Calculus Rules regarding the Dimensional Equations of some Electrodynamics Quantities Represented in Distributions Space

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Abstract: Calculus rules regarding the dimensional equations for some electrodynamics quantities represented in distributions are presented. The dimensional equations of the following quantities: electric potentials of a simple and a double layer, electric and magnetic fields in electrodynamics when the velocity of the charge tends to the speed light, are analyzed.

Key-Words: distributions theory, dimensional analysis, electric engineering

1 Introduction

The study of the most physical phenomena is mainly made by mathematical modelling, which leads to the establishing of the evolution equations of the respective phenomenon.

The distributions theory constitutes the adequate mathematical tool in the representation of the physical quantities and in the study of some phenomena with discontinuities, such phenomena we will find in electrodynamics.

The correctness of the mathematical modelling of some phenomena involves that the physical equations which describe the phenomena must satisfy the principle of homogeneity.

Consequently, an equality of the form $A = B$ between physical quantities involves the dimensional equality $[A] = [B]$ of the quantities A and B (both sides of the equation must have the same dimension).

In mechanics the fundamental quantities of measure in International System are the length L, the mass M and time T, therefore the dimensional equation of the mechanical quantities A is $[A] = L^{\alpha} M^{\beta} T^{\gamma}$.

In electro-magnetism 4 fundamental quantities of measure are considered, namely length, mass, time and the intensity of the electric current.

Consequently, an electro-magnetic quantity B will have the dimensional equation $[B] = L^{\alpha} M^{\beta} T^{\gamma} I^{\delta}$, where I represents the intensity of electric current.

The fundamental units of measure in IS for length, time, mass and intensity of the electric current are the meter (m), the second (s), the kilogram (kg) and ampere (A), respectively.

The ordained system of real numbers $(\alpha, \beta, \delta, \gamma) \in \mathbb{R}^4$ represents the dimension of the electro-magnetic quantity B. In the case when an exponent from the dimensional equation is zero, i.e. $\beta = 0$, then we shall write $[B] = L^{\alpha} T^{\gamma} I^{\delta}$ instead of $[B] = L^{\alpha} M^{0} T^{\gamma} I^{\delta}$.

Let $u = (u_1, u_2, ..., u_n)$ be a point from $\mathbb{R}^n$, then we shall denote $[u]^p = [u_1]^p [u_2]^p \cdots [u_n]^p$, $p \in \mathbb{Z}$.

We say that a quantity C is dimensionless if it has the dimension zero and we shall write $[C] = L^0 M^0 T^0 I^0 = 1$.

In the following we shall admit that the electro-magnetic quantities and also the considered equations are written in the distributions space $D'(\mathbb{R}^n)$, where $D(\mathbb{R}^n)$ is the Schwartz's space of functions of class $C^{\infty}(\mathbb{R}^n)$ and with compact support.

We shall establish the dimensional equations of some electro-magnetic quantities represented with the help of the derivative and the convolution product like as, the dimensional equations of the Dirac's distribution $\delta(x) \in D'(\mathbb{R}^n)$ concentrated at the origin ($\operatorname{supp} \delta(x) = \{0\}$) and the Dirac's distribution $\delta_S \in D'(\mathbb{R}^n)$ concentrated on the bounded hypersurface $S \subset \mathbb{R}^n, n \geq 3, \operatorname{supp} \delta_S = S$.

Calculus rules regarding the dimensional equations of some physical quantities represented in distributions have been used in [5] and [4].

We determine the dimensional equations of the following quantities: electric potentials of a simple and a double layer, electric and magnetic fields in electrodynamics when the velocity of the charge tends to the speed light.

2 General results

According to [5] we have the following results:

Proposition 1. If the physical quantity is represented by the distribution $f \in D'(\mathbb{R}^n)$ with respect to variable $u = (u_1,...,u_n) \in \mathbb{R}^n$, then the quantity represented by the distribution $\frac{\partial f}{\partial u_i} \in D'(\mathbb{R}^n)$ has the dimensional equation

$$\left[\frac{\partial f(u)}{\partial u_i}\right] = [u_i]^{-1}[f(u)] = \left[\frac{\partial}{\partial u_i}\right][f(u)], \quad (1)$$

where $[u_i]$, $[f(u)]$ represent the dimensional equations of the variable u_i and the distribution f, respectively.

