Stochastic Finite Element Based on Stochastic Linearization for Stochastic Nonlinear Ordinary Differential Equations with Random coefficients

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Abstract: We propose an algorithm for solving stochastic nonlinear ordinary differential equations with random coefficients. This algorithm is based on stochastic linearization and solution of the linearized equation by stochastic finite element method. The deterministic coefficient of the nonlinear term is divided into n levels and the coefficient of the linearization at each level depends on the moments of the solution at the previous level. Numerical examples are illustrated to choose the optimum n levels for each problem.

Key-Words: Stochastic linearization; stochastic finite element; Karhunen- Loeve expansion; chaos polynomials.

1 Introduction

The stochastic problem can be modeled as a differential or integral equation, linear or nonlinear [1]. The objective of solving a stochastic differential equation is to obtain the different moments of the solution process and to identify its transition probability density function. For nonlinear problems, approximate methods should be introduced to have an approximate form for the moments of the solution process. For example: the stochastic averaging [2-4], stochastic linearization [5-7], Adomian's decomposition method [8,9] and stochastic finite element method [10-13]. In this paper, the stochastic nonlinear ordinary differential with stochastic coefficient is replaced by multi-levels linearized equations. The linearized equation is solved using stochastic finite element method based on discretizing the stochastic coefficient by Karhunen-Loeve expansion and projecting the solution on two dimension-first order chaos polynomials.

2 Random Field

2.1 Definition

A random field can be viewed as the spatial extension of a random variable. In fact, it describes the spatial correlation of structural parameters that randomly fluctuate. The random field $V(x,\theta)$ is defined by its mean function $E(x)$ and its covariance function

$$C(x_1,x_2)=E\left[\left(V(x_1,\theta)-E[x_1]\right)\left(V(x_2,\theta)-E[x_2]\right)\right] \quad (1)$$

The θ dependence suggests the randomness. For instance, the first order autoregressive field (Markov process) is often used in numerical applications. Its covariance function is given by

$$C(x_1,x_2)=\sigma_v^2 \exp\left(-\frac{d(x_1,x_2)}{l_{cor}}\right), \quad 0\le x_1,x_2\le L \quad (2)$$

where $d(.,.)$ is an appropriate distance measure, $\sigma_v^{\ 2}$ is the point variance of the field and l_{cor} is its correlation length.

2.2 Karhunen-Loeve Expansion

The use of Karhunen-Loeve (K-L) expansion with orthogonal deterministic basis functions and uncorrelated random coefficients gained interest because of its bi-orthogonal property that is both the deterministic basis functions and the corresponding random coefficients are orthogonal. Let $\underline{\omega}(x)$ denotes the mean value of $\omega(x,\theta)$ and $C(x_1,x_2)$ denotes its covariance function which is bounded and positive definite. It has spectral decomposition as,

$$C(x_1,x_2)=\sum_{n=1}^{\infty}\lambda_n f_n(x_1) f_n(x_2), \quad (3)$$

where λ_n and $f_n(x)$ are the eigenvalues and the eigenvectors of the covariance kernel, respectively.

They are the solutions of the homogeneous Fredholm integral equation of second kind given by,

$$\int_D C(x_1,x_2) f_n(x_1) dx_1 = \lambda_n f_n(x_2). \quad (4)$$

Clearly, $\omega(x,\theta)$ can be written as,

$$\omega(x,\theta) = \bar{\omega}(x) + \alpha(x,\theta) \quad (5)$$

where $\alpha(x,\theta)$ is a process with zero mean and covariance function $C(x_1,x_2)$. Finally, the K-L decomposition of the field $\alpha(x,\theta)$ is given by,

$$\alpha(x,\theta) = \sum_{n=1}^{\infty} \zeta_n(\theta)\sqrt{\lambda_n} f_n(x), \quad (6)$$

where $\zeta_n(\theta)$ is a set of uncorrelated random variables. For example, the eigenfunctions of the exponential covariance described in (2) will be in the form:

$$f_i(x) = \frac{\omega_i Cos(\omega_i x) + Sin(\omega_i x)}{\sqrt{\frac{\omega^2}{2}\left(1+\frac{Sin(2\omega)}{2\omega}\right)+\frac{1}{2}\left(1-\frac{Sin(2\omega)}{2\omega}\right)+Sin^2(\omega)}}, \quad (7)$$

where ω_i is the solution of nonlinear equation

$$2\omega Cos(\omega) + \left(1-\omega^2\right) Sin(\omega) = 0, \quad (8)$$

Then, the eigenvalues becomes

$$\omega_i^2 = \frac{2-\lambda_i}{\lambda_i}\sigma_v^2 \quad (9)$$

2.3 The Chaos Polynomials

The chaos polynomials are a particular basis of the random variables space based on Hermite polynomials of independent standard random variables $\zeta_1, \zeta_2, \ldots\ldots, \zeta_\infty$. Classically, the one dimension Hermite polynomials are defined by

$$h_n(x) = (-1)^n \frac{d^n (e^{-\frac{1}{2}x^2})}{dx^n} e^{\frac{1}{2}x^2}. \quad (10)$$

The multivariable Hermite polynomials can be defined as tensor product of Hermite polynomials. Consider the multi-index,

$$\alpha = \{\alpha_1, \ldots\ldots, \alpha_m\} \qquad \alpha_i \geq 0, \qquad i = 1:m$$

The multivariable Hermite polynomials associated with this sequence are:

$$H_\alpha = \prod_{i=1}^{M} h_{\alpha_i}(\zeta_i)$$

Finally, any random variable $k(\theta)$ with finite variance can be expressed as

$$k(\theta) = \sum_{i=0}^{\infty} a_i H_i(\zeta) \quad (11)$$

where a_i are deterministic constants and H_i are enumeration of the H_α. The expansion is convergent in the mean square sense. In the application of chaos polynomials, one problem is to select the number of random variables and the order of chaos polynomials. The number of random variables can be selected according to the K-L expansion, but no formula can be utilized for selecting the order [14].

3 Stochastic Nonlinear Ordinary Differential Equation

Consider the following stochastic nonlinear ordinary differential equation

$$LU + \varepsilon U^m = f(x), \quad 0 \leq x \leq L, \quad (12)$$

where L is a stochastic linear differential operator, ε is an arbitrary parameter and m is a positive integer. The corresponding equivalent linearization equation is [12]

$$LU + \alpha_{equ.} U = f(x), \quad 0 \leq x \leq L, \quad (13)$$

where $\alpha_{equ.}$ is the equivalent linearization coefficient which can be found by minimizing the expectation of the mean square error. The error caused by the above replacement is

$$\Phi = \varepsilon U^m - \alpha_{equ.} U,$$

Minimizing the mean square error, we get

$$\alpha_{equ.} = \varepsilon \frac{E\left[U^{m+1}\right]}{E\left[U^2\right]}. \quad (14)$$

Then the stochastic finite element method is employed to calculate the moments of the solution process [15]. Let the solution be projected on 2-dimension-first order chaos polynomials, so the response at every node i of the finite element mesh will be in the form:

$$U_i = a_{0_i} H_0 + a_{1_i} H_1 + a_{2_i} H_2,$$

if $m = 3$, then we need to calculate U_i^2 and U_i^4;

$$U_i^2 = a_{0_i}^2 H_0^2 + a_{1_i}^2 H_1^2 + a_{2_i}^2 H_2^2 + 2a_{0_i} a_{1_i} H_0 H_1 + 2a_{0_i} a_{2_i} H_0 H_2 + 2a_{1_i} a_{2_i} H_1 H_2$$

$$
\begin{aligned}
U_i^4 = {} & a_{0_i}^4 H_0^4 + a_{1_i}^4 H_1^4 + a_{2_i}^4 H_2^4 + 6a_{0_i}^2 a_{1_i}^2 H_0^2 H_1^2 + 6a_{0_i}^2 a_{2_i}^2 H_0^2 H_2^2 \\
& + 6a_{1_i}^2 a_{2_i}^2 H_1^2 H_2^2 + 4a_{0_i}^3 a_{1_i} H_0^3 H_1 + 4a_{0_i}^3 a_{2_i} H_0^3 H_2 \\
& + 4a_{1_i}^3 a_{0_i} H_1^3 H_0 + 4a_{2_i}^3 a_{0_i} H_2^3 H_0 + 4a_{1_i}^3 a_{2_i} H_1^3 H_2 \\
& + 4a_{2_i}^3 a_{1_i} H_2^3 H_1 + 12a_0^2 a_{1_i} a_{2_i} H_0^2 H_1 H_2 \\
& + 12a_1^2 a_{0_i} a_{2_i} H_1^2 H_0 H_2 + 12a_2^2 a_{1_i} a_{0_i} H_2^2 H_1 H_0
\end{aligned}
$$

So the required moments at each node i of the mesh are:

$$E\left[U_i^2\right] = a_{0_i}^2 + a_{1_i}^2 + a_{2_i}^2 , \tag{15}$$

$$E\left[U_i^4\right] = a_{0_i}^4 + a_{1_i}^4 + a_{2_i}^4 + 6a_{0_i}^2 a_{1_i}^2 + 6a_{0_i}^2 a_{2_i}^2 + 6a_{1_i}^2 a_{2_i}^2 . \tag{16}$$

At this stage we know the second and fourth moments at every node of the mesh. Then $\alpha_{equ.}$ can be evaluated with the hint of the following proposed approach.

3.1 A Proposed Approach

1) Divide the ε value to n levels such that

$$\varepsilon = n\delta \quad \Rightarrow \quad \delta = \varepsilon/n \tag{17}$$

2) Solve the stochastic linear differential equation

$$LU = f(x), \tag{18}$$

and evaluate the second and fourth moments of the solution process using equations (15) and (16).

3) Transform the nonlinear equation

$$LU + \delta U^3 = f(x), \tag{19}$$

to the corresponding linear one,

$$LU + \alpha_1 U = f(x), \tag{20}$$

where α_1 can be evaluated using the obtained moments in step (2) from relation,

$$\alpha_1 = \delta \frac{E\left[U^4\right]}{E\left[U^2\right]}. \tag{21}$$

We expect that the difference in the solution moments between the equations (18) and (19) is very small for a sufficiently small value of δ and therefore an accurate result can be obtained. The proposed way to get the linearization coefficient is to discretize this coefficient over the domain. So the coefficient of linearization over the $i\,\underline{th}$ element $\alpha_{equ.}(i)$ can be calculated by using linear interpolation between the values of moments as in fig.(1)

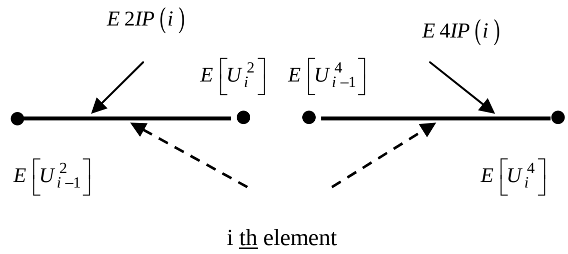

Fig.(1)

$$E\,2IP(i) = \frac{x - x(i)}{-h} E\left(U_{i-1}^2\right) + \frac{x - x(i-1)}{h} E\left(U_i^2\right) \tag{22}$$

$$E\,4IP(i) = \frac{x - x(i)}{-h} E\left(U_{i-1}^4\right) + \frac{x - x(i-1)}{h} E\left(U_i^4\right) \tag{23}$$

where $E\,2IP(i)$ and $E\,4IP(i)$ are the linear interpolating functions of second and fourth moments over $i\,\underline{th}$ element respectively, so

$$\alpha_1(i) = \delta \frac{E\,4IP(i)}{E\,2IP(i)}. \tag{24}$$

4) Solve the linearized differential equation (20) and calculate the second and fourth moments.

5) Repeat steps (3) and transform the nonlinear equation

$$LU + 2\delta U^3 = f(x), \tag{25}$$

to the corresponding linear one,

$$LU + \alpha_2 U = f(x), \tag{26}$$

where α_2 can be evaluated by using the moments obtained in step (4) from relation,

$$\alpha_2(i) = 2\delta \frac{E\,4IP(i)}{E\,2IP(i)}, \tag{27}$$

6) Solve the linearized differential equation (26) and calculate the second and fourth moments.

7) Repeat steps (5) and (6) until we reach the n levels where

$$\alpha_n(i) = n\delta \frac{E\,4IP(i)}{E\,2IP(i)}. \tag{28}$$

The main problem in this approach is to determine the n levels at which the problem converges. This problem is numerically studied in the following examples.

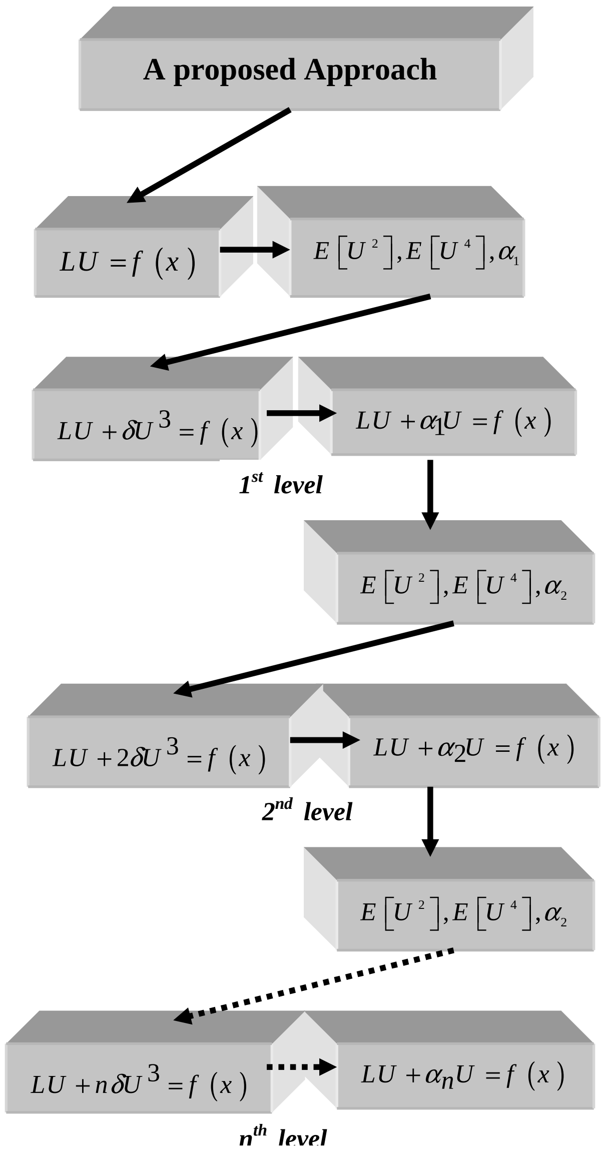

Fig.(2)

3.2 Numerical Examples

- ***Example (1)***

$$-\frac{d}{dx}\left(A\frac{dU}{dx}\right) - 0.5U + 0.5U^3 = 0, \quad 0 \le x \le 2 \qquad (29)$$

subjected to the following boundary conditions $U(0) = 0,\ A\, {dU}/{dx}\big|_{x=2} = 0.21$, where A is a stochastic process with mean one and exponential covariance as in (2). In the suggested algorithm we must choose a suitable value of δ or number of levels n. This choice was studied numerically by drawing the transition values of the mean of solution process at the end of domain $U(2)$ from its linear solution to the nonlinear solution at fixed n.This study is illustrated in figure (3).

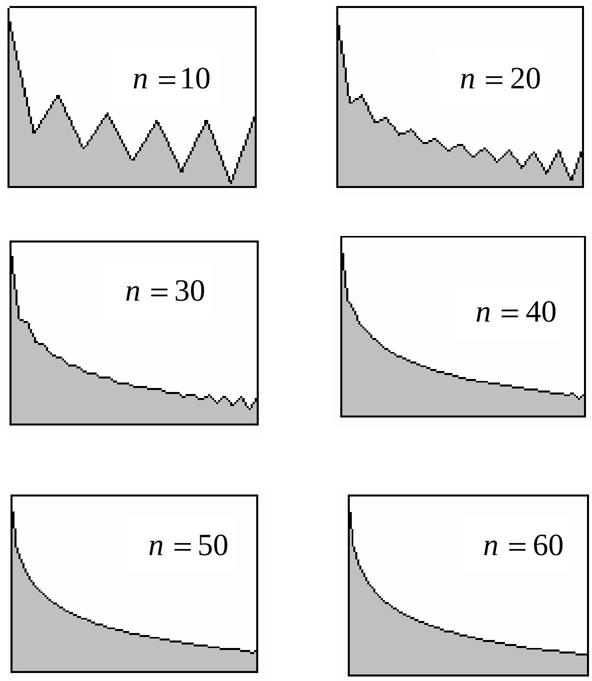

Fig.(3)
The transition of mean of solution at end point over the levels of nonlinearity

The optimum value of n makes the transition of solution to be increasing or decreasing monotonously. So the required value is $n \ge 50$, the mean and standard deviation of solution are illustrated at this level in the next figures.

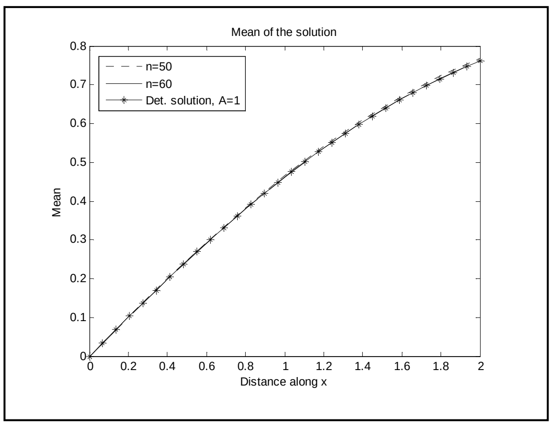

Fig.(4)

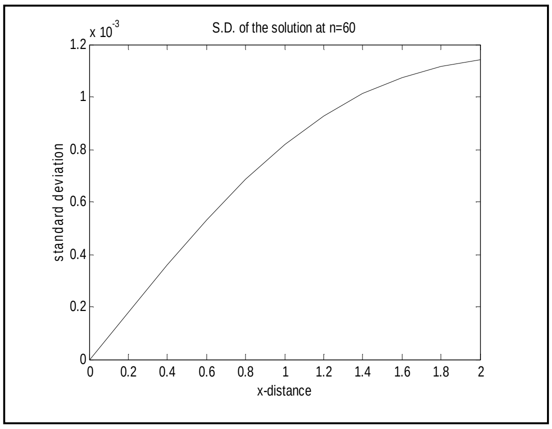

Fig.(5)

If $A = 1$ the equation (29) transformed to the corresponding deterministic one which takes the form

$$-\frac{d^2U}{dx^2} - 0.5U + 0.5U^3 = 0, \qquad (30)$$

which has exact deterministic solution $U = \tanh\left(\frac{x}{2}\right)$.

