New and Old Sequences of Orthogonal Polynomials

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Abstract: In this paper, two sequences of orthogonal polynomials on the closed interval $[0,1]$ with respect to the weight function $w(x)=1+\sin(1/x)$ and $w(x)=1+\cos(1/x)$ which were given by Walter Gautschi [3] in 2004 and four new sequences of orthogonal polynomials on the closed interval $[0,1]$ with respect to the weight function $w(x)=\ln(1+x)$, $w(x)=(1/(1+x)$, $w(x)=1+x$ and $w(x)=1-x$ will be introduced. We will show these orthogonal polynomials up to degree seven of each type of orthogonal polynomials. The Gauss-Quadrature formulas of all six sequences of orthogonal polynomials will be presented. One example will be shown.

Key-Word: Orthogonal-polynomial Gauss-Quadrature-Formula Weight-function Definite-integral

1. Introduction

The orthogonal polynomial of degree n on the interval $[a,b]$ with respect to the weight function $w(x)$ is in the form

$p_n(x)=a_0+a_1x+a_2x^2+...+a_nx^n$ which satisfies the following equation

$$\int_a^b w(x)p_n(x)q_m(x)dx=0$$

for all polynomials $q_m(x)$ of degree $m\le n-1$.

The sequence of orthogonal polynomials $\{p_0(x),p_1(x),p_2(x),...,p_n(x),...\}$ is said to be the sequence of orthogonal polynomials on the interval $[a,b]$ with respect to the weight function $w(x)$ if all of orthogonal polynomials $p_k(x)$'s are the orthogonal polynomials on the interval $[a,b]$ with respect to the weight function $w(x)$. Some classical sequences of orthogonal polynomials are the Legendre orthogonal polynomials, the Tschevbyshev orthogonal polynomials, the Laguerre orthogonal polynomials and the Hermite orthogonal polynomials.

Walter Gautshi, [3] in 2004, introduced five sequences of orthogonal polynomials. Two of them are;

First Sequence

$w(x)=1+\sin(1/x)$, a=0 and b=1. The first eight orthogonal polynomials are:

$p_0(x)=1$, $p_1(x)=x-0.5841158916$

$p_2(x)=x^2-1.0475504276x$
$+0.1997528133$

$p_3(x)=x^3-1.1545313388x^2+$
$0.6422638370x-0.0533316912$

$p_4(x)=x^4-2.0809723570x^3+$

$1.4145459603x^2 - 0.3392491803x + 0.017485542076$

$p_5(x) = x^5 - 2.5478870542x^4 + 2.3235876385x^3 - 0.9029927424x^2 + 0.13568352985x - 0.0048259647129$

$p_6(x) = x^6 - 3.0428985420x^5 + 3.5121950160x^4 - 1.9020626509x^3 + 0.479942251797x^2 - 0.047352564962x + 0.0011190032598$

$p_7(x) = x^7 - 3.5765134367x^6 + 5.0800707555x^5 - 3.6338966329x^4 + 1.3651153144x^3 - 0.25301565376x^2 + 0.018807724545x - 0.0003275374172.$

Second Sequence

Let $w(x) = 1 + \cos(1/x)$, $a = 0$ and $b = 1$. The first eight orthogonal polynomials are;

$p_0(x) = 1$, $p_1(x) = x - 0.5658844678$

$p_2(x) = x^2 - 1.0025531762x$ 0.14437123802

$p_3(x) = x^3 - 1.5824780708x^2 + 0.67308939494x - 0.053909500391$

$p_4(x) = x^4 - 2.0076378803x^3 + 1.2840862156x^2 - 0.27811299492x$ 0.013996674327

$p_5(x) = x^5 - 2.5449446465x^4 + 2.2864412946x^3 - 0.84721948143x^2 + 0.11202998832x - 0.0034038523254$

$p_6(x) = x^6 - 3.0923378873x^5 + 3.6348966522x^4 - 2.0092008317x^3 + 0.51848325821x^2 - 0.052316088766x + 0.0012385722196$

$p_7(x) = x^7 - 3.5473940562x^6 + 4.9742231281x^5 - 3.4905806059x^4 + 1.2776224130x^3 - 0.23076198036x^2 + 0.017442871855x - 0.00033263143225.$

2. Formulation

The four new sequences of orthogonal polynomials are as follow.

