The Differentiability of the Solution of a Nonlinear Integral Equation

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Abstract: Using Perov's theorem, a fiber generalized contractions theorem and its application given by Rus [7], a theorem of differentiability of the solution of the following nonlinear integral equation

$$x(t)=\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds+f(t),\quad t\in[\alpha,\beta],$$

is given.

Key-Words: nonlinear integral equation, data dependence

1 Introduction

We consider the nonlinear integral equation

$$x(t)=\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds+f(t)\,, \qquad (1)$$

where $\alpha,\beta\in\mathbf{R}$, $\alpha\le\beta$, $a,b\in[\alpha,\beta]$ and $x\in C([\alpha,\beta],\mathbf{R^m})$.

In the paper [2] have been studied the existence and uniqueness of the solution, continuous data dependence of the solution and the approximation of the solution of this nonlinear integral equation.

The integral equations of this type have been studied in [1], [3], [4], [5], [6], [7], [8], [9].

The purpose of this paper is to give a theorem of differentiability of the solution of the equation (1).

2 Notations and preliminaries

Let X be a nonempty set, d a metric on X and $A:X\to X$ an operator. In this paper we shall use the following notations:

$F_A:=\{x\in X \mid A(x)=x\}$ – the fixed point set of A

$A^{n+1}:=A\circ A^n$, $A^0:=1_X$, $A^1:=A$, $n\in\mathbf{N}$.

Definition 1 (Rus [4] or [5]) *Let* (X,d) *be a metric space. An operator* $A:X\to X$ *is* ***Picard operator*** *if there exists* $x^*\in X$ *such that*:

(a) $F_A=\{x^*\}$;

(b) *the sequence* $(A^n(x_0))_{n\in\mathbf{N}}$ *converges to* x^*, *for all* $x_0\in X$.

Definition 2 (Rus [4] or [5]) *Let* (X,d) *be a metric space. An operator* $A:X\to X$ *is* ***weakly Picard operator*** *if the sequence* $(A^n(x_0))_{n\in\mathbf{N}}$ *converges for all* $x_0\in X$ *and the limit* (*which may depend on* x_0) *is a fixed point of* A.

If A is a weakly Picard operator, then we consider the following operator

$$A^\infty : X\to X,\quad A^\infty(x)=\lim_{n\to\infty}A^n(x)\,,$$

and it is clear that $A^\infty(X)=F_A$.

In order to study the differentiability of the solution of the integral equation (1), we use in section 3 the following theorem of differentiability of the solution of the integral equation

$$x(t)=\int_a^b K(t,s,x(s))ds+f(t),\quad t\in[\alpha,\beta], \qquad (2)$$

Theorem 1 (Rus [7]) *We suppose that there exists* $Q\in M_{mm}(\mathbf{R}_+)$ *such that*

(i) $[(\beta-\alpha)Q]^n\to 0$, as $n\to\infty$;

(ii) $$\begin{pmatrix} |K_1(t,s,u)-K_1(t,s,v)| \\ \cdots\cdots\cdots\cdots \\ |K_m(t,s,u)-K_m(t,s,v)| \end{pmatrix} \le Q \begin{pmatrix} |u_1-v_1| \\ \cdots\cdots \\ |u_m-v_m| \end{pmatrix}$$

for all $t,s\in[\alpha,\beta]$, $u,v\in\mathbf{R^m}$.

Then

(a) *the integral equation* (2) *has a unique solution* $x^*(\cdot;a,b) \in C([\alpha,\beta],\mathbf{R^m})$;

(b) *for all* $x^0 \in C([\alpha,\beta],\mathbf{R^m})$ *the sequence* $(x^n)_{n\in\mathbf{N}}$, *defined by*

$$x^{n+1}(t;a,b) = \int_a^b K(t,s,x^n(s;a,b))ds + f(t) \qquad (3)$$

converges uniformly to x^*, *for all* $t,a,b \in [\alpha,\beta]$ *and*

$$\begin{pmatrix} |x_1^n(t;a,b) - x_1^*(t;a,b)| \\ \dots\dots\dots\dots\dots \\ |x_m^n(t;a,b) - x_m^*(t;a,b)| \end{pmatrix} \le$$

$$\le [I - (\beta-\alpha)Q]^{-1}[(\beta-\alpha)Q]^n \begin{pmatrix} |x_1^0(t;a,b) - x_1^1(t;a,b)| \\ \dots\dots\dots\dots\dots \\ |x_m^0(t;a,b) - x_m^1(t;a,b)| \end{pmatrix};$$

(c) *the function*

$$x^*:[\alpha,\beta]\times[\alpha,\beta]\times[\alpha,\beta] \to \mathbf{R^m},$$

$$(t;a,b) \mapsto x^*(t;a,b)$$

is continuous;

(d) *if* $K(t,s,\cdot) \in C^1(\mathbf{R^m},\mathbf{R^m})$ *for all* $t,s \in [\alpha,\beta]$, *then*

$$x^*(t;\cdot,\cdot) \in C^1([\alpha,\beta]\times[\alpha,\beta],\mathbf{R^m}),$$

for all $t \in [\alpha,\beta]$.

