Littlewood Paley Spline Wavelets

E. SERRANO and C.E. D'ATTELLIS
Escuela de Ciencia y Tecnología
Universidad Nacional de General San Martín
M. Irigoyen 3100 - (1650) San Martín - Pcia de Buenos Aires
ARGENTINE

Abstract: In this work we propose a simple and efficient method to compute the Semi-discrete Wavelet Transform using appropriate spline wavelets in the Littlewood Paley decomposition scheme. These wavelets are compactly supported and oscillating functions. Moreover, the scheme can be refined in the frequency domain and a fine approximate on the Continuous Transform is achieved.

Key-Words:- Littlewood Paley Analysis - Wavelet Transform - Spline functions - Spline wavelets

1 Introduction

Nowadays, wavelets play a significant role in pure and applied mathematics. In physics or signal processing applications wavelet analysis can be consider as a time-scale technique. Therefore,for many problems, it performs better than Fourier methods.

However, it is a multifaceted tool. For instance, the Discrete Wavelet Transform (DWT) is correlated with the atomic decomposition of the given signal, using an orthonormal wavelet basis in the context of a multiresolution scheme.

On the other hand, when the shift-invariance is needed or large scales phenomena are as important as small scales ones, the correct tool for analyzing is the Continuous (CWT) or the Semi-Discrete Wavelet Transform (SDWT).

Recently, Littlewood - Paley analysis seems to have the better performance for analizing turbulent phenomena, fractional Brownian processes or frequency modulated signals (*chirps*). We refer to [4,5] for details.

Littlewood-Paley analysis can be considered as a particular semi-discrete wavelet transform. Following the above mentioned reference [5], let us give a quick survey.

Let a function $\varphi \in \mathcal{S}$, the Schwartz class in $L^2(\mathcal{R})$, and assume that its Fourier transform $\widehat{\varphi}$ vanishes outside the interval $|\omega| \leq 2\pi$ and $\widehat{\varphi}(\omega) = 1$ on $|\omega| \leq \pi$.

Given a signal s, we define, in the frecuency domain, the convolution operator L_j, for each $j \in \mathcal{Z}$, as:

$$[\widehat{L_j f}](\omega) \quad = \quad \widehat{\varphi}(2^{-j}\omega)\widehat{s}(\omega)$$

and the difference operator:

$$D_j s \quad = \quad (L_{j+1} - L_j)s$$

Observe that

$$D_j s \quad = \quad (2^{(j+1)}\varphi(2^{j+1}.) - 2^j\varphi(2^j.)) * s$$

and defining the *Littlewood-Paley wavelet*:

$$\psi(x) \quad = \quad 2\varphi(2x) - \varphi(x)$$

we have:

$$D_j s = 2^j \ \psi(2^j.) * s$$

In summary, we have the remarkable decomposition:

$$s = L_0 s + \sum_{j=0}^{\infty} D_j s$$

where each component is a convolution and its Fourier transform is concentrated in the two side band $2^j\pi \leq |\omega| \leq 2^{j+1}\pi$.

The function ψ is and *admissible wavelet.* On the other hand, we remark that φ is not compactly supported because its Fourier transform has finite support. Furthemore, in general, the function φ doesn't verify a finite double-scale equation. These circunstances may be a drawback for the numerical applications.

For these reasons we explore alternatives. Here we propose to replace the function φ in the Schwartz class for an appropriate spline function ϕ with compact support and oscillating.

As we expect, using splines we can exploit self- similarity properties, particularly two scale relations. This give us numerical adventages.

Indeed, beyond the semidiscrete transform and the diadic scales, those properties led us to design a refinated scheme of analysis in the frecuency domain, in a tight approximate of the Continuous Wavelet Transform, [3].

2. Basic Functions and Wavelets

For integer odd numbers m, the *Q-spline functions* are defined from [1,7]:

$$\widehat{Q_m}(\omega) = \left(\frac{\text{sen}\omega/2}{\omega/2}\right)^{m+1}$$

or, in the time domain:

$$Q_m(x) = \underbrace{Q_0 * \ldots * Q_0}_{m+1\ times}(x)$$

being Q_0 the characteristic function of $[-1/2, 1/2]$. Let us enphatize that Q_m can be avaluated using closed formulas, [7].

The functions Q_m are symmetrical, supported on the intervall $[-(m+1)/2, (m+1)/2]$ and their Fourier transforms, $\widehat{Q_m}(\omega)$ are well concentrated on $[-\pi, \pi]$ and they decay as $|\omega|^{-(m+1)}$.

Moreover, we reacall that each family $\{Q_m(x-k); k \in \mathcal{Z}\}$ is a Riesz Basis for the subspace of spline functions of order m with integer knots and finite energy. We refer to [1,3] for details.

Next, for each m and any positive integer n we define $\phi = \phi_{m,n}$, the *basic spline function* as:

$$\widehat{\phi}(\omega) = \widehat{Q_m}(\omega) F_{m,n}(\omega)$$

where:

$$F_{m,n}(\omega) = f_{m,n}[0] + \sum_{k=1}^{n} f_{m,n}[k] \cos(k\omega)$$

and the coefficients $f_{m,n}$ give us the unique solution for the linear equations:

$$\widehat{\phi}(0) = 1,\ \widehat{\phi}^{(2i)}(0) = 0;\ i = 1, \ldots, n$$

The basic function $\phi \in V_0^m$ is a nice alternative for the function $\varphi \in \mathcal{S}$. Depending on m and n, it is supported on the interval $[-n-(m+1)/2,$
$n+(m+1)/2]$ and $\widehat{\phi}$ is an almost ideal low pass-filter on $[-\pi, \pi]$.

Following, we define $\psi = \psi_{m,n}$, the associate *Littlewood-Paley wavelet*, as:

$$\psi(x) = 2\phi(2x) - \phi(x)$$

This is a spline function of order m with knots in $\mathcal{Z}/2$. It is symmetric, centered on $x = 0$ and it is also supported on $[-n-(m+1)/2, n+(m+1)/2]$.

The Fourier transform $\widehat{\psi}(\omega)$ is well localized on the two side band $\pi/2 \leq |\omega| \leq \pi$ and it decays as $1/|\omega|^{m+1}$. It has a pic in
$\omega_{pic} \approx 1.5\pi$.

Since

$$\widehat{\psi}^{(l)}(0) = 0;\ l = 0, \ldots, 2n+1$$

the wavelet has $2n+1$ null moments.

In summary, the proposed spline wavelets seem similar to the original ones in the Schwartz class. We expect that they will constitute an efficient tool for the applications.

3. Semidiscrete Transform

We recall that, for odd integers m, the two scale relations for the Q-spline are:

$$\widehat{Q_m}(2\omega) = \widehat{Q_m}(\omega)\ \cos^{m+1}(\omega/2)$$

$$Q_m(x/2) = 2 \sum_{|k| \leq (m+1)/2} h_m[k] Q_m(x-k)$$

where:

$$h_m[k] = \frac{1}{2^{m+1}} \binom{m+1}{k+(m+1)/2}$$

Then, for each $\psi = \psi_{m,n}$ we can write:

$$\widehat{\psi}(\omega) = \widehat{Q_m}(\omega/2)G(\omega)$$

where

$$G(\omega) = F_{m,n}(\omega/2) - F_{m,n}(\omega)\cos^{m+1}(\omega/4)$$

This is a 4π-periodic trigonometrical polynomial of the form:

$$G(\omega) = \sum_{|p|\leq P} g_{m,n}[p]\exp(-ip\omega/2)$$

where $P = 2n + (m+1)/2$ and its coefficients can be easily computed from the above formulas. Finally, we simply obtain:

$$\psi(x) = 2\sum_{|p|\leq P} g_{m,n}[p]Q_m(2x-p)$$

Given m and n and a locally integrable signal s, the *Semidiscrete Transform*, $Ds = D_{m,n}s$, is defined as:

$$D_j s = 2^j\ \psi(2^j.) * s$$

for $j \in \mathcal{Z}$.

