)^2$$

If the variables are x_1 and y_1, the minimum of the functionals (17) is the solution of the system:

$$\frac{1}{2}\frac{\partial \operatorname{Re}[\underline{S}]}{\partial x_1} = -G_1(x_2 - x_1 + a) + G_2(x_1 - x_2) + G_3(x_1 - x_2) = 0$$

$$\frac{1}{2}\frac{\partial \operatorname{Re}[\underline{S}]}{\partial y_1} = -G_1(y_2 - y_1 + b) + G_2(y_1 - y_2) + G_3(y_1 - y_2) = 0$$

$$\frac{1}{2}\frac{\partial \operatorname{Im}[\underline{S}]}{\partial x_1} = -B_1(x_2 - x_1 + a) + B_2(x_1 - x_2) + B_3(x_1 - x_2) = 0$$

$$\frac{1}{2}\frac{\partial \operatorname{Im}[\underline{S}]}{\partial y_1} = -B_1(y_2 - y_1 + b) + G_2(y_1 - y_2) + G_3(y_1 - y_2) = 0.$$

If we sum up the first equation of the system and the third equation multiplied by (-1), and if we sum up the second equation multiplied by $(-j)$ with the fourth equation, we obtain the 1st theorem of Kirchhoff expressed in node 1:

$$-\underline{I}_1^* + \underline{I}_2^* + \underline{I}_3^* = \sum_{l_k \in n_1} \underline{I}_k^* = 0$$

5 Conclusions

The determination of the extreme of the power functional for linear circuits is a problem of utmost importance, with useful tutorial, theoretical and practical applications. Using variational principles, it has been established that the solutions of the linear electric d.c. and a.c. circuit represent a minimum of the dissipated power in the stationary regime of the circuit. Generally speaking, the circuit doesn't generate the maximum power for all the consumers.

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