In the proof of this theorem one uses:

– the following fiber generalized contractions theorem:

Theorem 2 (Rus [7]) *Let* (X,d) *be a metric space* (*generalized or not*) *and* (Y,ρ) *be a complet generalized metric space* ($\rho(x,y) \in \mathbf{R^m}$).

Let $A:X\times Y \to X\times Y$ *be a continuous operator*.

We suppose that:

(i) $A(x,y)=(B(x),C(x,y))$, *for all* $x\in X$, $y\in Y$;

(ii) $B:X \to X$ *is a weakly Picard operator*;

(iii) *there exists a matrix* $Q \in M_{mm}(\mathbf{R}_+)$, $Q^n \to 0$ *as* $n\to\infty$, *such that*

$$\rho(C(x,y_1),C(x,y_2)) \le Q\rho(y_1,y_2),$$

for all $x\in X$, $y_1, y_2 \in Y$.

Then, *the operator* A *is weakly Picard operator*. *Moreover*, *if* B *is Picard operator*, *then* A *is Picard operator*.

– and Perov's fixed point theorem:

Theorem 3 (Perov) *Let* (X,d) *be a complet generalized metric space with* $d(x,y) \in \mathbf{R^m}$ *and* $A:X \to X$ *an operator*. *We suppose that there exists a matrix* $Q \in M_{mm}(\mathbf{R}_+)$ *such that*

(i) $d(A(x),A(y)) \le Qd(x,y)$, *for all* $x,y \in X$;

(ii) $Q^n \to 0$ *as* $n\to\infty$.

Then

(a) $F_A = \{x^*\}$;

(b) $A^n(x) \to x^*$ *as* $n\to\infty$ *and*

$$d(A^n(x), x^*) \le (I - Q)^{-1}Q^n d(x_0,A(x_0)).$$

3 The main result

We consider the equation (1) in the following conditions:

(c_1) $K \in C([\alpha,\beta]\times[\alpha,\beta]\times\mathbf{R^m}\times\mathbf{R^m}\times\mathbf{R^m}\times\mathbf{R^m},\mathbf{R^m})$;

(c_2) $f \in C([\alpha,\beta],\mathbf{R^m})$;

(c_3) $g \in C([\alpha,\beta])$, $a \le g(s) \le b$, $s \in [a,b]$.

The main result of this paper is the following theorem:

Theorem 4 *We suppose that there exists* $Q \in M_{mm}(\mathbf{R}_+)$ *such that*

(i) $[4(\beta-\alpha)Q]^n \to 0$, *as* $n \to \infty$;

(ii) $$\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3,u_4) - K_1(t,s,v_1,v_2,v_3,v_4)| \\ \dots\dots\dots\dots\dots \\ |K_m(t,s,u_1,u_2,u_3,u_4) - K_m(t,s,v_1,v_2,v_3,v_4)| \end{pmatrix} \le$$

$$\le Q \begin{pmatrix} |u_{11}-v_{11}| + |u_{21}-v_{21}| + |u_{31}-v_{31}| + |u_{41}-v_{41}| \\ \dots\dots\dots\dots\dots \\ |u_{1m}-v_{1m}| + |u_{2m}-v_{2m}| + |u_{3m}-v_{3m}| + |u_{4m}-v_{4m}| \end{pmatrix}$$

for all $t,s \in [\alpha,\beta]$, $u_i,v_i \in \mathbf{R^m}$, $i = \overline{1,4}$.

Then

(a) *the integral equation* (1) *has a unique solution* $x^*(\cdot;a,b) \in C([\alpha,\beta],\mathbf{R^m})$;

(b) *for all* $x^0 \in C([\alpha,\beta],\mathbf{R^m})$, *the sequence* $(x^n)_{n\in\mathbf{N}}$, *defined by*

$$x^{n+1}(t;a,b) := = \int_a^b K(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))ds + +f(t) \qquad (4)$$

converges uniformly to x^*, *for all* $t,a,b \in [\alpha,\beta]$ *and*

$$\begin{pmatrix} \left|x_1^n(t;a,b) - x_1^*(t;a,b)\right| \\ \dots\dots\dots\dots\dots \\ \left|x_m^n(t;a,b) - x_m^*(t;a,b)\right| \end{pmatrix} \le$$

$$\le \left[I - 4(\beta-\alpha)Q\right]^{-1}\left[4(\beta-\alpha)Q\right]^n \begin{pmatrix} \left|x_1^0(t;a,b) - x_1^1(t;a,b)\right| \\ \dots\dots\dots\dots\dots \\ \left|x_m^0(t;a,b) - x_m^1(t;a,b)\right| \end{pmatrix};$$

(c) *the function*

$$x^*:[\alpha,\beta]\times[\alpha,\beta]\times[\alpha,\beta] \to \mathbf{R^m},$$
$$(t;a,b) \mapsto x^*(t;a,b)$$

is continuous;

(d) *if*

$$K(t,s,\cdot,\cdot,\cdot,\cdot) \in C^1(\mathbf{R^m}\times\mathbf{R^m}\times\mathbf{R^m}\times\mathbf{R^m},\mathbf{R^m})$$

for all $t,s \in [\alpha,\beta]$,

then

$$x^*(t;\cdot,\cdot) \in C^1([\alpha,\beta]\times[\alpha,\beta],\boldsymbol{R^m}),$$

for all $t \in [\alpha,\beta]$.