As usual, we will compute the transform for $j = -1, -2, \ldots, J_{min}$. Assume that m and n are chosen and denote:

$$S_j = Q_m(2^j.) * s$$

Then, recalling that:

$$2^j\psi(x) = 2^{j+1}\sum_{|p|\leq P} g_{m,n}[p]Q_m(2^{j+1}x-p)$$

we have:

$$D_j s(x) = 2^{j+1}\sum_{|p|\leq P} g_{m,n}[p]S_{j+1}(x-p2^{-(j+1)})$$

Particullarly, for integer values q:

$$D_j s(q) = 2^{j+1}\sum_{|p|\leq P} g_{m,n}[p]S_{j+1}(q-p2^{-(j+1)})$$

Now using the two-scale relation for the Q_m spline we deduce the recursive scheme:

$$S_j(q) = 2\sum_{|k|\leq(m+1)/2} h_m[k]S_{j+1}(q-2^{(-j+1)}k)$$

for any $q \in \mathcal{Z}$. At this point we recall that these discrete convolutions are computed intercaling $2^{(-j+1)}-1$ zeros in the filter $g_{m,n}$ or h_m.

In analogous form we deduce the recursive scheme to compute the convolutions $L_j s$:

$$L_j s(q) = 2^j\sum_{|p|\leq P} b_{m,n}[p]S_j(q-2^{-(j)}p)$$

where $b_{m,n}[0] = f_{m,n}[0]$ and $b_{m,n}[p] = 1/2 f_{m,n}[|p|]$.

Therefore, we can decompose the signal in the frequency components $D_j s$. Also we obtain the filtered signal $L_j s$.

In the first step we must compute the initial values $S_0[q]$. Many alternatives are opened for this purpose. One simple choice, assuming that $s[k]$ are the sampled valued of the signal, is to compute the discrete convolution:

$$S_0[q] = \sum_k Q_m[k]s[q-k]$$

Observe that, in this case, S_j, $L_j s$ and $D_j s$ are spline functions of order m. Particullarly, $L_0 s$ is a representation of the signal in V_0^m.

In the opposite, we can reconstruct this representation as:

$$L_0[q] = \sum_{j=J_{min}}^{-1} D_j[q] + L_{J_{min}}[q]$$

As it is well known the semidiscrete transform gives us a shift-invariant time-scale decomposition of the signal. It is a recommended tool for some applications, as turbulence analysis or pattern recognition, for instance.

Followig we seek for a refined scale providing a more precise transform in the frequency domain.

4. Extended Scheme

Assume that the sampled rate is R and we start with the sampled values $s[k/R]$.

Now, for $j \leq 0$ and $r = 0, \dots R-1$, define:

$$j_r = j + \log_2(R/(R+r))$$

Note that $j \leq j_r < j-1$ and

$$f(2^j_r x) = f(2^j(R/(R+r)x)$$

Also we remak that $(j+1)_r = j_r + 1$.

In accordance with these definitions, we have:

$$\phi_{j_r}(x) = 2^{j_r}\phi(2^{j_r}x)$$
$$\psi_{j_r}(x) = 2^{j_r}\psi(2^{j_r}x)$$

and the associated transforms $L_{j_r}s$ and $D_{j_r}s$.

Now the filters $\widehat{\psi_{j_r}}(\omega)$ are well localized on the two side bands $2^{j_r-1}\pi \leq |\omega| \leq 2^{j_r}\pi$ and their pics are $\omega_{j_r,pic} \approx 3.2^{j_r-1}\pi$.

