Applications of the Dirac Sequences in Electrodynamics

WILHELM W. KECS
Department of Mathematics
University of Petrosani
Universitatii Street 20
332006 Petrosani
ROMANIA

ANTONELA TOMA
Department of Mathematics II
University Politehnica of Bucharest
Splaiul Independentei Street 313
060042 Bucharest
ROMANIA

MARIA DOBRIȚOIU
Department of Mathematics
University of Petrosani
Universitatii Street 20
332006 Petrosani
ROMANIA

Abstract: We established a method of constructing the Dirac sequences. On the basis of this method we construct some Dirac sequences with application in electrical engineering.

Key-Words: distributions theory, generalized functions, Dirac sequences

1 Introduction

The theory of distributions (generalized functions) represents a general and unitary background regarding the mathematical representation of some physical quantities and the analysis of some discontinuous phenomena.

The Dirac sequences have important applications in the representation of the physical quantities with punctual support as well as in solving of boundary value problems from mathematical-physics.

We mention that [4] and [7] have applied the distribution theory in the mechanics of the deformable solid and electrical engineering and also application of Dirac sequences have been given by [3], [1], [6], [5], [2] and other in the electrical engineering.

We shall consider some application of the Dirac sequences in electrical engineering and electrodynamics.

We shall denote by $D(\mathbb{R}^n)$ the Schwartz's space of indefinitely differentiable functions with compact support, and by $D'(\mathbb{R}^n)$ the set of linear continuous functionals defined on $D(\mathbb{R}^n)$, named as distributions.

2 General results

Definition 1. Let $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}, \varepsilon > 0$ be a family of locally integrable functions $\left(f_\varepsilon \in L_{loc}(\mathbb{R}^n)\right)$. We say that the functions f_ε form a representative Dirac family or "Dirac sequence" if in the sense of the convergence of $D'(\mathbb{R}^n)$ we have

$$\lim_{\varepsilon \to 0} f_\varepsilon(x) = \delta(x) . \tag{1}$$

This means that $\forall \varphi \in D(\mathbb{R}^n)$ we have

$$\lim_{\varepsilon \to 0} \left(f_\varepsilon(x), \varphi(x)\right) = \left(\delta(x), \varphi(x)\right) = \varphi(0) . \tag{2}$$

If $f_\varepsilon \in C^\infty(\mathbb{R}^n)$, then from (1) we obtain

$$\lim_{\varepsilon \to 0} D^\alpha f_\varepsilon(x) = D^\alpha \delta(x) , \tag{3}$$

where $D^\alpha f_\varepsilon(x) = \dfrac{\partial^\alpha f_\varepsilon(x_1,...,x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$ represents the partial derivative of order $|\alpha| = \alpha_1 + ... + \alpha_n$ of the function f_ε.

If the real function $\psi, x \in \mathbb{R}^n$ is continuously in the vicinity of the point $a \in \mathbb{R}^n$, then

$$\psi(x)\delta(x-a) = \psi(a)\delta(x-a) , \tag{4}$$

which represents the filtrant property of the Dirac distribution $\delta(x-a) \in D'(\mathbb{R}^n)$ concentrated at the point $a \in \mathbb{R}^n$. Particularly, we have $x\delta(x) = 0$.

Continuous functions with certain properties allow the construction of "the Dirac sequences". Thus, according to [4], p. 163 we can state

Proposition 1. Let $f \in C^0(\mathbb{R}^n), f : \mathbb{R}^n \to \mathbb{R}$ be with the property $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, the family of functions f_ε, $\varepsilon > 0$ having the expression

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n} f\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^n, \varepsilon > 0 , \tag{5}$$

form a "Dirac sequences", hence $\lim_{\varepsilon \to 0} f_\varepsilon(x) = \delta(x)$.

For $f > 0$, we shall give the following generalization of the proposition 1:

Proposition 2. Let $g_\varepsilon : \mathbb{R}^n \to \mathbb{R}, \varepsilon > 0, g_\varepsilon(x) = = f_\varepsilon(x - h(\varepsilon)) = \frac{1}{\varepsilon^n} f\left(\frac{x - h(\varepsilon)}{\varepsilon}\right)$, be a family of functions, where $h(\varepsilon)$ is a continuously function with the finite limit $\lim_{\varepsilon \to 0} h(\varepsilon) = a$ and $f > 0$. Then we have

$$\lim_{\varepsilon \to 0} g_\varepsilon(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} f\left(\frac{x - h(\varepsilon)}{\varepsilon}\right) = \delta(x - a). \quad (6)$$

Proof. For any $\varphi \in D(\mathbb{R}^n)$ we have

$$\left(f_\varepsilon\left(x - h(\varepsilon)\right), \varphi(x)\right) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} f\left(\frac{x - h(\varepsilon)}{\varepsilon}\right) \varphi(x) dx.$$

Making the change of variable $x = \varepsilon u + h(\varepsilon)$, the Jacobian of the transformation is

$$J(u) = \frac{\partial\left(x_1, ..., x_n\right)}{\partial\left(u_1, ..., u_n\right)} = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} = \varepsilon^n,$$

and thus we have

$$\begin{aligned}\left(f_\varepsilon\left(x - h(\varepsilon)\right), \varphi(x)\right) &= \int_{\mathbb{R}^n} f(u) \varphi\left(\varepsilon u + h(\varepsilon)\right) du = \\ &= \int_{\mathbb{R}^n} f(u)\left[\varphi\left(\varepsilon u + h(\varepsilon)\right) - \varphi(a)\right] du + \varphi(a),\end{aligned} \quad (7)$$

where $a = \lim_{\varepsilon \to 0} h(\varepsilon)$.

On the other hand since $\int_{\mathbb{R}^n} f(u) du = 1$ is finite we have

$$\left|\int_{\mathbb{R}^n} f(u)\left[\varphi\left(\varepsilon u + h(\varepsilon)\right) - \varphi(a)\right] du\right| \leq$$

$$\leq \sup_{\mathbb{R}^n}\left|\varphi\left(\varepsilon u + h(\varepsilon)\right) - \varphi(a)\right|,$$

wherefrom on the basis of continuity of the function $\varphi(\varepsilon u + h(\varepsilon)) \in D(\mathbb{R}^n)$ it yields

$$\lim_{\varepsilon \to 0}\left|\int_{\mathbb{R}^n} f(u)\left[\varphi\left(\varepsilon u + h(\varepsilon)\right) - \varphi(a)\right] du\right| \leq 0,$$

hence

$$\lim_{\varepsilon \to 0}\left|\int_{\mathbb{R}^n} f(u)\left[\varphi\left(\varepsilon u + h(\varepsilon)\right) - \varphi(a)\right] du\right| = 0 \cdot$$

Consequently, from (7) we have

$$\lim_{\varepsilon \to 0}\left(f_\varepsilon\left(x - h(\varepsilon)\right), \varphi(x)\right) = \varphi(a) = \left(\delta(x - a), \varphi(x)\right),$$

hence $\lim_{\varepsilon \to 0} f_\varepsilon\left(x - h(\varepsilon)\right) = \delta(x - a)$ .Q.E.D.

Example 1. Let be the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2(x^2 + 1)^{3/2}}$. We note that $f \in C^0(\mathbb{R})$, $f(-x) = f(x)$ and we have

$$\int_{\mathbb{R}} f(x) dx = 2\int_0^\infty f(x) dx = \int_0^\infty \frac{dx}{(x^2 + 1)^{3/2}}.$$

Making the substitution $x = \sinh u, u \in \mathbb{R}$ we obtain

$$\int_{\mathbb{R}} f(x) dx = \int_0^\infty \frac{du}{\cosh^2 u} = \tanh u\Big|_0^\infty = 1.$$

Since the conditions from the proposition 1 are fulfilled we obtain the Dirac sequences

$$f_\varepsilon(x) = \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right) = \frac{\varepsilon^2}{2(x^2 + \varepsilon^2)^{3/2}}, \varepsilon > 0, x \in \mathbb{R},$$

and thus we can write

$$\lim_{\varepsilon \to 0} f_\varepsilon(x) = \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{2(x^2 + \varepsilon^2)^{3/2}} = \delta(x). \quad (8)$$

We consider the function $h(\varepsilon) = vt = ct(1 - \varepsilon^2)^{1/2}$, $0 \leq \varepsilon \leq 1$ where $c > 0, t \geq 0$ are constants.

We have $\lim_{\varepsilon \to 0} h(\varepsilon) = ct$ and $h \in C^0\left([0,1]\right)$.

On the basis of the proposition 2 it yields

$$\lim_{\varepsilon \to 0} f_\varepsilon(x - vt) = \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{2\left[(x - vt)^2 + \varepsilon^2\right]^{3/2}} = \delta(x - ct).$$

We remark that the family of Dirac sequences

$$f_\varepsilon(x - vt) = \frac{\varepsilon^2}{2\left[(x - vt)^2 + \varepsilon^2\right]^{3/2}}, \varepsilon > 0, x \in \mathbb{R}, \quad (9)$$

is used in electrodynamics [2] to the study of the electric and magnetic fields generated by a point charge q of mass m moving with constant velocity v along the x axis, if the particle is at the origin (0,0,0) at t=0.