Proof: We denote $X := C([\alpha,\beta]^3,\mathbf{R^m})$. Let $\|\cdot\|$ be a Chebyshev norm on X, i.e.

$$\|x\| := \begin{pmatrix} \|x\|_1^\infty \\ \dots \\ \|x\|_m^\infty \end{pmatrix}.$$

Let we consider the operator $B:X \to X$ defined by

$$B(x)(t;a,b) := = \int_a^b K(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))ds \qquad (5)$$

for all $t,a,b \in [\alpha,\beta]$.

From conditions (i), (ii) and Perov's theorem we have (a)+(b)+(c).

(d) Let we prove that there exists $\frac{\partial x^*}{\partial a}$, $\frac{\partial x^*}{\partial b} \in X$.

If we suppose that there exists $\frac{\partial x^*}{\partial a}$, then from (1) we have

$$\frac{\partial x^*(t;a,b)}{\partial a} =$$
$$= -K(t,a,x^*(a;a,b),x^*(g(a);a,b),x^*(a;a,b),x^*(b;a,b)) +$$
$$+\int_a^b \left[\frac{\partial K_j(t,s,x^*(s;a,b),x^*(g(s);a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i}\cdot\right.$$
$$\cdot\frac{\partial x^*(s;a,b)}{\partial a} +$$
$$+\frac{\partial K_j(t,s,x^*(s;a,b),x^*(g(s);a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i}\cdot$$
$$\cdot\frac{\partial x^*(g(s);a,b)}{\partial a} +$$
$$+\frac{\partial K_j(t,s,x^*(s;a,b),x^*(g(s);a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i}\cdot$$
$$\cdot\frac{\partial x^*(a;a,b)}{\partial a} +$$
$$+\frac{\partial K_j(t,s,x^*(s;a,b),x^*(g(s);a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i}\cdot$$
$$\left.\cdot\frac{\partial x^*(b;a,b)}{\partial a}\right]ds .$$

This relation sugest to consider the operator $C:X\times X \to X$, defined by

$$C(x,y)(t;a,b) :=$$
$$= -K(t,a,x(a;a,b),x(g(a);a,b),x(a;a,b),x(b;a,b)) +$$
$$+\int_a^b \left[\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))}{\partial x_i}\cdot\right.$$
$$\cdot y(s;a,b) +$$
$$+\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))}{\partial x_i}\cdot$$
$$\cdot y(g(s);a,b) +$$
$$+\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))}{\partial x_i}\cdot$$
$$\cdot y(a;a,b) +$$
$$+\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))}{\partial x_i}\cdot$$
$$\left.\cdot y(b;a,b)\right]ds . \qquad (6)$$

From condition (ii) it results that

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3,u_4)}{\partial x_i}\right|\right) \le Q \qquad (7)$$

for all $t,s \in [\alpha,\beta]$, $u_i \in \mathbf{R^m}$, $i = \overline{1,4}$.

From conditions (6) and (7) it results that

$$\| C(x,y_1)-C(x,y_2) \| \leq 4(\beta-\alpha)Q$$

for all $x,y_1,y_2\in X$.

Now, if we consider the operator

$$A:X\times X \to X\times X, \quad A=(B,C),$$

then the conditions of the fiber generalized contractions theorem are satisfied. By this theorem it results that the operator A is a Picard operator and the sequences

$$x^{n+1}(t;a,b) :=$$

$$= \int_a^b K(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))ds + f(t)$$

$$y^{n+1}(t;a,b) :=$$

$$= -K(t,a,x^n(a;a,b),x^n(g(a);a,b),x^n(a;a,b),x^n(b;a,b)) +$$

$$+\int_a^b \left[\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial x_i} \cdot \right.$$

$$\cdot y(s;a,b) +$$

$$+\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial x_i} \cdot$$

$$\cdot y(g(s);a,b) +$$

$$+\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial x_i} \cdot$$

$$\cdot y(a;a,b) +$$

$$+\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial x_i} \cdot$$

$$\left. \cdot\, y(b;a,b) \right] ds$$

converges uniformly (with respect to $t,a,b\in[\alpha,\beta]$) to $(x^*,y^*)\in F_A$ for all $x^0,y^0\in X$.

If we take $x^0 = y^0 = 0$, then $y^1 = \frac{\partial x^1}{\partial a}$ and one proofs by induction that $y^n = \frac{\partial x^n}{\partial a}$.

So we have

$$x^n \xrightarrow{uniformly} x^* \quad \text{as } n \to \infty$$

$$\frac{\partial x^n}{\partial a} \xrightarrow{uniformly} y^* \quad \text{as } n \to \infty$$

and so it results that there exists $\frac{\partial x^*}{\partial a}$ and $\frac{\partial x^*}{\partial a} = y^*$.

By a similar reasoning one proofs that there exists $\frac{\partial x^*}{\partial b}$. □

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