Denoting:

$$S_{j_r} = Q_m(2^{j_r}.) * s$$

in analogous way we deduce that:

$$\begin{aligned} S_{j_r}(x) &= 2 \sum_{|k|\leq(m+1)/2} h_m[k]. \\ &. \quad S_{j_r+1}(x - 2^{-(j_r+1)}k) \end{aligned}$$

and for integer values q:

$$\begin{aligned} D_{j_r}s(q/R) &= 2^{j_r+1} \sum_{|p|\leq P} g_{m,n}[p]. \\ &. \quad S_{j_r+1}(q/R - p2^{-(j_r+1)}) \end{aligned}$$

$$\begin{aligned} L_{j_r}s(q/R) &= 2^{j_r} \sum_{|p|\leq P} b_{m,n}[p]. \\ &. \quad S_{j_r}(q/R - 2^{-(j_r)}p) \end{aligned}$$

At this point let us summarize:

Alghoritm:
Given the signal s
decide the parameters m, n *and* R
decide $J_{min} < 0$, $q_{min} \leq q \leq q_{max}$.
Then:

- *For* $j = 0$ *and* $0 \leq r < R$,

  Compute $S_{0_r}[q/R]$

- *For* $j = -1, -2, \dots, J_{min}$, *and* $0 \leq r < R$,

  Compute $L_{j_r+1}s[q/R]$

  Compute $D_{j_r}s[q/R]$

  Compute $S_{j_r}[q/R]$

To compute the initial values $S_{0_r}[q/R]$, we propose the model:

$$s(x) = R\sum_k s[k/R]\delta(x/R-k)$$

Then:

$$\begin{aligned} S_{0_r}(x) &= Q_m(\frac{R}{R+r}.) * s(x) \\ &= R\sum_k s[k/R](\delta(./R-k) * Q_m(\frac{R}{R+r}.) \end{aligned}$$

and finally:

$$S_{0_r}(x) = R\sum_k s[k/R]Q_m(\frac{q-k}{R+r})$$

The values $Q_m((q-k)/(R+r))$ can be easely computed usig the closed formulas for the Q_m splines.

We also remark that, according with our proposal, S_{j-r}, $L_{j_r}s$ and $D_{j_r}s$ are spline functions of order m with knots in $\mathcal{Z}/R$.

we can reconstruct the averaged representation of the signal:

$$\begin{aligned} L_{0_r}[q/R] &= \frac{1}{R}\sum_{r=0}^{R-1}\sum_{j_r=J_{min,r}}^{-1} D_{j_r}[q/R] + \\ &+ L_{J_{min,r}}[q/R] \end{aligned}$$

5. Example

For instance, let us consider the case $m = 3$ and $n = 2$. First, the explicit formulas for the cubic Q_3-spline, are:

$$Q_3(x) = \begin{cases} \frac{1}{2}|x|^3 - |x|^2 + \frac{2}{3} & |x| \leq 1 \\ -\frac{1}{6}|x|^3 + |x|^2 - 2|x| + \frac{4}{3} & 1 \leq |x| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

It is supported on $[-2, 2]$. The two scale coefficientes are:

