The Differentiability of the Solution of a Nonlinear Integral Equation

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Abstract: Using Perov’s theorem, a fiber generalized contractions theorem and its application given by Rus [7], a theorem of differentiability of the solution of the following nonlinear integral equation

\[ x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t \in [\alpha, \beta], \]

is given.

Key-Words: nonlinear integral equation, data dependence

1 Introduction
We consider the nonlinear integral equation

\[ x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t \in [\alpha, \beta], \]

where \( \alpha, \beta \in \mathbb{R}, \alpha \leq \beta, a, b \in [\alpha, \beta] \) and \( x \in C([\alpha, \beta], \mathbb{R}^m) \).

In the paper [2] have been studied the existence and uniqueness of the solution, continuous data dependence of the solution and the approximation of the solution of this nonlinear integral equation.

The integral equations of this type have been studied in [1], [3], [4], [5], [6], [7], [8], [9].

The purpose of this paper is to give a theorem of differentiability of the solution of the integral equation (1).

2 Notations and preliminaries
Let \( X \) be a nonempty set, \( d \) a metric on \( X \) and \( A:X \rightarrow X \) an operator. In this paper we shall use the following notations:

\[ F_A := \{ x \in X \mid A(x) = x \} \]
\[ A^{n+1} := A^n, \quad A^0 := 1_X, \quad A^1 := A, \quad n \in \mathbb{N}. \]

Definition 1 (Rus [4] or [5]) Let \((X,d)\) be a metric space. An operator \( A:X \rightarrow X \) is a Picard operator if there exists \( x^* \in X \) such that:

(a) \( F_A = \{ x^* \} \); 
(b) the sequence \( (A^n(x_0))_{n \in \mathbb{N}} \) converges to \( x^* \), for all \( x_0 \in X \).

Definition 2 (Rus [4] or [5]) Let \((X,d)\) be a metric space. An operator \( A:X \rightarrow X \) is a weakly Picard operator if the sequence \( (A^n(x_0))_{n \in \mathbb{N}} \) converges for all \( x_0 \in X \) and the limit (which may depend on \( x_0 \)) is a fixed point of \( A \).

If \( A \) is a weakly Picard operator, then we consider the following operator

\[ A^\infty : X \rightarrow X, \quad A^\infty (x) = \lim_{n \rightarrow \infty} A^n(x), \]

and it is clear that \( A^\infty (X) = F_A \).

In order to study the differentiability of the solution of the integral equation (1), we use in section 3 the following theorem of differentiability of the solution of the integral equation

\[ x(t) = \int_a^b K(t,s,x(s))ds + f(t), \quad t \in [\alpha, \beta], \]

Theorem 1 (Rus [7]) We suppose that there exists \( Q \in M_{m,n}(\mathbb{R}) \) such that

(i) \( (\beta - \alpha)Q^n \rightarrow 0 \), as \( n \rightarrow \infty \); 
(ii) \( \begin{vmatrix} K_1(t,s,u) - K_1(t,s,v) \\ K_m(t,s,u) - K_m(t,s,v) \end{vmatrix} \leq Q \begin{vmatrix} u - v \\ u - v \end{vmatrix} \) 

for all \( t, s \in [\alpha, \beta], u, v \in \mathbb{R}^m \).
We suppose that the following fiber generalized contractions hold:

\[ x^{n+1}(t; a, b) = \frac{b}{a} K(t, s, x^n(s; a, b)) ds + f(t) \]  

for all \( x^0 \in C([\alpha, \beta], \mathbb{R}^m) \) the sequence \( (x^n)_{n \in \mathbb{N}} \), defined by

\[ x^n(t; a, b) = \frac{b}{a} K(t, s, x^n(s; a, b)) ds + f(t) \]  

converges uniformly to \( x^* \), for all \( t, a, b \in [\alpha, \beta] \) and

\[ \left| x^n(t; a, b) - x^*(t; a, b) \right| \leq \left[ (1 - \beta - \alpha) Q \right] \left| x^n(t; a, b) - x^*(t; a, b) \right| \]  

for all \( t, a, b \in [\alpha, \beta] \).

Then, the operator \( A \) is weakly Picard operator. Moreover, if \( B \) is Picard operator, then \( A \) is Picard operator.

– and Perov’s fixed point theorem:

**Theorem 3** (Perov) Let \((X, d)\) be a complete generalized metric space with \( d(x, y) \in \mathbb{R}^m \) and \( A : X \rightarrow X \) an operator. We suppose that there exists a matrix \( Q \in M_{mm}(\mathbb{R}) \) such that

(i) \( d(A(x), A(y)) \leq Q d(x, y) \), for all \( x, y \in X \);

(ii) \( Q^n \rightarrow 0 \) as \( n \rightarrow \infty \).

Then

(a) \( F_I = \{ x^* \} \);

(b) \( A^n(x) \rightarrow x^* \) as \( n \rightarrow \infty \) and

\[ d(A^n(x), x^*) \leq (I - Q)^n d(x_0, A(x_0)). \]

The main result

We consider the equation (1) in the following conditions:

(c) \( K \in C([\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \); \( R \);

(c) \( f \in C([\alpha, \beta], \mathbb{R}) \);

(c) \( g \in C([\alpha, \beta]), \alpha \leq g(s) \leq b, s \in [a, b] \).