Third Sequence

Let $w(x) = \ln(1 + x), a = 0$ and $b = 1$. The first eight orthogonal polynomials are;

$p_0(x) = 1$, $p_1(x) = x - 0.6471748624$

$p_2(x) = x^2 - 1.1784205854x$ 0.28549421076

$p_3(x) = x^3 - 1.6921433808x^2 + 0.83043195816x - 0.10754719556$

$p_4(x) = x^4 - 2.1998476752x^3 + 1.6281643893x^2 - 0.45683428963x + 0.037079671768$

$p_5(x) = x^5 - 2.7047767097x^4 + 2.6771330758x^3 - 1.1743702186x^2 + 0.21642930494x - 0.01207637154$

$p_6(x) = x^6 - 3.2081085490x^5 + 3.9765075057x^4 - 2.3854092067x^3 + 0.70653901574x^2 - 0.092676615261x + 0.0037785255799$

$p_7(x) = x^7 - 3.7077775822x^6 + 5.5172180908x^5 - 4.2038882437x^4 + 1.7317172498x^3 - 0.37257032326x^2 + 0.03660654653x - 0.0011358740095.$

Forth Sequence

Let $w(x) = 1(1 + x), a = 0$ and $b = 1$. The first eight orthogonal polynomials are;

$p_0(x) = 1$, $p_1(x) = x - 0.44269504089$

$p_2(x) = x^2 - 0.95420805838x$ 0.14377069586

$p_3(x) = x^3 - 1.4558602304x^2 + 0.55586023045x - 0.042643371741$

$p_4(x) = x^4 - 1.9564125501x^3 + 1.2203331109x^2 - 0.2595618158x$ 0.012106341793

$p_5(x) = x^5 - 2.4566592414x^4 + 2.1355407061x^3 - 0.77760950255x^2 + 0.10666454587x - 0.0033491003385$

$p_6(x) = x^6 - 2.9566246808x^5 + 3.3006526530x^4 - 1.7217923057x^3 + 0.4183994120x^2 - 0.04029083016x + 0.00091012750802$

$p_7(x) = x^7 - 3.4509412972x^6 + 4.6989785354x^5 - 3.1981410765x^4 + 1.1345804466x^3 - 0.19798103583x^2 + 0.014087184608x - 0.00023828540972.$

Fifth Sequence

Let $w(x)=1+x$, $a=0$ and $b=1$. The first eight orthogonal polynomials are;

$p_0(x)=1,\ p_1(x)=x-0.5555555556$

$p_2(x)=x^2-1.0461538462x$
0.19230769231

$p_3(x)=x^3-1.5442176871x^2+$
$0.64625850338x-0.058503401356$

$p_4(x)=x^4-2.0436137058x^3+$
$1.3530633416x^2-0.31390001394x+$
0.016837264417

$p_5(x)=x^5-2.5433478572x^4+$
$2.3108084950x^3-0.89198572618x^2+$
$0.13265451045x-0.0046981131515$

$p_6(x)=x^6-3.0432049788x^5+$
$3.5189761239x^4-1.9180200925x^3+$
$0.49308357123x^2-0.051185854645x+$
0.0012852314539

$p_7(x)=x^7-3.5439806493x^6+$
$4.9772551647x^5-3.5169794010x^4+$
$1.3064011368x^3-0.24152023596x^2+$
$0.018521849998x-0.00034673306861.$

Sixth Sequence

Let $w(x)=1-x, a=0$ and $b=1$. The first eight orthogonal polynomials are;

$p_0(x)=1,\ p_1(x)=x-0.3333333333$

$p_2(x)=x^2-0.8x+0.1$

$p_3(x)=x^3-1.2857142857x^2+$
$0.42857142854x-0.028571428567$

$p_4(x)=x^4-1.7777777777x^3+$
$0.9999999999x^2-0.19047619011x$
0.0079365079098

$p_5(x)=x^5-2.2727272727x^4+$
$1.8181818176x^3-0.60606060531x^2+$
$0.075757575497x-0.002164502145$

$p_6(x)=x^6-2.7692310541x^5+$
$2.8846160397x^4-1.3986019301x^3+$
$0.31468549482x^2-0.027972050936x+$
0.00058275125298

$p_7(x)=x^7-3.2666723575x^6+$
$4.2000161085x^5-2.6923249171x^4+$
$0.89744451892x^3-0.14685516492x^2+$
$0.0097903984629x-0.00015540435754.$

3. Gauss-Quadrature Formula

Gauss-Quadrature formula is the formula for finding the approximated value of the definite integral which is in the form

$$\int_a^b w(x)f(x)dx \cong \sum_{k=1}^{n} A_k f(x_k)$$

where x_k's are the roots of the orthogonal of degree n, $p_n(x)$, on the interval $[a,b]$ with respect to the weight function $w(x)$ and

$$A_k=\frac{1}{p_n'(x_k)}\int_a^b \frac{w(x)p_n(x)}{x-x_k}dx .$$

The following tables are the table of the points x_k's and the values A_k's of the above orthogonal polynomials.