Example 2. Let $g : \mathbb{R} \to \mathbb{R}$ be a function, where

$$g(x) = \frac{1}{2\ln a}\left(\frac{1}{\sqrt{x^2 + 1}} - \frac{1}{\sqrt{x^2 + a^2}}\right), a > 0, a \neq 1. \quad (10)$$

We note that $g \in C^0(\mathbb{R}), g(-x) = g(x)$ and thus we have

$$\int_{\mathbb{R}} g(x) dx = 2\int_0^\infty g(x) dx = \frac{1}{\ln a}\int_0^\infty\left(\frac{1}{\sqrt{x^2 + 1}} - \frac{1}{\sqrt{x^2 + a^2}}\right) dx.$$

Since $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln \frac{x+\sqrt{x^2+a^2}}{a} + C$ we obtain $\int_{\mathbb{R}} g(x)dx = \frac{1}{\ln a} \ln a \frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+a^2}}\Bigg|_0^\infty = 1$.

Consequently, the function g satisfies the conditions of the proposition 1, hence the family of functions

$$g_\varepsilon(x) = \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}\right) = \frac{1}{2\ln a}\left(\frac{1}{\sqrt{x^2+\varepsilon^2}} - \frac{1}{\sqrt{x^2+a^2\varepsilon^2}}\right),$$

represents a Dirac sequences. Thus, we can write

$$\lim_{\varepsilon\to 0} g_\varepsilon(x) = \frac{1}{2\ln a}\lim_{\varepsilon\to 0}\left(\frac{1}{\sqrt{x^2+\varepsilon^2}} - \frac{1}{\sqrt{x^2+a^2\varepsilon^2}}\right) = \delta(x).$$

Appling the proposition 2 we obtain

$$\lim_{\varepsilon\to 0} g_\varepsilon(x-vt) = \frac{1}{2\ln a}\lim_{\varepsilon\to 0}\left(\frac{1}{\sqrt{(x-vt)^2+\varepsilon^2}} - \frac{1}{\sqrt{(x-vt)^2+a^2\varepsilon^2}}\right) = \delta(x-ct).$$

We remark that this result was used in [1] regarding the study of the gravitational field of a massless particle in general relativity, when $v \to c$.

Example 3. We consider the function $f:\mathbb{R}\to\mathbb{R}$, where $f(x) = \frac{1}{2}\frac{1}{\cosh^2 x}$.

Obviously, $f \in C^0(\mathbb{R})$ and $f(-x) = f(x)$. We have $\int_{\mathbb{R}} f(x)dx = 2\int_0^\infty f(x)dx = \int_0^\infty \frac{dx}{\cosh^2 x} = \tanh x\big|_0^\infty = 1$.

Since the conditions of the proposition 1 are satisfied it results that the family of functions

$$f_\varepsilon(x) = \frac{1}{2\varepsilon}\frac{1}{\cosh^2 \frac{x}{\varepsilon}}, x\in\mathbb{R}, \varepsilon > 0,$$

represents a Dirac sequence. Consequently, we can write the relations

$$\lim_{\varepsilon\to 0} f_\varepsilon(x) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\frac{1}{\cosh^2\frac{x}{\varepsilon}} = \delta(x), \quad (11)$$

$$\lim_{\varepsilon\to 0} f_\varepsilon(x-ct) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\frac{1}{\cosh^2\frac{x-ct}{\varepsilon}} = \delta(x-ct). \quad (12)$$

The study of electromagnetic wave pulses [3] uses the formula (11) and (12) substituting $\varepsilon = \frac{1}{n}, n\in\mathbb{N}_0$.

Thus, we have

$$\lim_{n\to\infty}\frac{n}{2}\frac{1}{\cosh^2 nx} = \delta(x),$$

$$\lim_{n\to\infty}\frac{n}{2}\frac{1}{\cosh^2 n(x-ct)} = \delta(x-ct).$$

We remark that the Dirac sequence $h_n(x) = \frac{n}{2}\frac{1}{\cosh^2 nx}, n\in\mathbb{N}_0$ is obtained from the functions sequence $H_n(x) = \frac{1}{2}\left(1+\tanh nx\right), n\in\mathbb{N}_0$ by differentiation. Indeed, we have $H_n'(x) = \frac{n}{2}\frac{1}{\cosh^2 nx} = h_n(x)$.