$$h_3[0] = 6/8$$

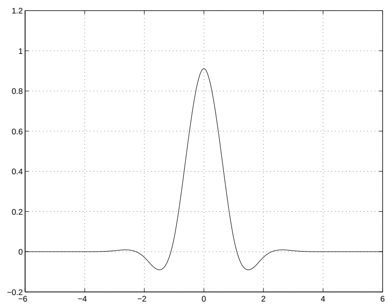

Figure 1: Function $\phi_{3,2}(x)$

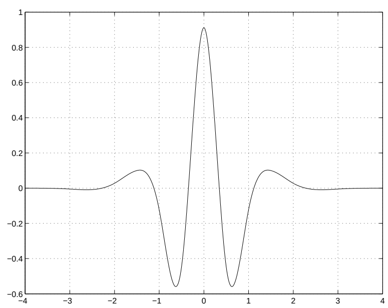

Figure 2: Wavelet $\psi_{3,2}(x)$

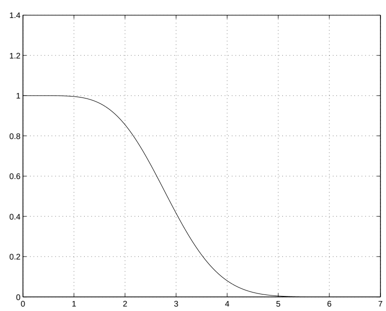

Figure 3: Fourier Transform $\widehat{\phi}_{3,2}(\omega)$

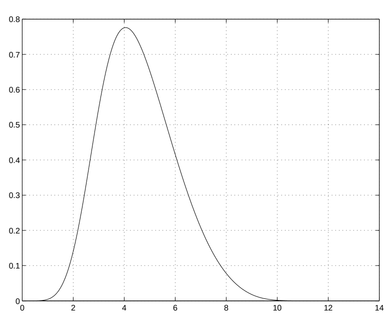

Figure 4: Fourier Transform $\widehat{\psi}_{3,2}(\omega)$

$$\begin{aligned} h_3[-1] &= h_3[1] = 1/2 \\ h_3[-2] &= h_3[2] = 1/8 \end{aligned}$$

In accordance we have the basic function:

$$\phi_{3,2}(x) = \sum_{|k|\le 2} b_{3,2}[k] Q_3(x-k)$$

where

$$\begin{aligned} b_{3,2}[0] &= 21/720 \\ b_{3,2}[-1] &= b_{3,2}[1] = -17/60 \\ b_{3,2}[-2] &= h_{3,2}[2] = 181/120 \end{aligned}$$

and

$$\psi_{3,2}(x) = \sum_{|k|\le 6} g_{3,2}[k] Q_3(2x-k)$$

where

$$\begin{aligned} g_{3,2}[0] &= 313/160 \\ g_{3,2}[-1] &= g_{3,2}[1] = -283/240 \\ g_{3,2}[-2] &= g_{3,2}[2] = 151/1920 \\ g_{3,2}[-3] &= g_{3,2}[3] = 61/480 \\ g_{3,2}[-4] &= g_{3,2}[4] = 13/960 \\ g_{3,2}[-5] &= g_{3,2}[5] = -7/480 \\ g_{3,2}[-6] &= g_{3,2}[6] = -7/1920 \end{aligned}$$

We illustrate these functions in the figures 1. and 2. The respective Fourier transforms are showed in the figures 3. and 4.

Finally, we give a simple application. Figure 5. shows the *chirp*:

$$s(x) = \cos(10\sqrt{|x|})$$

We choice $R = 7$ and compute the transform D_{j_r} for $j = -1, \ldots, -5$ and $r = 0, \ldots, 6$ on the interval $[0, 5000]$. See figure 5. As expected, the maxima modulus map give us a shape conciding with the instantaneous frequency og the chirp, [3].

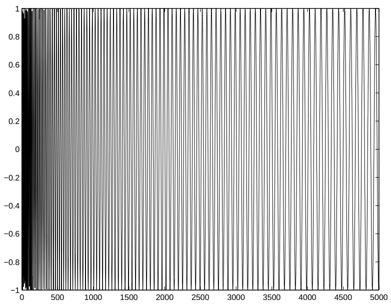

Figure 5: Function chirp $cos(10\sqrt{|x|})$

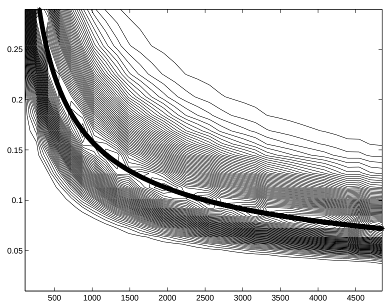

Figure 6: Wavelet transform and instantaneous frequencies (solid line)

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