From (1) results that the dimensional equation of the operator $\frac{\partial}{\partial u_i}$ is $\left[\frac{\partial}{\partial u_i}\right] = [u_i]^{-1}$.

To generalize the formula (1) we shall consider the derivative operator of the order $|\alpha| = \alpha_1 + ... + \alpha_n$, namely $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n}$ where $D_i = \frac{\partial}{\partial u_i}$, $i = \overline{1,n}$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$, $\mathbb{N}_0 = \{0,1,2,...,n,...\}$.

For any distribution $f(u) \in D'(\mathbb{R}^n)$ we can write $D^{\alpha+\beta} = D^\alpha\left(D^\beta f\right) = D^\beta\left(D^\alpha f\right)$, $\alpha, \beta \in \mathbb{N}_0^n$.

Thus we have:

Proposition 2. If a quantity is represented by the distribution $f \in D'(\mathbb{R}^n)$ with respect to variable $u = (u_1,...,u_n) \in \mathbb{R}^n$, then the dimensional equation of the quantity $D^\alpha f \in D'(\mathbb{R}^n)$ is

$$\left[D^\alpha f(u)\right] = \left[D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n} f(u)\right] =$$
$$= [D_1]^{\alpha_1} ... [D_n]^{\alpha_n} [f(u)] = [u_1]^{-\alpha_1} ... [u_n]^{-\alpha_n} [f(u)],$$

where $D_i = \frac{\partial}{\partial u_i}$.

For the dimensional equation of the Dirac distribution, which has important application in physics, we have:

Proposition 3. The dimensional equation of the Dirac's distribution $\delta \in D'(\mathbb{R}^n)$ with respect to the variable $u = (u_1,...,u_n) \in \mathbb{R}^n$ has the expression $\left[\delta(u_1,u_2,...,u_n)\right] = [u_1]^{-1}[u_2]^{-1}...[u_n]^{-1} = [u]^{-1}$.

We mention that the Dirac's distribution $\delta(u) \in D'(\mathbb{R}^n)$ is defined by the formula

$$\left(\delta(u), \varphi(u)\right) = \varphi(0), \quad \varphi \in D(\mathbb{R}^n).$$

Example 1. We consider the Dirac's distribution $\delta(u_1, u_2, ..., u_{n-1}, t) \in D'(\mathbb{R}^n)$, where the variables u_i, $i = \overline{1, n-1}$ have the length dimension, and the variable $t \in \mathbb{R}$ has the time dimension, hence $[t] = T$. In this case the Dirac's distribution has the dimensional equation

$$\left[\delta(u_1, u_2, ..., u_{n-1}, t)\right] = L^{-n+1} T^{-1}$$

Also, according to [2] the four-dimensional delta distribution in Minkowski space is defined as

$$\delta(X) = \frac{i}{c}\delta(x,t)$$

where $X = (x; ict), x \in \mathbb{R}^3, i^2 = -1$, t the temporal variable and c the speed of light.

It results that

$$\left[\delta(X)\right] = \frac{1}{[c]}\left[\delta(x,t)\right] = \frac{T}{L} L^{-3} T^{-1} = L^{-4}.$$

The Dirac's distribution $\delta_S \in D'(\mathbb{R}^n)$ concentrated on the hypersurface $S \subset \mathbb{R}^n$ piecewise smooth has a great importance to solve the mathematical physics problems.

For a real function f piecewise continuous defined on S, it is defined the distribution $f(x)\delta_S \in D'(\mathbb{R}^n)$ by the formula

$$\left(f(x)\delta_S, \varphi(x)\right) = \int_S f(x)\varphi(x) dS, \ \varphi \in D(\mathbb{R}^n).$$

If $S \subset \mathbb{R}^n$ is a bounded and piecewise smooth hypersurface, and g and f are piecewise continuous functions on $\mathbb{R}^n$ and S, respectively, then in accordance with [4] p.366 we have the formula

$$g * f\delta_S = \int_S f(y) g(x-y) dS(y),$$

where the symbol "$*$" represents the convolution product.

In this case the distribution $f\delta_S \in D'(\mathbb{R}^n)$ is with compact support and $\mathrm{supp}(f\delta_S) \subset S$, because for $x \notin S$, $f\delta_S = 0$.

Let $f, g \in L^1_{loc}(\mathbb{R}^n)$ be locally integrable functions with the propriety that one of them has compact support. Then the convolution product between them exists and it is defined by the formula

$$(f * g)(x) = \int_{\mathbb{R}^n} f(t)g(x-t)dt = \int_{\Omega} f(t)g(x-t)dt, \quad (2)$$

where $\mathrm{supp}\, f = \Omega \subset \mathbb{R}^n$ is a compact set.

Taking into account the definition of the multiple integral and the volume element dt has the expression $dt = dt_1 ... dt_n$, it results that the dimensional equation of a physical quantity given by (2) is

$$\begin{aligned}\left[(f * g)(x)\right] &= [f(x)][g(x)][x_1]...[x_n] = \\ &= [f(x)][g(x)][x].\end{aligned} \quad (3)$$

Since the convolution product for the distributions constitutes a generalization of the formula (2), the dimensional equation (3) can be used for the distributions.

Definition 1. Let be the distributions $f(x) \in D'(\mathbb{R}^n)$ and $g(x) \in D'(\mathbb{R}^n)$. If the convolution product $f * g \in D'(\mathbb{R}^n)$ exists, then its dimensional equation has the expression

$$\begin{aligned}\left[(f * g)(x)\right] &= [f(x)][g(x)][x_1]...[x_n] = \\ &= [f(x)][g(x)][x].\end{aligned} \quad (4)$$

Let $S \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded and smooth hypersurface, defined under explicit form by the equation

$$x_n = \psi(x_1, ..., x_{n-1}), \quad (x_1, x_2, ..., x_{n-1}) \in \Omega \subset \mathbb{R}^{n-1},$$

where $\psi \in C^1(\Omega)$.

With the notations $p_i = \dfrac{\partial \psi}{\partial x_i}$, $i = \overline{1, n-1}$, [1], the normal vector to the surface S is $\vec{N} = (p_1, p_2, ..., p_{n-1}, -1)$, and the surface element dS has the expression

$$dS = \sqrt{1 + p_1^2 + .. + p_{n-1}^2}\, dx_1 dx_2 ... dx_{n-1}. \quad (5)$$

Since the quantities p_i are dimensionless, from (5) results that dS has the dimension

$$[dS] = [x_1][x_2]...[x_{n-1}] = L^{n-1}.$$

Concerning the dimensional equation of $\delta_S \in D'(\mathbb{R}^n), n \geq 3$ we have

Proposition 4. Let $S \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded and piecewise smooth hypersurface. Then the dimensional equation of the Dirac's distribution δ_S concentrated on the surface S has the expression

$$[\delta_S] = L^{-1}.$$

3 Applications in electrical engineering

3.1 The dimensional equations for the electric potentials of a simple and a double layer

Let $\rho \in D'(\mathbb{R}^3)$ be the volume density of an electric charge q discrete or continuous distributed in space.

By definition the distribution

$$V_3(x, y, z) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r} * \rho \in D'(\mathbb{R}^3), \; r = \sqrt{x^2 + y^2 + z^2}, \quad (6)$$

is named the volume potential corresponding to the electric charge with the density ρ. In (6) ε_0 represents the dielectric constant of vacuum, farads per meter, whose dimensional equation in SI is

$$[\varepsilon_0] = L^{-3}M^{-1}T^4I^2.$$

We observe that the formula (6) exists if ρ has compact support. It is shown immediately that the volume potential V_3 verifies the Poisson's equation

$$\Delta V_3 = -\frac{\rho}{\varepsilon_0}, \; \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

due to the fact that $\Delta \dfrac{1}{r} = -4\pi\delta(x, y, z)$.

Applying the dimensional formula (4) we have

$$[V_3] = \frac{1}{[\varepsilon_0]} \cdot \frac{1}{[r]}[x][y][z][\rho] = L^2MT^{-3}I^{-1}, \quad (7)$$

where $[\rho] = \dfrac{[q]}{L^3} = L^{-3}TI$.

Let $\sigma = \dfrac{dq}{dS}$ be a piecewise continuous function on S expressing the surface electric density of the layer.

The electric volume density $\rho_0 \in D'(\mathbb{R}^3)$ of the charges distributed on the surface S has the expression

$$\rho_0(x, y, z) = \sigma(x, y, z)\delta_S.$$

The fact that this is a volume electric density results from its dimensional equation

$$[\rho_0]=[\sigma][\delta_S]=\frac{[q]}{L^2}L^{-1}=\frac{TI}{L^3}=L^{-3}TI.$$

Taking into account (6), we define the electric potential corresponding to an electric simple layer with the density σ, the distribution $V_3^{(0)}\in D'(\mathbb{R}^3)$, as

$$V_3^{(0)}(x,y,z)=\frac{1}{4\pi\varepsilon_0}\frac{1}{r}*\sigma\delta_S,\ r=\sqrt{x^2+y^2+z^2}. \quad (8)$$

Its dimensional equation is

$$\left[V_3^{(0)}\right]=\frac{1}{[\varepsilon_0]}\frac{1}{[r]}[\sigma][\delta_S][x][y][z]= \\ =L^2M^1T^{-3}I^{-1}=[V_3], \quad (9)$$

where $[\sigma]=\frac{[q]}{L^2}=L^{-2}TI$.

Consequently, $V_3^{(0)}$ has identical dimension with the volume potential V_3.

For the electric double layer, the volume density $\rho_1\in D'(\mathbb{R}^3)$ has the expression

$$\rho_1(x,y,z)=-\frac{\partial}{\partial n}\left(\mu(x,y,z)\delta_S\right), \quad (10)$$

where μ represents the moment of the electric double layer and it is defined by a piecewise continuous function on *S*, and $\frac{\partial}{\partial n}$ represents the derivative along the normal to the surface $S\subset\mathbb{R}^3$.

Using (6), the electric potential corresponding to an electric double layer has the expression

$$V_3^{(1)}(x,y,z)=-\frac{1}{4\pi\varepsilon_0}\frac{1}{r}*\frac{\partial}{\partial n}\left(\mu\delta_S\right). \quad (11)$$

We observe that the convolution product exists because the distribution $\mu\delta_S\in D'(\mathbb{R}^3)$ has compact support, $\operatorname{supp}\mu\delta_S\subset S$.

From (11) results that the dimensional equation of the double layer potential is

$$\left[V_3^{(1)}\right]=\frac{1}{[\varepsilon_0]}\frac{1}{[r]}\left[\frac{\partial}{\partial n}\right][\mu][\delta_S]L^3=L^2MT^{-3}I^{-1},$$

where

$$\left[\frac{\partial}{\partial n}\right]=[\vec{n}][\nabla]=[\nabla]=L^{-1},$$

$$[\mu]=[\sigma]L=\frac{[q]}{L^2}L=\frac{[q]}{L}=L^{-1}TI.$$

Hence $V_3^{(1)}$ has the same dimension as V_3, therefore $[V_3]=\left[V_3^{(0)}\right]=\left[V_3^{(1)}\right]=L^2MT^{-3}I^{-1}$.