3 Applications in electrodynamics

Let q be a point electrical charge in vacuum placed at the origin of the orthogonal reference $Oxyz$. If at the point $M \neq O$ is placed an electrical charge +1, then according to Coulomb law, the force which acts on the charge +1 is

$$E = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2},\ \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}, r = \overline{OM} = \sqrt{x^2+y^2+z^2}, \quad (13)$$

where ε_0 represents the dielectric constant of vacuum.

By definition, the vectorial function $\vec{E}:\mathbb{R}^3\setminus\{0\}\to\mathbb{R}^3$ given by the formula (13) is named the intensity of the electrostatic field corresponding to point electrical charge q placed at the origin of the reference $Oxyz$.

We note that the vectorial function $\vec{E}$ is a locally integrable function, since $\forall\Omega\subset\mathbb{R}^3$, the integral $\int_\Omega \frac{dv}{r^2}, dv = dxdydz$ exists and it is finite.

Consequently, the intensity $\vec{E}$ of the electrical field generates a function type distribution from $D'(\mathbb{R}^3)$.

By definition, the function $V:\mathbb{R}^3\setminus\{0\}\to\mathbb{R}$ given by the formula

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}, r = \sqrt{x^2+y^2+z^2}, \quad (14)$$

is named the electrostatic field potential corresponding to the charge q placed at the origin.

The potential function V, is also a locally integrable function which determines a function type distribution from $D'(\mathbb{R}^3)$.

Between the intensity $\vec{E}$ of the electrostatic field and the potential V we have the relation

$$\vec{E} = -\operatorname{grad} V . \qquad (15)$$

Generally, if we denote by ρ the volume density of the electrical charge, represented by a distribution from $D'(\mathbb{R}^3)$ and with V the field potential, then the quantities $\vec{E}, V, \rho$ considered distributions from $D'(\mathbb{R}^3)$ satisfy the Poisson equation

$$\Delta V = -\frac{\rho}{\varepsilon_0}, \qquad (16)$$

hence $\operatorname{div}\vec{E} = -\operatorname{div}(\operatorname{grad} V) = -\Delta V = \frac{\rho}{\varepsilon_0}$.

The Eqs. (15) and (16) considered in the distributions space constitute the fundamental equations of the electrostatics.

We remark that the Eqs. (15) and (16) can be particularly applied on $\mathbb{R}$ or $\mathbb{R}^2$. In these cases ρ represents the linear and the surface density of the electrical charge, respectively.

We will study the case when the electrical charge q is moving linear with constant velocity v.

Let q be a point electrical charge of mass m moving with constant velocity v along the x axis. We admit that the electrical charge q at $t = 0$ is at the origin $O \in \mathbb{R}^3$. Then [2], the electric field $\vec{E}$ and the magnetic field $\vec{B}$ at any point $M(x, y, z)$ and time $t > 0$ have the expressions

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\gamma q}{r^3}\left[(x - vt)\vec{i} + y\vec{j} + z\vec{k}\right], \qquad (17)$$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{\gamma q v}{r^3}\left[-z\vec{j} + y\vec{k}\right], \qquad (18)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, r^2 = \gamma^2(x - vt)^2 + y^2 + z^2, \qquad (19)$$

and μ_0 represents the magnetic permeability of vacuum.

From (18) and (19) we obtain

$$\vec{B} = (\mu_0\varepsilon_0)v\vec{i} \times \vec{E} . \qquad (20)$$

From above it results that the magnetic field $\vec{B}$ is perpendicular on the $\vec{i}$ direction of the axis x and also on the electric field $\vec{E}$.

A very important problem is the behaviour of the electric and magnetic fields when $v \to c$, c being the speed of light in vacuum.

The limit case $v \to c$ is equivalent with $\gamma \to \infty$. From the expressions of the fields $\vec{E}$ and $\vec{B}$ it results that $\lim\limits_{\gamma\to\infty}\vec{E}$ and $\lim\limits_{\gamma\to\infty}\vec{B}$ don't exist in the classical sense.

We shall show that these limits exist in the generalized sense that means in the distributions space.

Since the electrical charge q is moving along the Ox-axis with constant velocity v, the coordinates y and z being constants, the quantities $\vec{E}$ and $\vec{B}$ are vectorial distributions with components from $D'(\mathbb{R})$ with respect to the variable x.