This deterministic solution was illustrated in fig.(4) and coincide with the mean of stochastic solution at very small σ_v^2.

- ***Example (2)***

$$\frac{d^2}{dx^2}\left(A\frac{d^2U}{dx^2}\right) + U^3 = \sin^3(x) + \sin(x), \quad 0 \le x \le \pi \qquad (31)$$

subjected to the following boundary conditions

$$\begin{cases} U(0) = 0, \quad U(\pi) = 0 \\ A\left.\frac{d^2U}{dx^2}\right|_{x=0} = 0, \quad A\left.\frac{d^2U}{dx^2}\right|_{x=\pi} = 0 \end{cases}$$

Where A is the same as example (1).The transition of the mean of solution process at the middle point of the domain from linear to the nonlinear solution at fixed n is illustrated in figure (6).

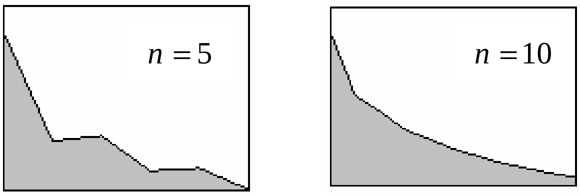

Fig.(6)
The transition of mean of solution at the middle point over the levels of nonlinearity

The required n is $n \ge 10$, the mean and standard deviation of solution are illustrated below at this level.

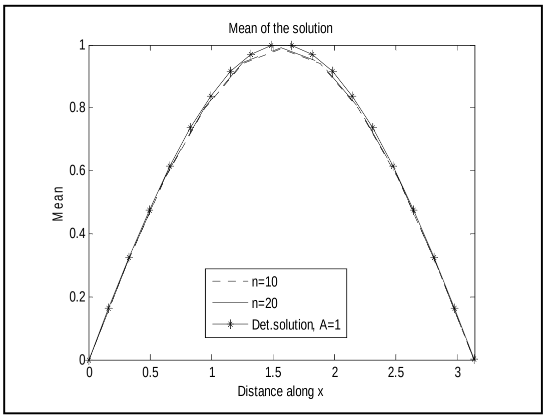

Fig.(7)

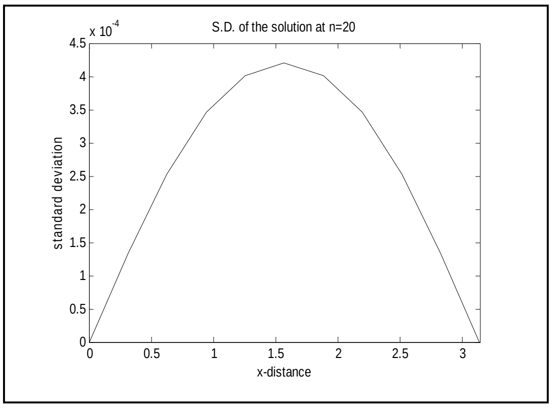

Fig.(8)

If $A = 1$ the equation (31) transformed to corresponding deterministic one which takes the form

$$\frac{d^4U}{dx^4} + U^3 = \sin^3(x) + \sin(x), \qquad (32)$$

which has exact deterministic solution $U = \sin(x)$.

The deterministic solution of (32) was illustrated in fig.(7) and coincide with the mean of stochastic solution at very small σ_v^2. We note from the previous examples that the mean of solution process of the given stochastic differential equations approach the exact solution of the corresponding deterministic equation ($A = 1$) using very small point variance ($\sigma_v^2 = .000001$).

4 Conclusion

The suggested algorithm is an effective tool for solving stochastic nonlinear ordinary differential equations with random coefficient. The choice of the optimum n levels is numerically studied for the selected examples. This choice is based on the transition of mean of solution process at any point from linear to nonlinear case. This transition must be increasing or decreasing monotonously. We can obtain the exact deterministic solution from the stochastic problem by assuming a small value of point variance σ_v^2 of the stochastic process.

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