n	x_k	A_k
1	0.5841115892	1.5040670620
2	0.2506677966	0.5859139672
	0.7968826310	0.9181530948
3	0.1101476325	0.2579889065
	0.5422504518	0.7486368341
	0.8929153036	0.4974413214
4	0.0698804533	0.1236260832
	0.3895370944	0.5134598445
	0.6882792319	0.5366523820
	0.9332755773	0.3303287956
5	0.0503983721	0.1286429474
	0.2493865580	0.1946130952
	0.5197200038	0.5104116655
	0.7745856658	0.4530025664
	0.9537964546	0.2173967874
6	0.0336947285	0.0819118792
	0.1554999121	0.1422192026
	0.4165030912	0.3478682108
	0.6345997598	0.4382646843
	0.8355396180	0.3355885032
	0.9670614325	0.1552611734
7	0.0244781324	0.0619784233
	0.1222551603	0.1345865254
	0.3439864043	0.2001624154
	0.5235375430	0.3708879611

	0.7129966707	0.3605139297
	0.8741396960	0.2586986999
	0.9751198301	0.1172391072

Table 1 $w(x) = 1 + \sin(1/x)$

n	x_k	A_k
1	0.5658844678	0.9155890495
2	0.1743102599	0.3673352602
	0.8282429163	0.5482537893
3	0.1037330178	0.2570373743
	0.5751256852	0.3154571744
	0.9036193679	0.3430945008
4	0.0712457030	0.1809064891
	0.2995827557	0.1612913429
	0.7000005130	0.3388785835
	0.9368089087	0.2345126340
5	0.0426068706	0.1495298167
	0.1865068964	0.1834926070
	0.5615813038	0.1937088884
	0.7959116453	0.2754284130
	0.9583379304	0.1580061594
6	0.0333643522	0.0828049217
	0.1614814683	0.1860180498
	0.4292401259	0.0810618444
	0.6535888991	0.2210418342
	0.8455077877	0.2262920526
	0.9691552541	0.1183703467
7	0.0278846339	0.0689129769
	0.1364725480	0.1552107210
	0.2588521809	0.0693047795
	0.5404326356	0.1391694855
	0.7264808632	0.2089587996
	0.8807735572	0.1831712350
	0.9764976373	0.0908611189

Table 2 $w(x) = 1.0 + \cos(1/x)$

n	x_k	A_k
1	0.6471748624	0.3862943611
2	0.3408666667	0.1480657213
	0.8375539221	0.2382286398
3	0.2042632302	0.0595467240
	0.5798240091	0.1793499142
	0.9080561415	0.1473977229
4	0.1350003060	0.0274837856
	0.4074525791	0.1061498039
	0.7162705667	0.1545294120
	0.9411242235	0.0981313597
5	0.0955571603	0.0287483303
	0.2978335287	0.0179725667
	0.5547190456	0.1759616439
	0.7975180625	0.0781862892
	0.9591489125	0.0854255310
6	0.0710768112	0.0080135200
	0.2258215399	0.0382559768
	0.4348406873	0.0806001676
	0.6574445037	0.1083067906
	0.8489051729	0.0994916844
	0.9700198341	0.0516262216
7	0.0546849180	0.0094624243
	0.1760572946	0.0081675490
	0.3465366686	0.0826893148
	0.5409289198	0.0545328509
	0.7295537332	0.1216094850
	0.8829891292	0.0652809768
	0.9770269159	0.0445517605

Table 3 $w(x) = \ln(1 + x)$

n	x_k	A_k
1	0.4426950409	0.6931471806
2	0.1875223329	0.3877545302
	0.7666857255	0.3053926503
3	0.1021075694	0.2546532993
	0.4755563610	0.2064240241
	0.8781963000	0.2320698572
4	0.0639745627	0.1518192987
	0.3125345828	0.2428050791
	0.6538160158	0.2027497061
	0.9260873888	0.0957730967
5	0.0437685530	0.1701542229
	0.2193891476	0.0117797338
	0.4844777785	0.3997131396
	0.7588888436	0.0670194024
	0.9505855593	0.1047981440
6	0.0317942186	0.0783992869
	0.1612524425	0.1497756924
	0.3678584337	0.1694242206
	0.6072566341	0.1477762854
	0.8237663695	0.1022394562
	0.9646965825	0.0455322390
7	0.0239326315	0.0729658933
	0.1226403421	0.0818894321
	0.2855588257	0.1960771316
	0.4870886498	0.0943236633
	0.6927598833	0.1486371376
	0.8655306220	0.0598570823
	0.9734303428	0.0393968413