From (17) and (18) it results that

$$\vec{E}_1 = \lim_{v\to c}\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[(x - ct)\vec{i} + y\vec{j} + z\vec{k}\right]\lim_{v\to c}\frac{\gamma}{r^3}, \quad (21)$$

$$\vec{B}_1 = \lim_{v\to c}\vec{B} = \frac{q\mu_0 c}{4\pi}\left[-z\vec{j} + y\vec{k}\right]\lim_{v\to c}\frac{\gamma}{r^3}. \qquad (22)$$

Proposition 3. Denoting $\rho^2 = y^2 + z^2 > 0$ we have

$$\lim_{v\to c}\frac{\gamma}{r^3} = \lim_{\gamma\to\infty}\frac{\gamma}{r^3} = \frac{2}{\rho^2}\delta(x - ct). \qquad (23)$$

Proof. Taking into account (19) we have

$$\frac{\gamma}{r^3} = \frac{\gamma}{\left[\gamma^2(x-vt)^2+\rho^2\right]^{3/2}} = \frac{1}{\gamma^2\left[(x-vt)^2+\frac{\rho^2}{\gamma^2}\right]^{3/2}}. \quad (24)$$

Denoting $\frac{\rho^2}{\gamma^2} = \varepsilon^2$ and since $\gamma \to +\infty$ it results $\varepsilon \to 0$.

Thus, the expression (24) becomes

$$\frac{\gamma}{r^3} = \frac{1}{\rho^2}\frac{\varepsilon^2}{\left[(x - vt)^2 + \varepsilon^2\right]^{3/2}}, \varepsilon > 0 .$$

Consequently, on the basis of the formula (10) we have

$$\lim_{v\to c}\frac{\gamma}{r^3} = \lim_{\gamma\to\infty}\frac{\gamma}{r^3} = \frac{1}{\rho^2}\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{\left[(x - vt)^2 + \varepsilon^2\right]^{3/2}} =$$

$$= \frac{2}{\rho^2}\delta(x - ct).$$

Q.E.D.

Applying this result and taking into account (21) and (22) we obtain

$$\vec{E}_1 = \lim_{v\to c}\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[(x - ct)\vec{i} + y\vec{j} + z\vec{k}\right]\frac{2}{\rho^2}\delta(x - ct),$$

$$\vec{B}_1 = \lim_{v\to c}\vec{B} = \frac{q\mu_0 c}{4\pi}\left[-z\vec{j} + y\vec{k}\right]\frac{2}{\rho^2}\delta(x - ct),$$

where $\rho^2 = y^2 + z^2$.

Since $u\delta(u) = 0$, it results that $(x-ct)\delta(x-ct) = 0$ and thus we can write

$$\vec{E}_1 = \lim_{v\to c} \vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{q}{\rho^2}\delta(x-ct)(y\vec{j} + z\vec{k}),$$

$$\vec{B}_1 = \lim_{v\to c} \vec{B} = \frac{q\mu_0 c}{2\pi}\frac{1}{\rho^2}\delta(x-ct)\left[-z\vec{j} + y\vec{k}\right].$$

Between the fields $\vec{E}_1$ and $\vec{B}_1$ we have the relation

$$\vec{B}_1 = \mu_0\varepsilon_0 c\vec{i} \times \vec{E}_1,$$

which shows that the magnetic field $\vec{B}_1$ is perpendicular on *Ox*-axis direction as well as on the electric field $\vec{E}_1$.

Taking into account the results shown in [3], the Dirac sequence $h_n(x-ct) = \frac{n}{2}\frac{1}{\cosh^2(x-ct)}$ is used for studying the electromagnetic wave pulses.

Thus, the electric fields of three pulses are given by the expressions

$$E_1(x,t) = Ah_n(x-ct),$$
$$E_2(x,t) = Bh_n'(x-ct),$$
$$E_3(x,t) = Ch_n''(x-ct).$$

The energy density of a plane electromagnetic wave in vacuum is given by $W = \varepsilon_0 E^2$ which obviously is written using the generalized functions $h_n(x-ct), h_n'(x-ct), h_n''(x-ct)$.

4 Conclusions

The distributions theory constitutes the adequate mathematical tool in the representation of the physical quantities and in the study of some phenomena with discontinuities.

The Dirac sequences allow the study of some physical quantities such as electric and magnetic fields, when the charges have the velocity which tends to the light speed.

These limit cases when $v \to c$ don't exist in the classical manner, but in the generalized manner of theory of distributions.

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