Like in the case of the simple layer it is verified that the distribution ρ_1 given by (10) represents a volume density because we have

$$[\rho_1]=\left[\frac{\partial}{\partial n}\right][\mu][\delta_S]=L^{-1}\frac{[q]}{L}L^{-1}=\frac{[q]}{L^3}.$$

3.2 Applications in electrodynamics

Let q be a point electrical charge of mass m moving with constant velocity v along the x axis. We admit that the electrical charge q at $t=0$ is at the origin $O\in\mathbb{R}^3$. Then [1], the electric field $\vec{E}$ and the magnetic field $\vec{B}$ at any point $M(x,y,z)$ and time $t>0$ has the expressions

$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\frac{\gamma q}{r^3}\left[(x-vt)\vec{i}+y\vec{j}+z\vec{k}\right], \quad (12)$$

$$\vec{B}=\frac{\mu_0}{4\pi}\frac{\gamma q v}{r^3}\left[-z\vec{j}+y\vec{k}\right], \quad (13)$$

where

$$\gamma=\left(1-\frac{v^2}{c^2}\right)^{-1/2},r^2=\gamma^2(x-vt)^2+y^2+z^2, \quad (14)$$

and μ_0 represents the magnetic permeability of vacuum.

We remark that the quantities $\vec{E}$ and $\vec{B}$ are vectorial distributions with components from $D'(\mathbb{R})$ with respect to the variable *x*.

A very important case is the study of the electric and magnetic fields when $v\to c$, *c* being the speed of light in vacuum.

The limit case $v\to c$ is equivalent with $\gamma\to\infty$.

From the expressions (12), (13) of the fields $\vec{E}$ and $\vec{B}$ we have

$$\vec{E}_1=\lim_{v\to c}\vec{E}=\frac{1}{2\pi\varepsilon_0}\frac{q}{\rho^2}\delta(x-ct)(y\vec{j}+z\vec{k}),$$

$$\vec{B}_1=\lim_{v\to c}\vec{B}=\frac{q\mu_0 c}{2\pi}\frac{1}{\rho^2}\delta(x-ct)\left[-z\vec{j}+y\vec{k}\right].$$

We shall show that the pairs of fields $(\vec{E},\vec{B})$ and $(\vec{E}_1,\vec{B}_1)$ represented in the distributions space $D'(\mathbb{R})$ have the same dimensional equation, hence $[\vec{E}]=[\vec{E}_1]$ and $[\vec{B}]=[\vec{B}_1]$.

Indeed, we have

$$\left[\vec{E}\right] = \frac{1}{[\varepsilon_0]}\frac{[\gamma][q]}{[r^3]}L = \frac{1}{[\varepsilon_0]}\frac{[\gamma]}{L^3}[q]L = \frac{1}{[\varepsilon_0]}[\gamma]\frac{[q]}{L^2}.$$

Taking into account the definition of γ, we observe that this quantity is dimensionless, hence $[E] = \frac{1}{[\varepsilon_0]}\frac{[q]}{L^2}$.

Similarly, we have

$$\left[\vec{B}\right] = [\mu_0][q]L^{-1}T^{-1}.$$

On the other hand from the definition of $\vec{E}_1$ and $\vec{B}_1$ we obtain the following dimensional equations

$$\left[\vec{E}_1\right] = \frac{1}{[\varepsilon_0]}\frac{[q]}{L^2}[\delta(x-ct)]L =$$

$$= \frac{1}{[\varepsilon_0]}\frac{[q]}{L^2}\frac{1}{L}L = \frac{1}{[\varepsilon_0]}\frac{[q]}{L^2} = \left[\vec{E}\right],$$

$$\left[\vec{B}_1\right] = [\mu_0]\frac{[q]}{L^2}LT^{-1}[\delta(x-ct)]L =$$

$$= [\mu_0][q]L^{-1}T^{-1},$$

where we take into account that $[\delta(x-ct)] = L^{-1}$.

4 Conclusions

The importance of the calculus rules regarding the dimensional equations for some electrodynamics quantities represented in distributions is presented.

The dimensional equation of the Dirac distribution is used to obtain the dimensional equations for the electric field $\vec{E}$ and for the magnetic field $\vec{B}$ in the distributions space, in the limit case when $v \to c$.

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