Table 4 $w(x) = 1/(1 + x)$

n	x_k	A_k

1	0.5555555556	1.5
2	0.2379422616 0.8082115846	0.6645702798 0.8354297202
3	0.1246626933 0.5240683241 0.8954816697	0.3388299020 0.6696384391 0.4915316589
4	0.0755251964 0.3479782555 0.6854382012 0.9346720527	0.2001481734 0.4467514911 0.3619374689 0.3169065886
5	0.0503722277 0.2341409180 0.5153615917 0.7790587816 0.9554141644	0.2129752348 0.0875054844 0.6942901643 0.2221680165 0.2830522478
6	0.0359033397 0.1779236795 0.3936723321 0.6308221731 0.8373197315 0.9675637229	0.0939612192 0.2203584891 0.3278152574 0.3760210773 0.3202922745 0.1615516824
7	0.0268513306 0.1352509100 0.3073502987 0.5112719338 0.7116468084 0.8752079132 0.9755014545	0.0906672885 0.1028309274 0.3440062814 0.2207641972 0.3939811069 0.2115427721 0.1362074265

Table 5 $w(x) = 1 + x$

n	x_k	A_k
1	0.3333333333	0.5
2	0.1550510257 0.6449489743	0.3180413817 0.1819586183
3	0.0885879595 0.4094668644 0.7876594618	0.2009319137 0.2292411064 0.0698269799
4	0.0571041925 0.2768432105 0.5835896594 0.8602407154	0.1355071099 0.2034641164 0.1298477753 0.0311809984
5	0.0398098568 0.1980134173 0.4379748101 0.6954642740 0.9014649144	0.1431357663 0.0513761414 0.2664934720 0.0052156286 0.0337789917
6	0.0293164380 0.1480786471 0.3369847689 0.5586715996 0.7692339096 0.9269456910	0.0723103561 0.1355425231 0.1407925506 0.0986611242 0.0439551482 0.0087382978
7	0.0224795866 0.1146798879 0.2657911819 0.4528477925 0.6473763629 0.8197599106 0.9437376352	0.0682108670 0.0761156802 0.1727707358 0.0668021095 0.0924636101 0.0155965816 0.0080404159

Table 6 $w(x) = 1 - x$

4 Examples

There will be one example in this section. We will use the above forty-two formulas to find the numerical value of the definite integral of this example.

Example

Find the value of the definite integral of $\int_0^1 e^x dx$. The exact value of this integral is $e - 1 \cong 1.7182818285$.

The numerical results of six sequences of orthogonal polynomials are in following table 7 to table 13.

Seq.	Cal. value	Error
1	1.3554399721	0.3628418563
2	2.0031504520	0.2848686235
3	1.4785249984	0.2397568300
4	1.5568974718	0.1613843566
5	1.6806622487	0.0376195798
6	1.0467093188	0.6715725097

Table 7 One Point

Seq.	Cal. Value	Error
1	20.903109934	19.184828106
2	1.1617610644	0.5565207641
3	1.6145728374	0.1037089910
4	1.7168510243	0.0014308041
5	1.7177701651	0.0005116634
6	1.4162739568	0.3020078716

Table 8 Two Points

Seq.	Cal. Value	Error
1	4.0532011032	2.33491927480
2	13.602074563	11.8837927340

3	1.6589743270	0.05930750145
4	1.8498363532	0.13155452469
5	1.5804932963	0.13778853214
6	1.1548389248	0.16989258026

Table 9 Three Points

Seq.	Cal. Value	Error
1	1.5104196718	0.20786215662
2	12.500166725	10.7818848970
3	1.6799305963	0.03835123215
4	1.7182818224	0.00000000060
5	1.5133506554	0.20493117310
6	1.6095525939	0.10872923461

Table10 Four Points

Seq.	Cal. Value	Error
1	11.335524364	9.61724253590
2	1.4789388530	0.23934297542
3	1.7502091195	0.03192729100
4	1.7206991395	0.00241731107
5	1.7185834399	0.00030161148
6	1.8467389015	0.12845707305

Table11 Five Points

Seq.	Cal. Value	Error
1	7.8105147499	6.09223292140
2	1.5816204889	0.13666133957
3	1.6984838139	0.19798014557
4	1.7182818285	0.00000000000
5	1.7182818016	0.00000002689
6	1.6628065746	0.05547525381

Table 12 Six Points

Seq.	Cal. Value	Error
1	1.5562276898	0.1620541386
2	1.7364558151	0.0181739867
3	1.7408927444	0.0226109159
4	1.7182858289	0.0000040004
5	1.7182911291	0.0000093006
6	1.7313799649	0.0130981364

Table 13 Seven Points

5. Conclusion

The results from the example, indicated that the forth and the fifth sequences of the orthogonal polynomials give the expected results but not the first, the second, the third and sixth sequences of the orthogonal polynomials. We strongly recommend the forth sequence and the fifth sequence. The first, the second and the third did not work well because of the term $1/x$ and $\ln(1+x)$.

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