The main result of this paper is the following theorem:

**Theorem 4** We suppose that there exists \( Q \in M_{mm}(\mathbb{R}) \) such that

(i) \( (4 - \beta - \alpha) Q^n \rightarrow 0 \), as \( n \rightarrow \infty \);

(ii) \[ \left| K_i(t, s, u_i, u_2, u_3, u_4) - K_i(t, s, v_i, v_2, v_3, v_4) \right| \leq \left| K_i(t, s, u_i, u_2, u_3, u_4) - K_i(t, s, v_i, v_2, v_3, v_4) \right| \]

\[ \leq Q \left| u_{i1} - v_{i1} \right| + \left| u_{i2} - v_{i2} \right| + \left| u_{i3} - v_{i3} \right| + \left| u_{i4} - v_{i4} \right| \]

for all \( t, s \in [\alpha, \beta], u_i, v_i \in \mathbb{R}^m, i = 1, 4 \).

Then

(a) the integral equation (1) has a unique solution \( x^* \in C([\alpha, \beta], \mathbb{R}^m) \);

(b) for all \( x^0 \in C([\alpha, \beta], \mathbb{R}^m) \), the sequence \( (x^n)_{n \in \mathbb{N}} \), defined by
\[
x^{a+1}(t,a,b) := \int_a^b K(t,s,x^a(s,a,b),x^a(g(s);a,b),x^a(a;a,b),x^a(b;a,b))ds + f(t)
\]
and converges uniformly to \(x^*\), for all \(t,a,b \in [\alpha, \beta]\).

(c) the function
\[
x^*([\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}^n,
(t,a,b) \to x^*(t,a,b)
\]
is continuous;

(d) if
\[
K(t,s,\cdots,\cdots) \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^m)
\]
for all \(t,s \in [\alpha, \beta]\),
then
\[
x^*(t,\cdots,\cdots) \in C^1([\alpha, \beta] \times [\alpha, \beta], \mathbb{R}^m),
\]
for all \(t \in [\alpha, \beta]\).

Proof: We denote \(X := C([\alpha, \beta]^3, \mathbb{R}^m)\). Let \(\|\cdot\|\) be a Chebyshev norm on \(X\), i.e.
\[
\|x\| = \left(\|x\|_1^2, \ldots, \|x\|_m^2\right).
\]

Let we consider the operator \(B: X \to X\) defined by
\[
B(x)(t,a,b) := \int_a^b K(t,s,x(s,a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))ds
\]
for all \(t,a,b \in [\alpha, \beta]\).

From conditions (i), (ii) and Perov’s theorem we have (a)+(b)+(c).

(d) Let we prove that there exists
\[
\frac{\partial x^*}{\partial a}, \frac{\partial x^*}{\partial b} \in X.
\]

If we suppose that there exists \(\frac{\partial x^*}{\partial a}\), then from (1) we have
\[
\frac{\partial x^*}{\partial a}(t,a,b) = -K(t,a,x^*(a,a,b),x^*(g(a);a,b),x^*(a;a,b),x^*(b;a,b)) +
\frac{\partial K}{\partial (g(a);a,b),x^*(a;a,b),x^*(b;a,b),x^*(b;a,b)}
\]

This relation suggest to consider the operator
\(C: X \times X \to X\), defined by
\[
C(x,y)(t,a,b) := -K(t,a,x(a;a,b),x(g(a);a,b),x(a;a,b),x(b;a,b)) +
\frac{\partial K}{\partial (g(a);a,b),x^*(a;a,b),x^*(b;a,b),x^*(b;a,b)}
\]

From condition (ii) it results that
\[
\left|\frac{\partial K}{\partial (u_i;u_j,u_k;u_k)}\right| \leq Q
\]
for all \(t,s \in [\alpha, \beta]\), \(u_i \in \mathbb{R}^m, i = 1,4\).
From conditions (6) and (7) it results that
\[ \| C(x_1,y_1) - C(x_2,y_2) \| \leq 4(\beta-\alpha)Q \]
for all \( x_1, y_1, y_2 \in X \).

Now, if we consider the operator
\[ A : X \times X \to X \times X, \quad A = (B,C), \]
then the conditions of the fiber generalized contractions theorem are satisfied. By this theorem it results that the operator \( A \) is a Picard operator and the sequences

For all \( x_0, y_0 \in X \),

If we take \( x_0 = y_0 = 0 \), then \( y^n = \frac{\partial x^n}{\partial a} \) and one proofs by induction that \( y^n = \frac{\partial x^n}{\partial a} \).

So we have
\[ x^n \stackrel{\text{uniformly}}{\to} x^* \quad \text{as} \quad n \to \infty \]
\[ \frac{\partial x^n}{\partial a} \stackrel{\text{uniformly}}{\to} y^* \quad \text{as} \quad n \to \infty \]
and so it results that there exists \( \frac{\partial x^*}{\partial a} \) and \( \frac{\partial x^*}{\partial a} = y^* \).

References: